$$\underline{1-a^2x^2}\,\underline{1-a^2y^2} = 1-a^4 \Leftrightarrow x^2+y^2 = a^2\,\underline{1+x^2y^2}$$

$$\underline{1-a^2x^2}\,\underline{1-a^2y^2} = 1-a^2\,\underline{x^2+y^2-a^2x^2y^2} = 1-a^4 \Leftrightarrow x^2+y^2-a^2\,x^2\,y^2 = a^2 \Leftrightarrow x^2+y^2 = a^2\,\underline{1+x^2y^2}$$

$$\underline{1-x^2u^2}\,\underline{1-x^2v^2} = \underline{1-a^2x^2}\,\underline{1-x^2y^2u^2v^2}$$
$$\underline{1-y^2u^2}\,\underline{1-y^2v^2} = \underline{1-a^2y^2}\,\underline{1-x^2y^2u^2v^2}$$

$$\text{LHS} = 1-x^2\,\underline{u^2+v^2} + x^4\,u^2\,v^2 \underset{\text{uv}}{\overset{\text{Eq}}{=}} 1-x^2\,a^2\,\underline{1+u^2v^2} + x^4\,u^2\,v^2$$
$$= 1-x^2\,a^2 + x^2\,u^2\,v^2\,\underline{x^2-a^2} \underset{\text{xy}}{\overset{\text{Eq}}{=}} 1-x^2\,a^2 + x^2\,u^2\,v^2\,y^2\,\underline{a^2x^2-1} = \text{RHS}$$

$$x|y+u|v = \frac{xv+yu}{a\,(1+xyuv)}|\frac{yv-xu}{a\,(1-xyuv)}\colon \quad az = \frac{xv+yu}{1+xyuv}\colon \quad aw = \frac{yv-xu}{1-xyuv}$$

$$1+az = \frac{(1+xv)\,(1+yu)}{1+xyuv}\colon \quad 1-az = \frac{(1-xv)\,(1-yu)}{1+xyuv}$$
$$1+aw = \frac{(1+yv)\,(1-xu)}{1-xyuv}\colon \quad 1-aw = \frac{(1-yv)\,(1+xu)}{1-xyuv}$$

$$\underline{1-a^2z^2}\,\underline{1-a^2w^2} = 1-a^4$$

$$\text{LHS} = \underline{1+az}\,\underline{1-az}\,\underline{1+aw}\,\underline{1-aw}$$
$$= \frac{(1+xv)\,(1+yu)}{1+xyuv}\,\frac{(1-xv)\,(1-yu)}{1+xyuv}\,\frac{(1+yv)\,(1-xu)}{1-xyuv}\,\frac{(1-yv)\,(1+xu)}{1-xyuv}$$
$$= \frac{\left(1-x^2u^2\right)\left(1-x^2v^2\right)}{1-x^2y^2u^2v^2}\,\frac{\left(1-y^2u^2\right)\left(1-y^2v^2\right)}{1-x^2y^2u^2v^2} \underset{\text{Lem}}{=} \underline{1-a^2x^2}\,\underline{1-a^2y^2} = 1-a^4$$

$$\underline{x|y+u|v}+s|t=x|y+\underline{u|v+s|t}$$

$$\text{LHS}\ =\frac{xv+yu}{a\left(1+xyuv\right)}|\frac{yv-xu}{a\left(1-xyuv\right)}+s|t=$$

$$\frac{\frac{xv+yu}{1+xyuv}\frac{t}{a}+\frac{yv-xu}{1-xyuv}\frac{s}{a}}{a\left(1+\frac{xv+yu}{1+xyuv}\frac{yv-xu}{1-xyuv}\frac{s}{a}\frac{t}{a}\right)}|\frac{\frac{yv-xu}{1-xyuv}\frac{t}{a}-\frac{xv+yu}{1+xyuv}\frac{s}{a}}{a\left(1-\frac{xv+yu}{1+xyuv}\frac{yv-xu}{1-xyuv}\frac{s}{a}\frac{t}{a}\right)}=\frac{\frac{xv+yu}{1+xyuv}t+\frac{yv-xu}{1-xyuv}s}{a^2+\frac{xv+yu}{1+xyuv}\frac{yv-xu}{1-xyuv}st}|\frac{\frac{yv-xu}{1-xyuv}t-\frac{xv+yu}{1+xyuv}s}{a^2-\frac{xv+yu}{1+xyuv}\frac{yv-xu}{1-xyuv}st}$$

$$=\frac{\left(xv+yu\right)\left(1-xyuv\right)t+\left(yv-xu\right)\left(1+xyuv\right)s}{a^2\left(1-x^2y^2u^2v^2\right)+\left(xv+yu\right)\left(yv-xu\right)st}|\frac{\left(yv-xu\right)\left(1+xyuv\right)t-\left(xv+yu\right)\left(1-xyuv\right)s}{a^2\left(1-x^2y^2u^2v^2\right)-\left(xv+yu\right)\left(yv-xu\right)st}$$

$$\text{RHS}\ =x|y+\frac{ut+vs}{a\left(1+uvst\right)}|\frac{vt-us}{a\left(1-uvst\right)}=$$

$$\frac{\frac{x}{a}\frac{vt-us}{1-uvst}+\frac{y}{a}\frac{ut+vs}{1+uvst}}{a\left(1+\frac{x}{a}\frac{y}{a}\frac{ut+vs}{1+uvst}\frac{vt-us}{1-uvst}\right)}|\frac{\frac{y}{a}\frac{vt-us}{1-uvst}-\frac{x}{a}\frac{ut+vs}{1+uvst}}{a\left(1-\frac{x}{a}\frac{y}{a}\frac{ut+vs}{1+uvst}\frac{vt-us}{1-uvst}\right)}=\frac{x\frac{vt-us}{1-uvst}+y\frac{ut+vs}{1+uvst}}{a^2+xy\frac{ut+vs}{1+uvst}\frac{vt-us}{1-uvst}}|\frac{y\frac{vt-us}{1-uvst}-x\frac{ut+vs}{1+uvst}}{a^2-xy\frac{ut+vs}{1+uvst}\frac{vt-us}{1-uvst}}$$

$$=\frac{x\left(vt-us\right)\left(1+uvst\right)+y\left(ut+vs\right)\left(1-uvst\right)}{a^2\left(1-u^2v^2s^2t^2\right)+xy\left(ut+vs\right)\left(vt-us\right)}|\frac{y\left(vt-us\right)\left(1+uvst\right)-x\left(ut+vs\right)\left(1-uvst\right)}{a^2\left(1-u^2v^2s^2t^2\right)-xy\left(ut+vs\right)\left(vt-us\right)}$$

$$\frac{\left(xv+yu\right)\left(1-xyuv\right)t+\left(yv-xu\right)\left(1+xyuv\right)s}{a^2\left(1-x^2y^2u^2v^2\right)+\left(xv+yu\right)\left(yv-xu\right)st}=\frac{x\left(vt-us\right)\left(1+uvst\right)+y\left(ut+vs\right)\left(1-uvst\right)}{a^2\left(1-u^2v^2s^2t^2\right)+xy\left(ut+vs\right)\left(vt-us\right)}$$

$$\left(\left(xv+yu\right)\left(1-xyuv\right)t+\left(yv-xu\right)\left(1+xyuv\right)s\right)\left(a^2\left(1-u^2v^2s^2t^2\right)+xy\left(ut+vs\right)\left(vt-us\right)\right)$$

$$=\left(x\left(vt-us\right)\left(1+uvst\right)+y\left(ut+vs\right)\left(1-uvst\right)\right)\left(a^2\left(1-x^2y^2u^2v^2\right)+\left(xv+yu\right)\left(yv-xu\right)st\right)$$

$$\left(\left(xvt+yut+yvs-xus\right)+xyuv\left(yvs-xus-xvt-yut\right)\right)$$

$$\left(\left(a^2+\left(v^2-u^2\right)xyst\right)+uv\left(xy\left(t^2-s^2\right)-a^2uvs^2t^2\right)\right)$$

$$=\left(\left(xvt-xus+yut+yvs\right)+uvst\left(xvt-xus-yut-yvs\right)\right)$$

$$\left(\left(a^2+\left(v^2-u^2\right)xyst\right)+uv\left(\left(y^2-x^2\right)st-a^2x^2y^2uv\right)\right)$$