$$ {}^{\tau}G_k = \underbrace{\mathbb{Z}\overset{\times}{+}\tau\mathbb{Z}}_{-k} = \sum_{m:n}^{\mathbb{Z}^2\llcorner 0:0} \overbrace{m+\tau n}^{-k} = \sum_{m}^{\mathbb{Z}^\times} \overline{m}^k + \sum_{n}^{\mathbb{Z}^\times}\sum_{m}^{\mathbb{Z}} \overbrace{m+\tau n}^{-k} = \sum_{n}^{\mathbb{Z}^\times}\left(\overline{n}^k + \sum_{m}^{\mathbb{Z}} \overbrace{m+\tau n}^{-k}\right) $$

$$ {}^{\tau+1}G_k = {}^{\tau}G_k $$

$$ \text{LHS} = \sum_{n}^{\mathbb{Z}^\times}\left(\overline{n}^k + \sum_{m}^{\mathbb{Z}} \overbrace{m+\underbrace{1+\tau}n}^{-k}\right) = \sum_{n}^{\mathbb{Z}^\times}\left(\overline{n}^k + \sum_{m}^{\mathbb{Z}} \overbrace{\underbrace{m+n}+\tau n}^{-k}\right) = \sum_{n}^{\mathbb{Z}^\times}\left(\overline{n}^k + \sum_{\ell}^{\mathbb{Z}} \overbrace{\ell+\tau n}^{-k}\right) = \text{RHS} $$

$$ {}^{-1/\tau}G_k = \tau^k\, {}^{\tau}G_k $$

$$ \text{LHS} = \sum_{m:n}^{\mathbb{Z}^2\llcorner 0:0} \overbrace{m-n/\tau}^{-k} = \tau^k \sum_{m:n}^{\mathbb{Z}^2\llcorner 0:0} \overbrace{\tau m-n}^{-k} = \tau^k \sum_{m:\ell}^{\mathbb{Z}^2\llcorner 0:0} \overbrace{\tau m+\ell}^{-k} = \text{RHS} $$

$$ {}^{\infty}G_k = 2\,\overline{\pi}^{2k}\,\zeta_{2k} \in \mathbb{Q} $$

$$ {}^{\infty}G_2 = \frac{1}{315\cdot 16} $$

$$ {}^{\infty}G_3 = \frac{1}{8\cdot 27\cdot 140} $$

$$ {}^{\tau}\Delta = \overbrace{60\,{}^{\tau}G_2}^{3} - 27\,\overbrace{140\,{}^{\tau}G_3}^{2} $$