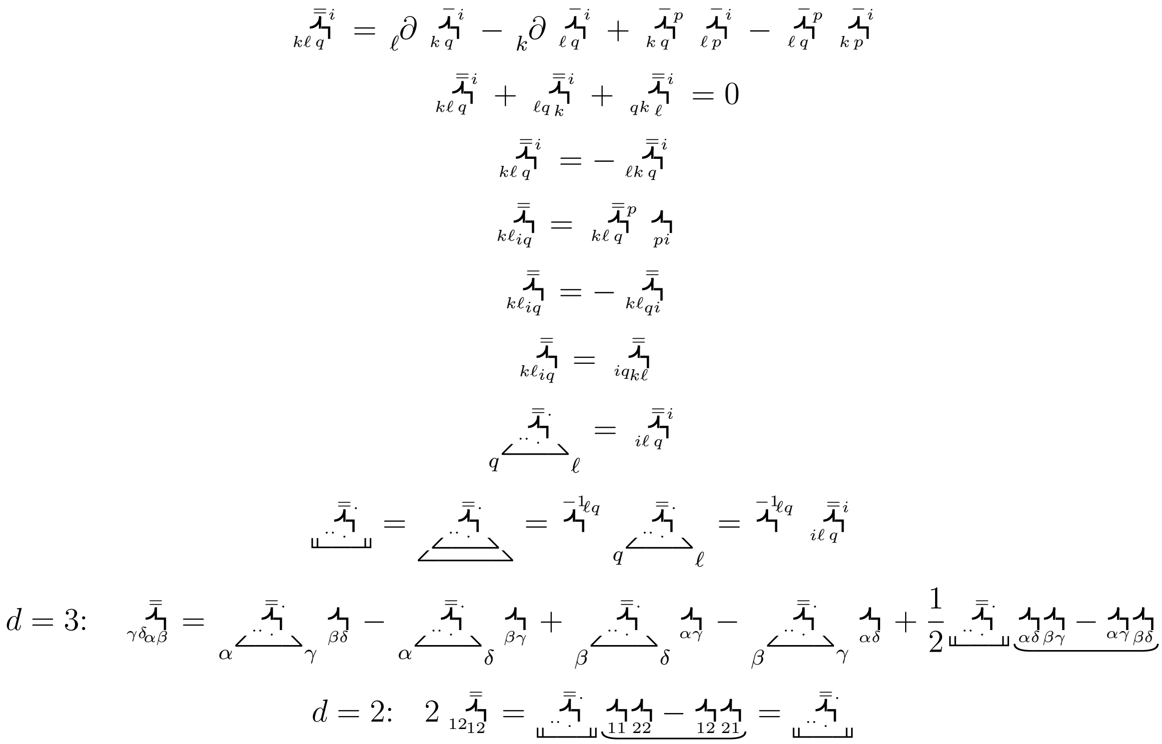

d = 3:
d = 2:

$$ {}_{i}\mathcal{S}_{-}\;{}_{\mu}\mathcal{P} = {}_{\mu}\mathcal{P}_{i}^{\,j}\;{}_{j}\mathcal{S}_{-} $$

matrix-valued column
$$ {}_{\cdot}\mathcal{P} = \left({}_{\mu}\mathcal{P}_{i}^{\,j}\right) = \begin{array}{c} {}_{1}\mathcal{P}_{\cdot}^{\,\cdot} \\ \vdots \\ {}_{n}\mathcal{P}_{\cdot}^{\,\cdot} \end{array} = \begin{array}{c|c|c} \begin{array}{c} {}_{1}\mathcal{P}_{1}^{\,1} \\ \vdots \\ {}_{1}\mathcal{P}_{n}^{\,1} \end{array} & \cdots & \begin{array}{c} {}_{1}\mathcal{P}_{1}^{\,n} \\ \vdots \\ {}_{1}\mathcal{P}_{n}^{\,n} \end{array} \\ \hline \vdots & \ddots & \vdots \\ \hline \begin{array}{c} {}_{n}\mathcal{P}_{1}^{\,1} \\ \vdots \\ {}_{n}\mathcal{P}_{n}^{\,1} \end{array} & \cdots & \begin{array}{c} {}_{n}\mathcal{P}_{1}^{\,n} \\ \vdots \\ {}_{n}\mathcal{P}_{n}^{\,n} \end{array} \end{array} $$

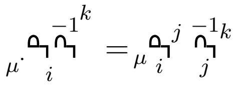
global transpose
$$ \overset{T}{\mathcal{P}} = \begin{array}{c|c|c} \begin{array}{c} {}_{1}\mathcal{P}_{1}^{\,1} \\ \vdots \\ {}_{1}\mathcal{P}_{n}^{\,1} \end{array} & \cdots & \begin{array}{c} {}_{n}\mathcal{P}_{1}^{\,1} \\ \vdots \\ {}_{n}\mathcal{P}_{n}^{\,1} \end{array} \\ \hline \vdots & \ddots & \vdots \\ \hline \begin{array}{c} {}_{1}\mathcal{P}_{1}^{\,n} \\ \vdots \\ {}_{1}\mathcal{P}_{n}^{\,n} \end{array} & \cdots & \begin{array}{c} {}_{n}\mathcal{P}_{1}^{\,n} \\ \vdots \\ {}_{n}\mathcal{P}_{n}^{\,n} \end{array} \end{array} $$

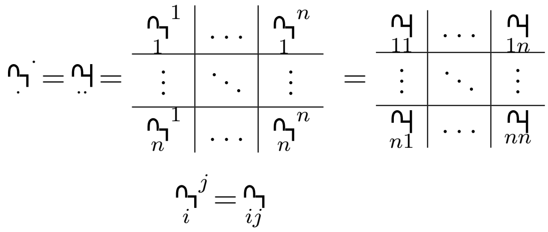
$$ {}_{\mu}\overset{T}{\mathcal{P}}{}_{i}^{\,j} = {}_{j}\mathcal{P}_{i}^{\,\mu} $$

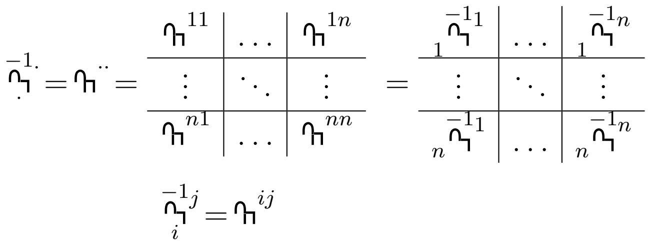
local transpose
$$ \overset{t}{\mathcal{P}} = \begin{array}{c} {}_{1}\overset{t}{\mathcal{P}}{}_{\cdot}^{\,\cdot} \\ \vdots \\ {}_{n}\overset{t}{\mathcal{P}}{}_{\cdot}^{\,\cdot} \end{array} = \begin{array}{c|c|c} \begin{array}{c} {}_{1}\mathcal{P}_{1}^{\,1} \\ \vdots \\ {}_{1}\mathcal{P}_{n}^{\,n} \end{array} & \cdots & \begin{array}{c} {}_{1}\mathcal{P}_{n}^{\,1} \\ \vdots \\ {}_{1}\mathcal{P}_{n}^{\,n} \end{array} \\ \hline \vdots & \ddots & \vdots \\ \hline \begin{array}{c} {}_{n}\mathcal{P}_{1}^{\,1} \\ \vdots \\ {}_{n}\mathcal{P}_{1}^{\,n} \end{array} & \cdots & \begin{array}{c} {}_{n}\mathcal{P}_{n}^{\,1} \\ \vdots \\ {}_{n}\mathcal{P}_{n}^{\,n} \end{array} \end{array} $$

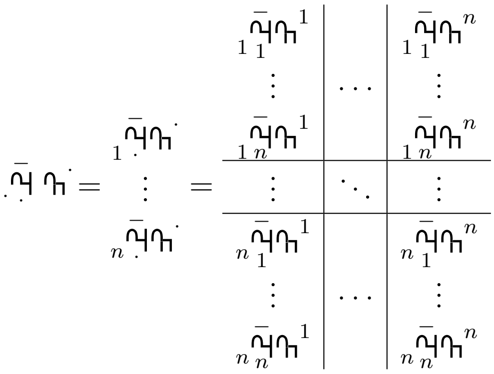
$$ {}_{\mu}\overset{t}{\mathcal{P}}{}_{i}^{\,j} = {}_{\mu}\mathcal{P}_{j}^{\,i} $$

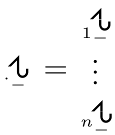
right multiple
$$ \mathcal{P}\,\mathcal{Q}^{-1} = \begin{array}{c} {}_{1}\mathcal{P}_{\cdot}^{\,\cdot}\,\mathcal{Q}_{\cdot}^{-1\cdot} \\ \vdots \\ {}_{n}\mathcal{P}_{\cdot}^{\,\cdot}\,\mathcal{Q}_{\cdot}^{-1\cdot} \end{array} $$

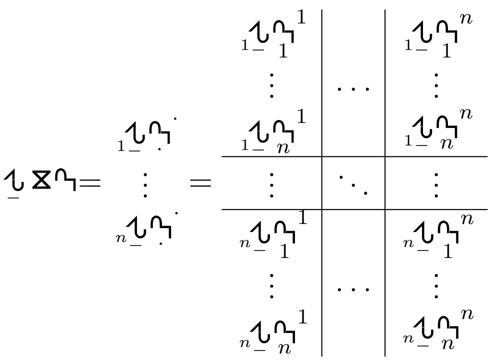


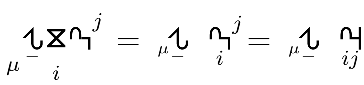
metric 

inverse metric 

Christoffel 

 column

 matrix-valued column



$$\text{global transpose } \langle \boxtimes^{T} \rangle = \left[\begin{array}{c|c|c} {}_{1}\langle\rangle_{1}^{1} & & {}_{n}\langle\rangle_{1}^{1} \\ \vdots & \cdots & \vdots \\ {}_{1}\langle\rangle_{n}^{1} & & {}_{n}\langle\rangle_{n}^{1} \\ \hline \vdots & \ddots & \vdots \\ \hline {}_{1}\langle\rangle_{1}^{n} & & {}_{n}\langle\rangle_{1}^{n} \\ \vdots & \cdots & \vdots \\ {}_{1}\langle\rangle_{n}^{n} & & {}_{n}\langle\rangle_{n}^{n} \end{array}\right]$$

$${}_{\mu}\langle \boxtimes^{T} \rangle_{i}^{j} = {}_{j}\langle \boxtimes \rangle_{i}^{\mu} = {}_{j}\langle\ \rangle_{i}^{\mu} = {}_{j}\langle\ \rangle_{i\mu}$$

$$\text{local transpose } \langle \boxtimes^{Tt} \rangle = \left[\begin{array}{c|c|c} {}_{1}\langle\rangle_{1}^{1} & & {}_{1}\langle\rangle_{n}^{1} \\ \vdots & \cdots & \vdots \\ {}_{n}\langle\rangle_{1}^{1} & & {}_{n}\langle\rangle_{n}^{1} \\ \hline \vdots & \ddots & \vdots \\ \hline {}_{1}\langle\rangle_{1}^{n} & & {}_{1}\langle\rangle_{n}^{n} \\ \vdots & \cdots & \vdots \\ {}_{n}\langle\rangle_{1}^{n} & & {}_{n}\langle\rangle_{n}^{n} \end{array}\right]$$

$${}_{\mu}\langle \boxtimes^{Tt} \rangle_{i}^{j} = {}_{\mu}\langle \boxtimes^{T} \rangle_{j}^{i} = {}_{i}\langle \boxtimes \rangle_{j}^{\mu} = {}_{i}\langle\ \rangle_{j}^{\mu} = {}_{i}\langle\ \rangle_{j\mu}$$

$$/ \qquad 2\ \bar{\rangle}\rangle = \underbrace{\langle \boxtimes \rangle - \langle \boxtimes^{T} \rangle + \langle \boxtimes^{Tt} \rangle}\ \rangle^{-1}$$

$${}_{\mu}\mathrm{RHS}_{i}^{k} = {}_{\mu}\underbrace{\langle \boxtimes \rangle - \langle \boxtimes^{T} \rangle + \langle \boxtimes^{Tt} \rangle}_{i}^{j}\ \rangle_{j}^{-1k} = \left({}_{\mu}\langle \boxtimes \rangle_{i}^{j} - {}_{\mu}\langle \boxtimes^{T} \rangle_{i}^{j} + {}_{\mu}\langle \boxtimes^{Tt} \rangle_{i}^{j} \right) \rangle_{j}^{-1k}$$

$$= \left({}_{\mu}\langle\rangle_{ij} - {}_{j}\langle\rangle_{i\mu} + {}_{i}\langle\rangle_{j\mu} \right) \rangle^{jk} = {}_{\mu}\bar{\rangle}_{i}\ \rangle^{j}$$

$$\text{matrix-valued matrix}\quad \mathbb{A} = \left({}_{\mu\nu}\mathbb{A}_{\cdot}^{\,\cdot}\right) = \left({}_{\mu\nu}\mathbb{A}_{i}^{\,j}\right) = \begin{array}{c|c|c} {}_{11}\mathbb{A}_{\cdot}^{\,\cdot} & \cdots & {}_{1n}\mathbb{A}_{\cdot}^{\,\cdot} \\ \hline \vdots & \ddots & \vdots \\ \hline {}_{n1}\mathbb{A}_{\cdot}^{\,\cdot} & \cdots & {}_{nn}\mathbb{A}_{\cdot}^{\,\cdot} \end{array} = \begin{array}{c|c|c} \begin{array}{c|c|c} {}_{11}\mathbb{A}_{1}^{\,1} & \cdots & {}_{11}\mathbb{A}_{1}^{\,n} \\ \hline \vdots & \ddots & \vdots \\ \hline {}_{11}\mathbb{A}_{n}^{\,1} & \cdots & {}_{11}\mathbb{A}_{n}^{\,n} \end{array} & \cdots & \begin{array}{c|c|c} {}_{1n}\mathbb{A}_{1}^{\,1} & \cdots & {}_{1n}\mathbb{A}_{1}^{\,n} \\ \hline \vdots & \ddots & \vdots \\ \hline {}_{1n}\mathbb{A}_{n}^{\,1} & \cdots & {}_{1n}\mathbb{A}_{n}^{\,n} \end{array} \\ \hline \vdots & \ddots & \vdots \\ \hline \begin{array}{c|c|c} {}_{n1}\mathbb{A}_{1}^{\,1} & \cdots & {}_{n1}\mathbb{A}_{1}^{\,n} \\ \hline \vdots & \ddots & \vdots \\ \hline {}_{n1}\mathbb{A}_{n}^{\,1} & \cdots & {}_{n1}\mathbb{A}_{n}^{\,n} \end{array} & \cdots & \begin{array}{c|c|c} {}_{nn}\mathbb{A}_{1}^{\,1} & \cdots & {}_{\lrcorner}\mathbb{A}_{1}^{\,n} \\ \hline \vdots & \ddots & \vdots \\ \hline {}_{nn}\mathbb{A}_{n}^{\,1} & \cdots & {}_{nn}\mathbb{A}_{n}^{\,n} \end{array} \end{array}$$

$$\text{global transpose}\quad \overset{T}{\mathbb{A}} = \begin{array}{c|c|c} {}_{11}\mathbb{A}_{\cdot}^{\,\cdot} & \cdots & {}_{n1}\mathbb{A}_{\cdot}^{\,\cdot} \\ \hline \vdots & \ddots & \vdots \\ \hline {}_{1n}\mathbb{A}_{\cdot}^{\,\cdot} & \cdots & {}_{nn}\mathbb{A}_{\cdot}^{\,\cdot} \end{array} = \begin{array}{c|c|c} \begin{array}{c|c|c} {}_{11}\mathbb{A}_{1}^{\,1} & \cdots & {}_{11}\mathbb{A}_{1}^{\,n} \\ \hline \vdots & \ddots & \vdots \\ \hline {}_{11}\mathbb{A}_{n}^{\,1} & \cdots & {}_{11}\mathbb{A}_{n}^{\,n} \end{array} & \cdots & \begin{array}{c|c|c} {}_{n1}\mathbb{A}_{1}^{\,1} & \cdots & {}_{n1}\mathbb{A}_{1}^{\,n} \\ \hline \vdots & \ddots & \vdots \\ \hline {}_{n1}\mathbb{A}_{n}^{\,1} & \cdots & {}_{n1}\mathbb{A}_{n}^{\,n} \end{array} \\ \hline \vdots & \ddots & \vdots \\ \hline \begin{array}{c|c|c} {}_{1n}\mathbb{A}_{1}^{\,1} & \cdots & {}_{1n}\mathbb{A}_{1}^{\,n} \\ \hline \vdots & \ddots & \vdots \\ \hline {}_{1n}\mathbb{A}_{n}^{\,1} & \cdots & {}_{1n}\mathbb{A}_{n}^{\,n} \end{array} & \cdots & \begin{array}{c|c|c} {}_{nn}\mathbb{A}_{1}^{\,1} & \cdots & {}_{\lrcorner}\mathbb{A}_{1}^{\,n} \\ \hline \vdots & \ddots & \vdots \\ \hline {}_{nn}\mathbb{A}_{n}^{\,1} & \cdots & {}_{nn}\mathbb{A}_{n}^{\,n} \end{array} \end{array}$$

$${}_{\mu\nu}\overset{T}{\mathbb{A}}{}_{i}^{\,j} = {}_{\nu\mu}\mathbb{A}_{i}^{\,j}$$

$$\text{local transpose}\quad \overset{t}{\mathbb{A}} = \begin{array}{c|c|c} {}_{11}\overset{t}{\mathbb{A}}{}_{\cdot}^{\,\cdot} & \cdots & {}_{1n}\overset{t}{\mathbb{A}}{}_{\cdot}^{\,\cdot} \\ \hline \vdots & \ddots & \vdots \\ \hline {}_{n1}\overset{t}{\mathbb{A}}{}_{\cdot}^{\,\cdot} & \cdots & {}_{nn}\overset{t}{\mathbb{A}}{}_{\cdot}^{\,\cdot} \end{array} = \begin{array}{c|c|c} \begin{array}{c|c|c} {}_{11}\mathbb{A}_{1}^{\,1} & \cdots & {}_{11}\mathbb{A}_{n}^{\,1} \\ \hline \vdots & \ddots & \vdots \\ \hline {}_{11}\mathbb{A}_{1}^{\,n} & \cdots & {}_{11}\mathbb{A}_{n}^{\,n} \end{array} & \cdots & \begin{array}{c|c|c} {}_{1n}\mathbb{A}_{1}^{\,1} & \cdots & {}_{1n}\mathbb{A}_{n}^{\,1} \\ \hline \vdots & \ddots & \vdots \\ \hline {}_{1n}\mathbb{A}_{1}^{\,n} & \cdots & {}_{1n}\mathbb{A}_{n}^{\,n} \end{array} \\ \hline \vdots & \ddots & \vdots \\ \hline \begin{array}{c|c|c} {}_{n1}\mathbb{A}_{1}^{\,1} & \cdots & {}_{n1}\mathbb{A}_{n}^{\,1} \\ \hline \vdots & \ddots & \vdots \\ \hline {}_{n1}\mathbb{A}_{1}^{\,n} & \cdots & {}_{n1}\mathbb{A}_{n}^{\,n} \end{array} & \cdots & \begin{array}{c|c|c} {}_{nn}\mathbb{A}_{1}^{\,1} & \cdots & {}_{\lrcorner}\mathbb{A}_{n}^{\,1} \\ \hline \vdots & \ddots & \vdots \\ \hline {}_{nn}\mathbb{A}_{1}^{\,n} & \cdots & {}_{nn}\mathbb{A}_{n}^{\,n} \end{array} \end{array}$$

$${}_{\mu\nu}\overset{t}{\mathbb{A}}{}_{i}^{\,j} = {}_{\mu\nu}\mathbb{A}_{j}^{\,i}$$

$$\text{differential}\quad {}_{\cdot\cdot}\bar{\varphi}^{\cdot}_{\cdot} = \begin{array}{c|c|c} {}_{11}\bar{\varphi}^{\cdot}_{\cdot} & \cdots & {}_{1n}\bar{\varphi}^{\cdot}_{\cdot} \\ \hline \vdots & \ddots & \vdots \\ \hline {}_{n1}\bar{\varphi}^{\cdot}_{\cdot} & \cdots & {}_{nn}\bar{\varphi}^{\cdot}_{\cdot} \end{array} = \begin{array}{c|c|c} \begin{array}{c|c|c} {}_{11}\bar{\varphi}^{1}_{1} & \cdots & {}_{11}\bar{\varphi}^{n}_{1} \\ \hline \vdots & \ddots & \vdots \\ \hline {}_{11}\bar{\varphi}^{1}_{n} & \cdots & {}_{11}\bar{\varphi}^{n}_{n} \end{array} & \cdots & \begin{array}{c|c|c} {}_{1n}\bar{\varphi}^{1}_{1} & \cdots & {}_{1n}\bar{\varphi}^{n}_{1} \\ \hline \vdots & \ddots & \vdots \\ \hline {}_{1n}\bar{\varphi}^{1}_{n} & \cdots & {}_{1n}\bar{\varphi}^{n}_{n} \end{array} \\ \hline \vdots & \ddots & \vdots \\ \hline \begin{array}{c|c|c} {}_{n1}\bar{\varphi}^{1}_{1} & \cdots & {}_{n1}\bar{\varphi}^{n}_{1} \\ \hline \vdots & \ddots & \vdots \\ \hline {}_{n1}\bar{\varphi}^{1}_{n} & \cdots & {}_{n1}\bar{\varphi}^{n}_{n} \end{array} & \cdots & \begin{array}{c|c|c} {}_{nn}\bar{\varphi}^{1}_{1} & \cdots & {}_{nn}\bar{\varphi}^{n}_{1} \\ \hline \vdots & \ddots & \vdots \\ \hline {}_{nn}\bar{\varphi}^{1}_{n} & \cdots & {}_{nn}\bar{\varphi}^{n}_{n} \end{array} \end{array}$$

$$\varphi^{\sharp} = \left({}_{1}\varphi^{\cdot}_{\cdot} \quad \cdots \quad {}_{n}\varphi^{\cdot}_{\cdot} \right) \text{ matrix-valued row}$$

$$\underline{\psi} \bowtie \varphi^{\sharp} = \begin{pmatrix} {}_{1}\underline{\psi} \\ \vdots \\ {}_{n}\underline{\psi} \end{pmatrix} \left({}_{1}\varphi^{\cdot}_{\cdot} \quad \cdots \quad {}_{n}\varphi^{\cdot}_{\cdot} \right) = \begin{array}{c|c|c} {}_{1}\underline{\psi}_{1}\varphi^{\cdot}_{\cdot} & \cdots & {}_{1}\underline{\psi}_{n}\varphi^{\cdot}_{\cdot} \\ \hline \vdots & \ddots & \vdots \\ \hline {}_{n}\underline{\psi}_{1}\varphi^{\cdot}_{\cdot} & \cdots & {}_{n}\underline{\psi}_{n}\varphi^{\cdot}_{\cdot} \end{array} = \begin{array}{c|c|c} \begin{array}{c|c|c} {}_{1}\underline{\psi}_{1}\varphi^{1}_{1} & \cdots & {}_{1}\underline{\psi}_{1}\varphi^{n}_{1} \\ \hline \vdots & \ddots & \vdots \\ \hline {}_{1}\underline{\psi}_{1}\varphi^{1}_{n} & \cdots & {}_{1}\underline{\psi}_{1}\varphi^{n}_{n} \end{array} & \cdots & \begin{array}{c|c|c} {}_{1}\underline{\psi}_{n}\varphi^{1}_{1} & \cdots & {}_{1}\underline{\psi}_{n}\varphi^{n}_{1} \\ \hline \vdots & \ddots & \vdots \\ \hline {}_{1}\underline{\psi}_{n}\varphi^{1}_{n} & \cdots & {}_{1}\underline{\psi}_{n}\varphi^{n}_{n} \end{array} \\ \hline \vdots & \ddots & \vdots \\ \hline \begin{array}{c|c|c} {}_{n}\underline{\psi}_{1}\varphi^{1}_{1} & \cdots & {}_{n}\underline{\psi}_{1}\varphi^{n}_{1} \\ \hline \vdots & \ddots & \vdots \\ \hline {}_{n}\underline{\psi}_{1}\varphi^{1}_{n} & \cdots & {}_{n}\underline{\psi}_{1}\varphi^{n}_{n} \end{array} & \cdots & \begin{array}{c|c|c} {}_{n}\underline{\psi}_{n}\varphi^{1}_{1} & \cdots & {}_{n}\underline{\psi}_{n}\varphi^{n}_{1} \\ \hline \vdots & \ddots & \vdots \\ \hline {}_{n}\underline{\psi}_{n}\varphi^{1}_{n} & \cdots & {}_{n}\underline{\psi}_{n}\varphi^{n}_{n} \end{array} \end{array} \text{ matrix-val}$$

$${}_{\mu\nu}\underline{\psi} \bowtie \varphi^{\sharp}{}^{j}_{i} = {}_{\mu}\underline{\psi} \; {}_{\nu}\varphi^{j}_{i}$$

$$\text{global transpose}\quad \underline{\psi} \overset{T}{\bowtie} \varphi^{\sharp} = \begin{array}{c|c|c} \begin{array}{c|c|c} {}_{1}\underline{\psi}_{1}\varphi^{1}_{1} & \cdots & {}_{1}\underline{\psi}_{1}\varphi^{n}_{1} \\ \hline \vdots & \ddots & \vdots \\ \hline {}_{1}\underline{\psi}_{1}\varphi^{1}_{n} & \cdots & {}_{1}\underline{\psi}_{1}\varphi^{n}_{n} \end{array} & \cdots & \begin{array}{c|c|c} {}_{n}\underline{\psi}_{1}\varphi^{1}_{1} & \cdots & {}_{n}\underline{\psi}_{1}\varphi^{n}_{1} \\ \hline \vdots & \ddots & \vdots \\ \hline {}_{n}\underline{\psi}_{1}\varphi^{1}_{n} & \cdots & {}_{n}\underline{\psi}_{1}\varphi^{n}_{n} \end{array} \\ \hline \vdots & \ddots & \vdots \\ \hline \begin{array}{c|c|c} {}_{1}\underline{\psi}_{n}\varphi^{1}_{1} & \cdots & {}_{1}\underline{\psi}_{n}\varphi^{n}_{1} \\ \hline \vdots & \ddots & \vdots \\ \hline {}_{1}\underline{\psi}_{n}\varphi^{1}_{n} & \cdots & {}_{1}\underline{\psi}_{n}\varphi^{n}_{n} \end{array} & \cdots & \begin{array}{c|c|c} {}_{n}\underline{\psi}_{n}\varphi^{1}_{1} & \cdots & {}_{n}\underline{\psi}_{n}\varphi^{n}_{1} \\ \hline \vdots & \ddots & \vdots \\ \hline {}_{n}\underline{\psi}_{n}\varphi^{1}_{n} & \cdots & {}_{n}\underline{\psi}_{n}\varphi^{n}_{n} \end{array} \end{array}$$

$${}_{\mu\nu}\underline{\psi} \overset{T}{\bowtie} \varphi^{\sharp}{}^{j}_{i} = {}_{\nu\mu}\underline{\psi} \bowtie \varphi^{\sharp}{}^{j}_{i} = {}_{\nu}\underline{\psi} \; {}_{\mu}\varphi^{j}_{i}$$

$$\mathsf{P}\bowtie\mathsf{P}^{\sharp}=\begin{pmatrix}{}_{1}\mathsf{P}^{\cdot}_{\cdot}\\ \vdots\\ {}_{n}\mathsf{P}^{\cdot}_{\cdot}\end{pmatrix}\begin{pmatrix}{}_{1}\mathsf{P}^{\cdot}_{\cdot} & \cdots & {}_{n}\mathsf{P}^{\cdot}_{\cdot}\end{pmatrix}=\left[\begin{array}{c|c|c}{}_{1}\mathsf{P}^{\cdot}_{\cdot}\,{}_{1}\mathsf{P}^{\cdot}_{\cdot} & \cdots & {}_{1}\mathsf{P}^{\cdot}_{\cdot}\,{}_{n}\mathsf{P}^{\cdot}_{\cdot}\\ \hline \vdots & \ddots & \vdots\\ \hline {}_{n}\mathsf{P}^{\cdot}_{\cdot}\,{}_{1}\mathsf{P}^{\cdot}_{\cdot} & \cdots & {}_{n}\mathsf{P}^{\cdot}_{\cdot}\,{}_{n}\mathsf{P}^{\cdot}_{\cdot}\end{array}\right]=\left[\begin{array}{c|c|c}\begin{array}{c|c|c}{}_{1}\mathsf{P}_{1}^{i}\,{}_{1}\mathsf{P}_{i}^{1} & \cdots & {}_{1}\mathsf{P}_{1}^{i}\,{}_{1}\mathsf{P}_{i}^{n}\\ \hline \vdots & \ddots & \vdots\\ \hline {}_{1}\mathsf{P}_{n}^{i}\,{}_{1}\mathsf{P}_{i}^{1} & \cdots & {}_{1}\mathsf{P}_{n}^{i}\,{}_{1}\mathsf{P}_{i}^{n}\end{array} & \cdots & \begin{array}{c|c|c}{}_{1}\mathsf{P}_{1}^{i}\,{}_{n}\mathsf{P}_{i}^{1} & \cdots & {}_{1}\mathsf{P}_{1}^{i}\,{}_{n}\mathsf{P}_{i}^{n}\\ \hline \vdots & \ddots & \vdots\\ \hline {}_{1}\mathsf{P}_{n}^{i}\,{}_{n}\mathsf{P}_{i}^{1} & \cdots & {}_{1}\mathsf{P}_{n}^{i}\,{}_{n}\mathsf{P}_{i}^{n}\end{array}\\ \hline \vdots & \ddots & \vdots\\ \hline \begin{array}{c|c|c}{}_{n}\mathsf{P}_{1}^{i}\,{}_{1}\mathsf{P}_{i}^{1} & \cdots & {}_{n}\mathsf{P}_{1}^{i}\,{}_{1}\mathsf{P}_{i}^{n}\\ \hline \vdots & \ddots & \vdots\\ \hline {}_{n}\mathsf{P}_{n}^{i}\,{}_{1}\mathsf{P}_{i}^{1} & \cdots & {}_{n}\mathsf{P}_{n}^{i}\,{}_{1}\mathsf{P}_{i}^{n}\end{array} & \cdots & \begin{array}{c|c|c}{}_{n}\mathsf{P}_{1}^{i}\,{}_{n}\mathsf{P}_{i}^{1} & \cdots & {}_{n}\mathsf{P}_{1}^{i}\,{}_{n}\mathsf{P}_{i}^{n}\\ \hline \vdots & \ddots & \vdots\\ \hline {}_{n}\mathsf{P}_{n}^{i}\,{}_{n}\mathsf{P}_{i}^{1} & \cdots & {}_{n}\mathsf{P}_{n}^{i}\,{}_{n}\mathsf{P}_{i}^{n}\end{array}\end{array}\right]\ \text{matrix}$$

$${}_{\mu\nu}\mathsf{P}\underset{i}{\bowtie}\mathsf{P}^{\sharp\,j}={}_{\mu}\mathsf{P}_{i}^{k}\,{}_{\nu}\mathsf{P}_{k}^{j}$$

$$\text{global transpose}\ \mathsf{P}\overset{T}{\bowtie}\mathsf{P}^{\sharp}=\left[\begin{array}{c|c|c}\begin{array}{c|c|c}{}_{1}\mathsf{P}_{1}^{i}\,{}_{1}\mathsf{P}_{i}^{1} & \cdots & {}_{1}\mathsf{P}_{1}^{i}\,{}_{1}\mathsf{P}_{i}^{n}\\ \hline \vdots & \ddots & \vdots\\ \hline {}_{1}\mathsf{P}_{n}^{i}\,{}_{1}\mathsf{P}_{i}^{1} & \cdots & {}_{1}\mathsf{P}_{n}^{i}\,{}_{1}\mathsf{P}_{i}^{n}\end{array} & \cdots & \begin{array}{c|c|c}{}_{n}\mathsf{P}_{1}^{i}\,{}_{1}\mathsf{P}_{i}^{1} & \cdots & {}_{n}\mathsf{P}_{1}^{i}\,{}_{1}\mathsf{P}_{i}^{n}\\ \hline \vdots & \ddots & \vdots\\ \hline {}_{n}\mathsf{P}_{n}^{i}\,{}_{1}\mathsf{P}_{i}^{1} & \cdots & {}_{n}\mathsf{P}_{n}^{i}\,{}_{1}\mathsf{P}_{i}^{n}\end{array}\\ \hline \vdots & \ddots & \vdots\\ \hline \begin{array}{c|c|c}{}_{1}\mathsf{P}_{1}^{i}\,{}_{n}\mathsf{P}_{i}^{1} & \cdots & {}_{1}\mathsf{P}_{1}^{i}\,{}_{n}\mathsf{P}_{i}^{n}\\ \hline \vdots & \ddots & \vdots\\ \hline {}_{1}\mathsf{P}_{n}^{i}\,{}_{n}\mathsf{P}_{i}^{1} & \cdots & {}_{1}\mathsf{P}_{n}^{i}\,{}_{n}\mathsf{P}_{i}^{n}\end{array} & \cdots & \begin{array}{c|c|c}{}_{n}\mathsf{P}_{1}^{i}\,{}_{n}\mathsf{P}_{i}^{1} & \cdots & {}_{n}\mathsf{P}_{1}^{i}\,{}_{n}\mathsf{P}_{i}^{n}\\ \hline \vdots & \ddots & \vdots\\ \hline {}_{n}\mathsf{P}_{n}^{i}\,{}_{n}\mathsf{P}_{i}^{1} & \cdots & {}_{n}\mathsf{P}_{n}^{i}\,{}_{n}\mathsf{P}_{i}^{n}\end{array}\end{array}\right]\ \text{matrix-valued matrix}$$

$${}_{\mu\nu}\mathsf{P}\underset{i}{\overset{T}{\bowtie}}\mathsf{P}^{\sharp\,j}={}_{\nu\mu}\mathsf{P}\underset{i}{\bowtie}\mathsf{P}^{\sharp\,j}={}_{\nu}\mathsf{P}_{i}^{k}\,{}_{\mu}\mathsf{P}_{k}^{j}$$

$$1/\ {}_{\cdot\cdot}\bar{\mathsf{P}}^{\cdot}_{\cdot}=\underline{\wr}\bowtie{}_{\cdot}\mathsf{P}^{\sharp\cdot}_{\cdot}-\underline{\wr}\overset{T}{\bowtie}{}_{\cdot}\mathsf{P}^{\sharp\cdot}_{\cdot}+{}_{\cdot}\mathsf{P}^{\cdot}_{\cdot}\bowtie{}_{\cdot}\mathsf{P}^{\sharp\cdot}_{\cdot}-{}_{\cdot}\mathsf{P}^{\cdot}_{\cdot}\overset{T}{\bowtie}{}_{\cdot}\mathsf{P}^{\sharp\cdot}_{\cdot}$$

$${}_{\mu\nu}\underset{i}{\mathrm{RHS}}^{j}={}_{\mu\nu\cdot}\underline{\wr}\underset{i}{\bowtie}{}_{\cdot}\mathsf{P}^{\sharp\,j}_{\cdot}-{}_{\mu\nu\cdot}\underline{\wr}\underset{i}{\overset{T}{\bowtie}}{}_{\cdot}\mathsf{P}^{\sharp\,j}_{\cdot}+{}_{\mu\nu\cdot}\mathsf{P}^{\cdot}_{\cdot}\underset{i}{\bowtie}{}_{\cdot}\mathsf{P}^{\sharp\,j}_{\cdot}-{}_{\mu\nu\cdot}\mathsf{P}^{\cdot}_{\cdot}\underset{i}{\overset{T}{\bowtie}}{}_{\cdot}\mathsf{P}^{\sharp\,j}_{\cdot}$$

$$={}_{\mu}\underline{\wr}\ {}_{\nu}\mathsf{P}_{i}^{j}-{}_{\nu}\underline{\wr}\ {}_{\mu}\mathsf{P}_{i}^{j}+{}_{\mu}\mathsf{P}_{i}^{k}\,{}_{\nu}\mathsf{P}_{k}^{j}-{}_{\nu}\mathsf{P}_{i}^{k}\,{}_{\mu}\mathsf{P}_{k}^{j}={}_{\mu\nu}\bar{\mathsf{P}}_{i}^{j}$$

$$ {}_{\cdot\cdot}\overset{=}{\text{ኀ}}{}^{\cdot}_{\cdot} = {}_{\cdot}\underset{-}{\text{ᔑ}}\,⧖\,{}_{\cdot}\bar{\text{ዛ}}{}_{\cdot}\text{ኪ}{}^{\cdot} - {}_{\cdot}\underset{-}{\text{ᔑ}}\,\overset{T}{⧖}\,{}_{\cdot}\bar{\text{ዛ}}{}_{\cdot}\text{ኪ}{}^{\cdot} + {}_{\cdot}\bar{\text{ዛ}}{}_{\cdot}\text{ኪ}{}^{\cdot}\,⧖\,{}_{\cdot}\bar{\text{ዛ}}{}^{\sharp}_{\cdot}\text{ኪ}{}^{\cdot} - {}_{\cdot}\bar{\text{ዛ}}{}_{\cdot}\text{ኪ}{}^{\cdot}\,\overset{T}{⧖}\,{}_{\cdot}\bar{\text{ዛ}}{}^{\sharp}_{\cdot}\text{ኪ}{}^{\cdot} $$

$$ {}_{\nu}\underset{i}{\mathrm{Ric}} = {}_{\cdot\nu}\overset{=}{\underset{i}{\text{ኀ}}}{}^{\cdot} = {}_{j\nu}\overset{=}{\underset{i}{\text{ኀ}}}{}^{j} $$

$$ R = {}_{\nu}\underset{i}{\mathrm{Ric}}\ \text{ካ}^{\nu i} = {}_{j\nu}\bar{\underset{i}{\text{ኆ}}}{}^{j}\ \text{ካ}^{\nu i} $$