$$ {}^{z}\mathrm{cot} = z - \frac{z}{3} - \frac{z^3}{72} + \cdot\cdot $$

$$ {}^{z}\mathrm{cot} = \frac{a}{z} + bz + c\,z^3 + \cdot\cdot $$

$$ 1 - \frac{z^2}{2} + \frac{z^4}{24} - \cdot\cdot = {}^{z}\mathfrak{c} = {}^{z}\mathfrak{s}\,{}^{z}\mathrm{cot} = \left(z - \frac{z^3}{6} + \cdot\cdot\right)\left(\frac{a}{z} + bz + cz^3 + \cdot\cdot\right) $$

$$ = a + b\,z^2 + c\,z^4 - \frac{az^2}{6} - \frac{bz^4}{6} - \cdot\cdot = a + \left(b - \frac{a}{6}\right) z^2 + \left(c - \frac{b}{6}\right) z^4 + \cdot\cdot $$

$$ \Longrightarrow a = 1: \quad b - \frac{1}{6} = -\frac{1}{2} \Longrightarrow b = -\frac{1}{3}: \quad c + \frac{1}{18} = \frac{1}{24} \Longrightarrow c = -\frac{1}{72} $$

$$ \mathfrak{o} = \cosh $$

$$ z \in \underset{\vee}{\mathbb{C}} \xrightarrow{\underset{\vee}{\eta}} \mathbb{C} \ni {}^{z}\underset{\vee}{\eta} = \prod_{n \geqslant 1} \underbrace{1 - z^n} $$

$$ \tau \in \underset{\wedge}{\mathbb{C}} \xrightarrow[\text{hol}]{\mathbb{C}^{\mathfrak{e}}} \underset{\vee}{\mathbb{C}} \ni z = {}^{2\pi i\tau}\mathfrak{e} = {}^{\tau}_{\mathbb{C}}\mathfrak{e} $$

$$ {}^{\tau}\eta = {}^{\pi i\tau/12}\mathfrak{e}\; {}^{{}^{\tau}_{\mathbb{C}}\mathfrak{e}}\underset{>}{\eta} = {}^{\pi i\tau/12}\mathfrak{e} \prod_{n \geqslant 1} \underbrace{1 - {}^{\tau}_{\mathbb{C}}\mathfrak{e}^{n}} = {}^{\pi i\tau/12}\mathfrak{e} \prod_{n \geqslant 1} \underbrace{1 - {}^{2\pi i\tau}\mathfrak{e}^{n}} $$

$$ -i\,{}^{z}\mathrm{cot} = -i\frac{{}^{z}\mathfrak{c}}{{}^{z}\mathfrak{s}} = \frac{\mathfrak{e}^{iz} + \mathfrak{e}^{-iz}}{\mathfrak{e}^{iz} - \mathfrak{e}^{-iz}} = \frac{\mathfrak{e}^{i\,(x+iy)} + \mathfrak{e}^{-i\,(x+iy)}}{\mathfrak{e}^{i\,(x+iy)} - \mathfrak{e}^{-i\,(x+iy)}} = \frac{\mathfrak{e}^{ix}\mathfrak{e}^{-y} + \mathfrak{e}^{-ix}\mathfrak{e}^{y}}{\mathfrak{e}^{ix}\mathfrak{e}^{-y} - \mathfrak{e}^{-ix}\mathfrak{e}^{y}} = \begin{cases} \dfrac{\mathfrak{e}^{-2y} + \mathfrak{e}^{-2ix}}{\mathfrak{e}^{-2y} - \mathfrak{e}^{-2ix}} & y > 0 \\ \dfrac{\mathfrak{e}^{2ix} + \mathfrak{e}^{2y}}{\mathfrak{e}^{2ix} - \mathfrak{e}^{2y}} & y < 0 \end{cases} $$

$$ N = (n + \ 1/2)\,\pi $$

$$ {}^{iN\,(\alpha i + \beta r)}\mathrm{cot} = {}^{-N\alpha + iN\beta r}\mathrm{cot} = \begin{cases} \dfrac{\mathfrak{e}^{-2N\beta r} + \mathfrak{e}^{2iN\alpha}}{\mathfrak{e}^{-2N\beta r} - \mathfrak{e}^{2iN\alpha}} \rightsquigarrow -1 & \beta > 0 \\ \dfrac{\mathfrak{e}^{-2iN\alpha} + \mathfrak{e}^{2N\beta r}}{\mathfrak{e}^{-2iN\alpha} - \mathfrak{e}^{2N\beta r}} \rightsquigarrow 1 & \beta < 0 \end{cases} $$

$$ {}^{N\,(\alpha i + \beta r)\,/r}\mathrm{cot} = {}^{N\beta + iN\alpha/r}\mathrm{cot} = \begin{cases} \dfrac{\mathfrak{e}^{-2N\alpha/r} + \mathfrak{e}^{-2iN\beta}}{\mathfrak{e}^{-2N\alpha/r} - \mathfrak{e}^{-2iN\beta}} \rightsquigarrow -1 & \alpha > 0 \\ \dfrac{\mathfrak{e}^{2iN\beta} + \mathfrak{e}^{2N\alpha/r}}{\mathfrak{e}^{2iN\beta} - \mathfrak{e}^{2N\alpha/r}} \rightsquigarrow 1 & \alpha < 0 \end{cases} $$

$$ {}^{\alpha i+\beta r}\text{℩}_n = {}^{iN(\alpha i+\beta r)}\cot\ {}^{N(\alpha i+\beta r)/r}\cot $$

$$ 2\,\overbracket{{}^{z}\mathfrak{s}}^{2} = {}^{2y}\mathfrak{o} - {}^{2x}\mathfrak{c} $$

$$ 4\,\overbracket{{}^{z}\mathfrak{s}}^{2} = \overbracket{{}^{ix-y}\mathfrak{e} - {}^{y-ix}\mathfrak{e}}^{2} = \overbracket{{}^{ix}\mathfrak{e}^{-y}\mathfrak{e} - {}^{y}\mathfrak{e}^{-ix}\mathfrak{e}}^{2} = \underbrace{{}^{ix}\mathfrak{e}^{-y}\mathfrak{e} - {}^{y}\mathfrak{e}^{-ix}\mathfrak{e}}\underbrace{{}^{-ix}\mathfrak{e}^{-y}\mathfrak{e} - {}^{y}\mathfrak{e}^{ix}\mathfrak{e}} $$
$$ = {}^{-2y}\mathfrak{e} + {}^{2y}\mathfrak{e} - {}^{-2ix}\mathfrak{e} - {}^{2ix}\mathfrak{e} = 2\,{}^{2y}\mathfrak{o} - 2\,{}^{2x}\mathfrak{c} $$

$$ z = i\left(n+\ 1/2\right)\pi\left(tr+(1-t)\,i\right) \Longrightarrow 2\,\overbracket{{}^{z}\mathfrak{s}}^{2} = {}^{(2n+1)\,\pi tr}\mathfrak{o} + {}^{(2n+1)\,\pi t}\mathfrak{c} $$
$$ z = \left(n+\ 1/2\right)\pi\frac{(1-t)\,r+ti}{r} \Longrightarrow 2\,\overbracket{{}^{z}\mathfrak{s}}^{2} = {}^{(2n+1)\,\pi t/r}\mathfrak{o} + {}^{(2n+1)\,\pi t}\mathfrak{c} $$

$$ 2z = i\,(2n+1)\,\pi\left(tr+(1-t)\,i\right) = -\,(2n+1)\,\pi\,(1-t) + i\,(2n+1)\,\pi tr $$
$$ 2\,\overbracket{{}^{z}\mathfrak{s}}^{2} = {}^{(2n+1)\,\pi tr}\mathfrak{o} - {}^{(2n+1)\,\pi\,(1-t)}\mathfrak{c} = {}^{(2n+1)\,\pi tr}\mathfrak{o} - \underbrace{{}^{(2n+1)\,\pi}\mathfrak{c}}_{=-1}{}^{(2n+1)\,\pi t}\mathfrak{c} - \underbrace{{}^{(2n+1)\,\pi}\mathfrak{s}}_{=0}{}^{(2n+1)\,\pi t}\mathfrak{s} = \text{ RHS} $$
$$ 2z = (2n+1)\,\pi\frac{(1-t)\,r+ti}{r} = (2n+1)\,\pi\,(1-t) + i\,(2n+1)\,\pi t/r $$
$$ 2\,\overbracket{{}^{z}\mathfrak{s}}^{2} = {}^{(2n+1)\,\pi t/r}\mathfrak{o} - {}^{(2n+1)\,\pi\,(1-t)}\mathfrak{c} = {}^{(2n+1)\,\pi t/r}\mathfrak{o} - \underbrace{{}^{(2n+1)\,\pi}\mathfrak{c}}_{=-1}{}^{(2n+1)\,\pi t}\mathfrak{c} - \underbrace{{}^{(2n+1)\,\pi}\mathfrak{s}}_{=0}{}^{(2n+1)\,\pi t}\mathfrak{s} = \text{ RHS} $$

$$ \begin{cases} {}^{sr}\mathfrak{o} + {}^{s}\mathfrak{c} \geqslant \dfrac{\pi^2 r^2}{8} \curlywedge 1 \\ {}^{s/r}\mathfrak{o} + {}^{s}\mathfrak{c} \geqslant \dfrac{\pi^2}{8r^2} \curlywedge 1 \end{cases} $$

$$ s \geqslant \frac{\pi}{2} \Longrightarrow {}^{sr}\mathfrak{o} + \underbrace{{}^{s}\mathfrak{c}}_{\geqslant -1} \geqslant 1 + \frac{\left(sr\right)^2}{2} - 1 = \frac{\left(sr\right)^2}{2} \geqslant \frac{\pi^2 r^2}{8} $$
$$ s \leqslant \frac{\pi}{2} \Longrightarrow \underbrace{{}^{sr}\mathfrak{o}}_{\geqslant 1} + {}^{s}\mathfrak{c} \geqslant 1 + \underbrace{{}^{s}\mathfrak{c}}_{\geqslant 0} \geqslant 1 $$

$$ {}^{(\tau c \overset{-1}{+} d)\,(\tau a+b)}\eta = {}^{\tau}\eta\sqrt{-i\,(\tau c+d)}\exp\left(\pi i\frac{a+d}{12c}\right)\exp\left(\frac{1}{4c}\sum_{1\,\leqslant\, j\,\leqslant\, c-1}\cot\left(-\frac{\pi dj}{c}\right)\cot\left(\frac{\pi j}{c}\right)\right) $$