

$$\begin{aligned}
\mathbb{C}^{\frac{2}{\omega}} &= \frac{\mathbb{J} \in \mathbb{C}^{\frac{1}{\omega}}}{\mathbb{J} \times \mathbb{J} < +\infty} \\
T_{\mathbb{Z}\ell} S_{\mathbb{Z}\ell} \mathbb{J} &= \mathbb{J} \\
{}_z \mathbb{J} &= \frac{\sum_n \mathbb{Z} c_n \exp\left(\pi i \left(\frac{n}{\ell}\right)^2 \tau + 2\pi i \frac{n}{\ell} z\right)}{\mathbb{K} \in \mathbb{Z} \curvearrowright c_n = c_{n+\ell^2 \mathbb{K}}} \\
&= \sum_n \mathbb{Z} c_n \exp\left(\pi i \left(\frac{n}{\ell}\right)^2 \tau + 2\pi i \frac{n}{\ell} z\right) \\
&= \sum_{m \in \ell^2} c_m \sum_{\mathbb{K}} \exp\left(\pi i \left(\frac{m + \ell^2 \mathbb{K}}{\ell}\right)^2 \tau + 2\pi i \frac{m + \ell^2 \mathbb{K}}{\ell} z\right) \\
&= \sum_{m \in \ell^2} c_m \sum_{\mathbb{K}} \exp\left(\pi i \left(\frac{m}{\ell} + \ell \mathbb{K}\right)^2 \tau + 2\pi i \left(\frac{m}{\ell} + \ell \mathbb{K}\right) z\right) \\
{}_z \mathbb{J} &= \sum_n \mathbb{Z} c_n \exp\left(\pi i \left(\frac{n}{\ell}\right)^2 \tau + 2\pi i \frac{n}{\ell} z\right) = [c_n]
\end{aligned}$$

$$T_{\ell\chi} S_{\ell k} [c_n] = [c_{m - \ell^2 \chi}]$$

$$\begin{aligned}
& \underbrace{T_{\ell\chi} S_{\ell k}}_z \mathbf{J} = \exp\left(\pi i \ell^2 \chi^2 \tau + 2\pi i \ell \chi z\right) \underbrace{S_{\ell k}}_{z + \ell\chi\tau} \mathbf{J} \\
& = \exp\left(\pi i \ell^2 \chi^2 \tau + 2\pi i \ell \chi z\right)_{z + \ell\chi\tau + \ell k} \mathbf{J} \\
& = \exp\left(\pi i \ell^2 \chi^2 \tau + 2\pi i \ell \chi z\right) \sum_n^{\mathbb{Z}} c_n \exp\left(\pi i \left(\frac{n}{\ell}\right)^2 \tau + 2\pi i \frac{n}{\ell} (z + \ell\chi\tau + \ell k)\right) \\
& = \sum_n^{\mathbb{Z}} c_n \exp\left(\pi i \left(\frac{n}{\ell} + \ell\chi\right)^2 \tau + 2\pi i \left(\frac{n}{\ell} + \ell\chi\right) z\right) \underbrace{\exp(2\pi i n k)}_{=1} \\
& = \sum_n^{\mathbb{Z}} c_n \exp\left(\pi i \left(\frac{n + \ell^2 \chi}{\ell}\right)^2 \tau + 2\pi i \frac{n + \ell^2 \chi}{\ell} z\right) \\
& = \sum_m^{\mathbb{Z}} c_{m - \ell^2 \chi} \exp\left(\pi i \left(\frac{m}{\ell}\right)^2 \tau + 2\pi i \frac{m}{\ell} z\right) = [c_{m - \ell^2 \chi}]
\end{aligned}$$