$$\hat{\underset{\sim}{F}}_{\cdot} = \frac{1}{4}\, {}_{i}F^{j}\, \eta_{jj}\, \hat{q}^{i}\, \hat{q}^{j}$$

$$\underset{\sim}{F}_{\cdot} = \frac{1}{4}\, {}_{i}F^{j}\, \eta_{jj}\, q^{i} \times q^{j}$$

$${}_{i}F^{j}\, \eta_{jj} = -\, {}_{j}F^{i}\, \eta_{ii}$$

$$F \between G = -\operatorname{tr}\, \underset{\sim}{F}_{\cdot}\, \underset{\sim}{G}_{\cdot}$$

$$\text{LHS} = {}_{a}F_{b}\, {}^{a}G^{b} = {}_{a}F^{b}\, \eta_{bb}\, \eta^{aa}\, {}_{a}G^{b} = -\, {}_{b}F^{a}\, \eta_{aa}\, \eta^{aa}\, {}_{a}G^{b} = -\, {}_{b}F^{a}\, {}_{a}G^{b} = \text{RHS}$$

$$\operatorname{tr}_{S}\, \hat{\underset{\sim}{F}}_{\cdot}\, \hat{\underset{\sim}{G}}_{\cdot} = \frac{1}{8}\operatorname{tr}\, \underset{\sim}{F}_{\cdot}\, \underset{\sim}{G}_{\cdot} = -\frac{1}{8}\, \underset{\sim}{F}_{\cdot} \between \underset{\sim}{G}_{\cdot}$$

$$\operatorname{tr}_{S}\, \hat{\underset{\sim}{F}}_{\cdot}\, \hat{\underset{\sim}{G}}_{\cdot} = \operatorname{tr}_{S}\, {}_{i}F^{j}\, \eta_{jj}\, \hat{q}^{i}\, \hat{q}^{j}\, {}_{k}G^{\ell}\, \eta_{\ell\ell}\, \hat{q}^{k}\, \hat{q}^{\ell} = 3S\, \hat{q}^{i}\, \hat{q}^{j}\, \hat{q}^{k}\, \hat{q}^{\ell}\, {}_{i}F^{j}\, \eta_{jj}\, {}_{k}G^{\ell}\, \eta_{\ell\ell} =$$

$$\left(\eta^{ij}\eta^{kl} - \eta^{ik}\eta^{j\ell} + \eta^{i\ell}\eta^{jk}\right)\, {}_{i}F^{j}\, \eta_{jj}\, {}_{k}G^{\ell}\, \eta_{\ell\ell} = -\, \eta^{ii}\, \eta^{jj}\, {}_{i}F^{j}\, \eta_{jj}\, {}_{i}G^{j}\, \eta_{jj} + \eta^{ii}\, \eta^{jj}\, {}_{i}F^{j}\, \eta_{jj}\, {}_{j}G^{i}\, \eta_{ii} = 2\, {}_{i}F^{j}\, {}_{j}G^{i}$$

$${}^{*}\hat{\underset{\sim}{F}}_{\cdot} = -\, \hat{\underset{\sim}{F}}_{\cdot}$$

$$\text{LHS} = \frac{1}{4}\, {}_{i}\bar{F}^{j}\, \eta_{jj}\, {}^{*}\hat{q}^{j}\, {}^{*}\hat{q}^{i} = \frac{1}{4}\, {}_{i}F^{j}\, \eta_{jj}\, \hat{q}^{j}\, \hat{q}^{i} = -\frac{1}{4}\, {}_{j}F^{i}\, \eta_{ii}\, \hat{q}^{j}\, \hat{q}^{i} = \text{RHS}$$

$$\hat{\underset{\sim}{F}}_{\cdot} \between \hat{\underset{\sim}{G}}_{\cdot} = \operatorname{tr}_{S}\, {}^{*}\hat{\underset{\sim}{F}}_{\cdot}\, \hat{\underset{\sim}{G}}_{\cdot} = -\operatorname{tr}_{S}\, \hat{\underset{\sim}{F}}_{\cdot}\, \hat{\underset{\sim}{G}}_{\cdot} = \frac{1}{8}\, \underset{\sim}{F}_{\cdot} \between \underset{\sim}{G}_{\cdot}$$

$$\operatorname{tr}_{S}\, \hat{\underset{\sim}{F}}_{\cdot} = 0$$

$$4\operatorname{tr}_{S}\, \hat{\underset{\sim}{F}}_{\cdot} = {}_{i}F^{j}\, \eta_{jj}\operatorname{tr}_{S}\, \hat{q}^{i}\, \hat{q}^{j} = {}_{i}F^{j}\, \eta_{jj}\operatorname{tr}_{S}\frac{\hat{q}^{i}\hat{q}^{j} + \hat{q}^{j}\hat{q}^{i}}{2} = {}_{i}F^{j}\, \eta_{jj}\, \eta^{ij} = {}_{j}F^{i}\, \eta_{ii}\, \eta^{ji} = 0$$