difference relation $R$ on $\mathbb{N}\times\mathbb{N}$

$$m{:}p \sim n{:}q \Leftrightarrow m+q=n+p \text{ equ rel}$$

$$m+p=m+p \Rightarrow m{:}p\sim m{:}p \Rightarrow \text{ refl}$$

$$m{:}p\sim n{:}q \Rightarrow m+q=n+p \Rightarrow n+p=m+q \Rightarrow n{:}q\sim m{:}p \Rightarrow \text{ symm}$$

$$m{:}p\sim n{:}q\sim r{:}s \Rightarrow \begin{cases} m+q=n+p \\ n+s=r+q \end{cases} \Rightarrow$$

$$\underbrace{m+s}+q=\underbrace{m+q}+s \underset{\text{Vor}}{=} \underbrace{n+p}+s=\underbrace{n+s}+p \underset{\text{Vor}}{=} \underbrace{r+q}+p=\underbrace{r+p}+q$$

$$\overset{\text{add}}{\underset{\text{cancel}}{\Longrightarrow}} m+s=r+p \Rightarrow m{:}p\sim r{:}s \Rightarrow \text{ trans}$$

difference class $m\ominus p$

$$m{:}p\sim n{:}q \Leftrightarrow m\ominus p\sim n\ominus q$$

$$\mathbb{Z}=\mathbb{N}\ominus\mathbb{N}=\underbrace{\mathbb{N}\times\mathbb{N}} \ulcorner R=\frac{m\ominus p}{m\in\mathbb{N}\ni p}=\{0{:}\pm 1{:}\pm 2{:}\cdots\} \text{ integers}$$

$$m\ominus p\in\mathbb{Z}=\mathbb{N}\ominus\mathbb{N} \underset{\text{surj}}{\overset{\ominus}{\longleftarrow}} \mathbb{N}\times\mathbb{N}\ni m{:}p$$

$$\mathbb{Z}\times\mathbb{Z}\xrightarrow{+}\mathbb{Z}$$

$$\underbrace{m\ominus p}+\underbrace{n\ominus q}=\underbrace{m+n}\ominus\underbrace{p+q} \text{ easy well-def}$$

$$\begin{cases} m\ominus p=\acute{m}\ominus\acute{p} & \Rightarrow m+\acute{p}=\acute{m}+p \\ n\ominus q=\acute{n}\ominus\acute{q} & \Rightarrow n+\acute{q}=\acute{n}+q \end{cases} \Rightarrow$$

$$\underbrace{m+n}+\underbrace{\acute{p}+\acute{q}}=\underbrace{m+\acute{p}}+\underbrace{n+\acute{q}} \underset{\text{Vor+Vor}}{=} \underbrace{\acute{m}+p}+\underbrace{\acute{n}+q}=\underbrace{\acute{m}+\acute{p}}+\underbrace{p+\acute{q}}$$

$$\underset{\text{def}}{\Longrightarrow} \underbrace{m+n}{:}\underbrace{p+q}\sim\underbrace{\acute{m}+\acute{n}}{:}\underbrace{\acute{p}+\acute{q}} \Rightarrow \underbrace{m+n}\ominus\underbrace{p+q}=\underbrace{\acute{m}+\acute{n}}\ominus\underbrace{\acute{p}+\acute{q}}$$

$$\mathbb{Z}\times\mathbb{Z} \overset{\cdot}{\to} \mathbb{Z}$$

$$\underbrace{m\ominus p}_{}\cdot\underbrace{n\ominus q}_{} = \underbrace{m\cdot n+p\cdot q}_{}\ominus\underbrace{m\cdot q+p\cdot n}_{} \text{ hard} \quad \text{well-def}$$

$$\begin{cases} m\ominus p = \acute{m}\ominus\acute{p} & \Longrightarrow m+\acute{p} = \acute{m}+p \\ n\ominus q = \acute{n}\ominus\acute{q} & \Longrightarrow n+\acute{q} = \acute{n}+q \end{cases}$$

$$\Longrightarrow \overbrace{\underbrace{m\cdot n+p\cdot q}_{} + \underbrace{\acute{m}\cdot\acute{q}+\acute{p}\cdot\acute{n}}_{}} + \underbrace{\acute{m}+\acute{p}}_{}\cdot\underbrace{n+q}_{}$$

$$= m\underset{1}{\cdot}n + p\underset{2}{\cdot}q + \acute{m}\underset{3}{\cdot}\acute{q} + \acute{p}\underset{4}{\cdot}\acute{n} + \acute{m}\underset{5}{\cdot}n + \acute{m}\underset{6}{\cdot}q + \acute{p}\underset{7}{\cdot}n + \acute{p}\underset{8}{\cdot}q$$

$$\underset{\text{komm}}{=} m\underset{1}{\cdot}n + \acute{p}\underset{7}{\cdot}n + \acute{m}\underset{6}{\cdot}q + p\underset{2}{\cdot}q + \acute{m}\underset{5}{\cdot}n + \acute{m}\underset{3}{\cdot}\acute{q} + \acute{p}\underset{4}{\cdot}\acute{n} + \acute{p}\underset{8}{\cdot}q$$

$$= \underbrace{m+\acute{p}}_{}n + \underbrace{\acute{m}+p}_{}q + \acute{m}\,\underbrace{n+\acute{q}}_{} + \acute{p}\,\underbrace{\acute{n}+q}_{} \underset{\text{Vor}}{=} \underbrace{\acute{m}+p}_{}n + \underbrace{m+\acute{p}}_{}q + \acute{m}\,\underbrace{\acute{n}+q}_{} + \acute{p}\,\underbrace{n+\acute{q}}_{}$$

$$= \acute{m}\underset{1}{\cdot}n + p\underset{2}{\cdot}n + m\underset{3}{\cdot}q + \acute{p}\underset{4}{\cdot}q + \acute{m}\underset{5}{\cdot}\acute{n} + \acute{m}\underset{6}{\cdot}q + \acute{p}\underset{7}{\cdot}n + \acute{p}\underset{8}{\cdot}\acute{q}$$

$$\underset{\text{komm}}{=} \acute{m}\underset{5}{\cdot}\acute{n} + \acute{p}\underset{8}{\cdot}\acute{q} + m\underset{3}{\cdot}q + p\underset{2}{\cdot}n + \acute{m}\underset{1}{\cdot}n + \acute{m}\underset{6}{\cdot}q + \acute{p}\underset{7}{\cdot}n + \acute{p}\underset{4}{\cdot}q$$

$$= \overbrace{\underbrace{\acute{m}\cdot\acute{n}+\acute{p}\cdot\acute{q}}_{} + \underbrace{m\cdot q+p\cdot n}_{}} + \underbrace{\acute{m}+\acute{p}}_{}\cdot\underbrace{n+q}_{}$$

$$\underset{\text{add cancel}}{\Longrightarrow} \underbrace{m\cdot n+p\cdot q}_{} + \underbrace{\acute{m}\cdot\acute{q}+\acute{p}\cdot\acute{n}}_{} = \underbrace{\acute{m}\cdot\acute{n}+\acute{p}\cdot\acute{q}}_{} + \underbrace{m\cdot q+p\cdot n}_{}$$

$$\underset{\text{def}}{\Longrightarrow} \underbrace{m\cdot n+p\cdot q}_{}:\underbrace{m\cdot q+p\cdot n}_{} \sim \underbrace{\acute{m}\cdot\acute{n}+\acute{p}\cdot\acute{q}}_{}:\underbrace{\acute{m}\cdot\acute{q}+\acute{p}\cdot\acute{n}}_{}$$

$$\Longrightarrow \underbrace{m\cdot n+p\cdot q}_{}\ominus\underbrace{m\cdot q+p\cdot n}_{} = \underbrace{\acute{m}\cdot\acute{n}+\acute{p}\cdot\acute{q}}_{}\ominus\underbrace{\acute{m}\cdot\acute{q}+\acute{p}\cdot\acute{n}}_{}$$

triv difference $m\ominus 0 \in \mathbb{Z} = \mathbb{N}\ominus\mathbb{N} \xleftarrow[\ominus]{} \mathbb{N}\times\mathbb{N} \xleftarrow[:u]{} \mathbb{N} \ni m$

$\ominus 0$

inj

$$\underbrace{m\ominus 0}_{} + \underbrace{n\ominus 0}_{} = \underbrace{m+n}_{}\ominus 0$$

$$\underbrace{m\ominus 0}_{}\times\underbrace{n\ominus 0}_{} = \underbrace{m\times n}_{}\ominus 0$$