$$\mathbb{T}\!\triangleleft_{\!1}^{\,m}$$

$$\Big\downarrow d$$

$$\mathbb{T}\!\triangleleft_{\!1}^{\,m+1}$$

$$\begin{bmatrix} {}_0\flat \\ + \\ {}_m\flat \end{bmatrix} d\,\natural = \sum_{0 \leqslant i \leqslant m} (-1)^{i}\; {}_i\flat \bowtie \begin{bmatrix} {}_0\flat \\ + \\ {}_{\widehat{i}}\flat \\ + \\ {}_m\flat \end{bmatrix}\natural + \sum_{0 \leqslant i < j \leqslant m} (-1)^{\,j-i} \begin{bmatrix} {}_i\flat \bowtie {}_j\flat \\ {}_0\flat \\ + \\ {}_{\widehat{i}}\flat \\ + \\ {}_{\widehat{j}}\flat \\ + \\ {}_m\flat \end{bmatrix}\natural$$

$$_M\flat\, d\,\natural = \sum_{i}^{M} \overset{i}{\underset{M\llcorner i}{\vee}}\; {}_i\flat \bowtie\; {}_{M\llcorner i}\flat\,\natural + \sum_{i<j} \overset{i}{\underset{M\llcorner i}{\vee}}\,\overset{j}{\underset{M\llcorner j}{\vee}} \begin{bmatrix} {}_i\flat \bowtie {}_j\flat \\ {}_{M\llcorner i:j}\flat \end{bmatrix}\natural$$

$$\flat\bowtie \;=\; \underline{\flat \vDash}\, d + d\,\underline{\flat \vDash}$$

$$_M\flat\, \overline{\underline{\flat\vDash d\natural}} \;+\; {}_M\flat\, \overline{\underline{\flat\vDash\natural}} = \begin{bmatrix} \flat \\ {}_M\flat \end{bmatrix}\underline{d\natural} + \sum_{i}^{M} \overset{i}{\underset{M\llcorner i}{\vee}}\; {}_i\flat \bowtie \overline{{}_{M\llcorner i}\flat\,\underline{\flat\vDash\natural}} + \sum_{i<j} \overset{i}{\underset{M\llcorner i}{\vee}}\,\overset{j}{\underset{M\llcorner j}{\vee}} \begin{bmatrix} {}_i\flat \bowtie {}_j\flat \\ {}_{M\llcorner i:j}\flat \end{bmatrix}\underline{\flat\vDash\natural}$$

$$= \flat \bowtie \overline{{}_M\flat\natural} - \sum_{i}^{M} \overset{i}{\underset{M\llcorner i}{\vee}}\; {}_i\flat \bowtie \begin{bmatrix} \flat \\ {}_{M\llcorner i}\flat \end{bmatrix}\natural - \sum_{i}^{M} \overset{i}{\underset{M\llcorner i}{\vee}} \begin{bmatrix} \flat \bowtie {}_i\flat \\ {}_{M\llcorner i}\flat \end{bmatrix}\natural + \sum_{i<j} \overset{i}{\underset{M\llcorner i}{\vee}}\,\overset{j}{\underset{M\llcorner j}{\vee}} \begin{bmatrix} {}_i\flat \bowtie {}_j\flat \\ \flat \\ {}_{M\llcorner i:j}\flat \end{bmatrix}\natural$$

$$+ \sum_{i}^{M} \overset{i}{\underset{M\llcorner i}{\vee}}\; {}_i\flat \bowtie \begin{bmatrix} \flat \\ {}_{M\llcorner i}\flat \end{bmatrix}\natural + \sum_{i<j} \overset{i}{\underset{M\llcorner i}{\vee}}\,\overset{j}{\underset{M\llcorner j}{\vee}} \begin{bmatrix} \flat \\ {}_i\flat \bowtie {}_j\flat \\ {}_{M\llcorner i:j}\flat \end{bmatrix}\natural = \flat \bowtie \overline{\flat_M\natural} + \sum_{i}^{M} \overset{i}{\underset{M\llcorner i}{\vee}} \begin{bmatrix} {}_i\flat \bowtie \flat \\ {}_{M\llcorner i}\flat \end{bmatrix}\natural = \,{}_M\flat\, \underline{\flat\bowtie\natural}$$

$$\underline{\flat\bowtie}d = d\underline{\flat\bowtie},$$

$$\underline{\flat\bowtie}d = \widehat{\flat\vDash d + d\underline{\flat\vDash}}\,d = d\left(-\flat\vDash\right)d = d\widehat{\flat\vDash d + d\underline{\flat\vDash}} = d\underline{\flat\bowtie},$$

$$dd = 0$$

$$\flat\vDash\widehat{d\underline{d\,\mathsf{q}}} = \flat\bowtie\underline{d\,\mathsf{q}} - d\widehat{\flat\vDash\underline{d\,\mathsf{q}}} = d\underline{\flat\bowtie\mathsf{q}} - d\widehat{\flat\vDash\underline{d\,\mathsf{q}}} = d\widehat{\flat\bowtie\mathsf{q} - \flat\vDash\underline{d\,\mathsf{q}}} = d\widehat{d\underline{\flat\vDash\mathsf{q}}} = dd\underline{\flat\vDash\mathsf{q}} \underset{\mathrm{Ind}}{=} 0$$