$$
\begin{array}{ccc}
\utilde{\mathbb{Z}} & \sqsubset & \utilde{\mathbb{Q}} \\
\cup & & \cup \\
\mathbb{Z} & \sqsubset & \mathbb{Q} \\
\cap & & \cap \\
\utilde{p}^{\mathbb{N}}p & \sqsubset & \utilde{p}^{\mathbb{Z}}p
\end{array}
$$

$$\utilde{p} = \mathbb{Z} \ulcorner \tilde{p} = \mathbb{Z} \ulcorner \widehat{\mathbb{Z}p} \text{ arithmetic residue field}$$

$$\mathbb{Z} \xrightarrow[\text{inj}]{} \utilde{p}^{\mathbb{N}}p$$

$$\text{arithmetic local ring } \utilde{p}^{\mathbb{N}}p = \frac{\sum_i^{\mathbb{N}} a_i p^i}{0 \leqslant a_i < p} \sqsubset \utilde{p}^{\mathbb{Z}}p = \frac{\sum_{i \geqslant m}^{\mathbb{Z}} a_i p^i}{0 \leqslant a_i < p} \text{ arithmetic local field}$$

$$\utilde{p}^{\mathbb{N}}_{\times}p = \frac{\sum_i^{\mathbb{N}} a_i p^i}{0 \leqslant a_i < p\colon \quad a_0 \neq 0}$$

$$\utilde{p}^{\mathbb{N}}_{+}p = \frac{\sum_i^{\mathbb{N}} a_i p^i}{0 \leqslant a_i < p\colon \quad a_0 = 0} = \frac{\sum_{i \geqslant 1} a_i p^i}{0 \leqslant a_i < p}$$

$$\utilde{p}^{\mathbb{N}}p = \frac{x \in \utilde{p}^{\mathbb{Z}}p}{\ulcorner x \urcorner_p \leqslant 1}$$

$$
\begin{array}{ccc}
k|\frac{1}{x} & \sqsubset & k||\frac{1}{x} \\
\cup & & \cup \\
k|x & \sqsubset & k||x \\
\cap & & \cap \\
\utilde{\daleth}^{\mathbb{N}}\daleth & \sqsubset & \utilde{\daleth}^{\mathbb{Z}}\daleth
\end{array}
$$

$$\utilde{\daleth} = k|x \ulcorner \tilde{\daleth} = k|x \ulcorner \widehat{k|x\daleth} \text{ geometric residue field}$$

$$\text{geometric local ring } \utilde{\daleth}^{\mathbb{N}}\daleth = \frac{\sum_i^{\mathbb{N}} a_i \daleth^i}{a_i \in \utilde{\daleth}} \sqsubset \utilde{\daleth}^{\mathbb{Z}}\daleth = \frac{\sum_{i \geqslant m}^{\mathbb{Z}} a_i \daleth^i}{a_i \in \utilde{\daleth}} \text{ geometric local field}$$

$$q = p^e$$

$$\begin{array}{ccc} q|x & \sqsubset & q|X \quad \text{field ext} \\ \cap & & \cap \\ q||x & \sqsubset & q||X \end{array}$$

$$\text{max id } \mathfrak{p} \triangleleft q|X$$

$$q||X \sqsubset q \underset{\mathfrak{p}}{||} X \text{ completion}$$

$$\begin{array}{ccc} q\underset{\mathfrak{p}}{|}X & \sqsubset & q\underset{\mathfrak{p}}{||}X \\ \cap & & \cap \\ q\overset{\times}{\underset{\mathfrak{p}}{|}}X & \sqsubset & q\overset{\times}{\underset{\mathfrak{p}}{||}}X \end{array}$$

$$\mathfrak{p} \to q\underset{\mathfrak{p}}{|}X \to q\underset{\mathfrak{p}}{|}X \ulcorner \mathfrak{p} = q||\mathfrak{p} \text{ residue field}$$

$${}^{s}X = \exp\left(\sum_{n \geqslant 1} \frac{\overbrace{q^n \cap X}}{n} q^{-ns}\right) = \prod_{\mathfrak{p} \triangleleft q|X} \left(1 - \overbrace{q\underset{\mathfrak{p}}{|}X \ulcorner \mathfrak{p}}^{-s}\right)^{-1} = \sum_{\mathfrak{a} \triangleleft q|X} \overbrace{q\underset{\mathfrak{p}}{|}X \ulcorner \mathfrak{p}}^{-s}$$