$$n_1 \cdots n_\ell \cdots \in \underbrace{2\mathbb{N}+1} \ast \cdots \ast \underbrace{2\mathbb{N}+1} \cdots \xrightarrow{\Sigma^{\text{odd}}} \mathbb{N}_{>} \ni n_1 + \cdots + n_\ell$$

$$\Sigma_0^{\text{odd}} = 1$$

$$\Sigma_n^{\text{odd}} = \sharp \frac{n_1 \cdots n_\ell \in \underbrace{2\mathbb{N}+1} \ast \cdots \ast \underbrace{2\mathbb{N}+1}}{n_1 + \cdots + n_\ell = n}$$

$$\sum_{n}^{\mathbb{N}} q^n \, \Sigma_n^{\text{odd}} = \prod_{n \geqslant 1} \frac{1}{1 - q^{2n-1}}$$

$$n_1 \cdots n_\ell \cdots \in \underbrace{\mathbb{N}+1} \overset{\neq}{\ast} \cdots \overset{\neq}{\ast} \underbrace{\mathbb{N}+1} \cdots \xrightarrow{\Sigma^{\neq}} \mathbb{N}_{>} \ni n_1 + \cdots + n_\ell$$

$$\Sigma_0^{\neq} = 1$$

$$\Sigma_n^{\neq} = \sharp \frac{n_1 \cdots n_\ell \in \underbrace{\mathbb{N}+1} \overset{\neq}{\ast} \cdots \overset{\neq}{\ast} \underbrace{\mathbb{N}+1}}{n_1 + \cdots + n_\ell = n}$$

$$\sum_{n}^{\mathbb{N}} q^n \, \Sigma_n^{\neq} = \prod_{n \geqslant 1} (1 + q^n)$$

$$\prod_{n \geqslant 1} (1 + q^n) = \prod_{n \geqslant 1} \frac{1}{1 - q^{2n-1}}$$

$$1 + q^n = \frac{1 - q^{2n}}{1 - q^n}$$

$$(1+q)\left(1+q^2\right)\left(1+q^3\right) \cdots = \frac{1-q^2}{1-q} \frac{1-q^4}{1-q^2} \frac{1-q^6}{1-q^3} \cdots = \frac{1}{1-q} \frac{1}{1-q^3} \cdots$$

$$\Sigma_n^{\text{odd}} = \Sigma_n^{\neq}$$