$$X^n_{\mathbb{C}} \text{ normal variety}$$

$$\deg_X = K_X | K_X = d$$

$$\mathrm{Pic}_X = \mathbb{Z}^{1|k} = \mathbb{Z}^{C_0|C_1 \cdot\cdot C_k}$$

$$-K = 3C_0 - C_1 - \cdot\cdot - C_k$$

$$\mathrm{gen}_X = 1 + \frac{C|(K+C)}{2}$$

$$\mathrm{rat}^1 \begin{cases} -K|C = 1 \\ C|C = -1 \end{cases}$$

$$\text{exceptional curves} \begin{cases} C|C = -1 \\ -K|C = 1 \end{cases} \quad : \quad \text{perms of}$$

$1 \leqslant k \leqslant 8$: $C_{\not 1}$ blowup through -1 points

$$\left(3C_0 - C_{\not k}\right) | C_{\not 1} = -C_{\not 1} | C_{\not 1} = 1$$

$$C_{\not 1} | C_{\not 1} = -1$$

$2 \leqslant k \leqslant 8$: $C_0 - C_{\not 2}$ line through 2 simple points

$$\left(3C_0 - C_{\not k}\right) | \left(C_0 - C_{\not 2}\right) = 3C_0|C_0 + C_{\not 2}|C_{\not 2} = 3 - 2 = 1$$

$$\left(C_0 - C_{\not 2}\right) | \left(C_0 - C_{\not 2}\right) = C_0|C_0 + C_{\not 2}|C_{\not 2} = 1 - 2 = -1$$

$5 \leqslant k \leqslant 8$: $2C_0 - C_{\not 5}$ conic through 5 simple points

$$\left(3C_0 - C_{\not k}\right) | \left(2C_0 - C_{\not 5}\right) = 6C_0|C_0 + C_{\not 5}|C_{\not 5} = 6 - 5 = 1$$

$$\left(2C_0 - C_{\not 5}\right) | \left(2C_0 - C_{\not 5}\right) = 4C_0|C_0 + C_{\not 5}|C_{\not 5} = 4 - 5 = -1$$

$$7 \leqslant k \leqslant 8: \quad 3C_0 - 2C_{\not{1}} - C_{\not{6}} \text{ cubic through 1 double/6 simple points}$$

$$\left(3C_0 - C_{\not{k}}\right) | \left(3C_0 - 2C_{\not{1}} - C_{\not{6}}\right) = 9C_0|C_0 + 2C_{\not{1}}|C_{\not{1}} + C_{\not{6}}|C_{\not{6}} = 9 - 2 - 6 = 1$$

$$\left(3C_0 - 2C_{\not{1}} - C_{\not{6}}\right) | \left(3C_0 - 2C_{\not{1}} - C_{\not{6}}\right) = 9C_0|C_0 + 4C_{\not{1}}|C_{\not{1}} + C_{\not{6}}|C_{\not{6}} = 9 - 4 - 6 = -1$$

$$k = 8: \quad 4C_0 - 2C_{\not{3}} - C_{\not{5}} \text{ quartic through 3 double/5 simple points}$$

$$\left(3C_0 - C_{\not{8}}\right) | \left(4C_0 - 2C_{\not{3}} - C_{\not{5}}\right) = 12C_0|C_0 + 2C_{\not{3}}|C_{\not{3}} + C_{\not{5}}|C_{\not{5}} = 12 - 2*3 - 1*5 = 1$$

$$\left(4C_0 - 2C_{\not{3}} - C_{\not{5}}\right) | \left(4C_0 - 2C_{\not{3}} - C_{\not{5}}\right) = 16C_0|C_0 + 4C_{\not{3}}|C_{\not{3}} + C_{\not{5}}|C_{\not{5}} = 16 - 4*3 - 1*5 = -1$$

$$k = 8: \quad 5C_0 - 2C_{\not{6}} - C_{\not{2}} \text{ quintic through 6 double/2 simple points}$$

$$\left(3C_0 - C_{\not{8}}\right) | \left(5C_0 - 2C_{\not{6}} - C_{\not{2}}\right) = 15C_0|C_0 + 2C_{\not{6}}|C_{\not{6}} + C_{\not{2}}|C_{\not{2}} = 15 - 2*6 - 1*2 = 1$$

$$\left(5C_0 - 2C_{\not{6}} - C_{\not{2}}\right) | \left(5C_0 - 2C_{\not{6}} - C_{\not{2}}\right) = 25C_0|C_0 + 4C_{\not{6}}|C_{\not{6}} + C_{\not{2}}|C_{\not{2}} = 25 - 4*6 - 1*2 = -1$$

$$k = 8: \quad 6C_0 - 3C_{\not{1}} - 2C_{\not{7}} \text{ sextic through 1 triple/7 double points}$$

$$\left(3C_0 - C_{\not{8}}\right) | \left(6C_0 - 3C_{\not{1}} - 2C_{\not{7}}\right) = 18C_0|C_0 + 3C_{\not{1}}|C_{\not{1}} + 2C_{\not{7}}|C_{\not{7}} = 18 - 3*1 - 2*7 = 1$$

$$\left(6C_0 - 3C_{\not{1}} - 2C_{\not{7}}\right) | \left(6C_0 - 3C_{\not{1}} - 2C_{\not{7}}\right) = 36C_0|C_0 + 9C_{\not{1}}|C_{\not{1}} + 4C_{\not{7}}|C_{\not{7}} = 36 - 9*1 - 4*7 = -1$$

$$\begin{cases} D=3 \\ k=8 \end{cases} \quad \begin{bmatrix}8\\1\end{bmatrix} \begin{cases} C_{\cancel{1}} \\ 6C_0 - 3C_{\cancel{1}} - 2C_{\cancel{7}} \end{cases} + \begin{bmatrix}8\\2\end{bmatrix} \begin{cases} C_0 - C_{\cancel{2}} \\ 5C_0 - 2C_{\cancel{6}} - C_{\cancel{2}} \end{cases} + \begin{bmatrix}8\\3\end{bmatrix} \begin{cases} 2C_0 - C_{\cancel{5}} \\ 4C_0 - 2C_{\cancel{3}} - C_{\cancel{5}} \end{cases} + 8 * 7 \, \underbrace{3C_0 - 2C_{\cancel{1}} - C_{\cancel{6}}}$$

$$= \; 240 \;\; E_8 \text{ roots}$$

$$\begin{cases} D=4 \\ k=7 \end{cases} \quad \begin{bmatrix}7\\1\end{bmatrix} \begin{cases} C_{\cancel{1}} \\ 3C_0 - 2C_{\cancel{1}} - C_{\cancel{6}} \end{cases} + \begin{bmatrix}7\\2\end{bmatrix} \begin{cases} C_0 - C_{\cancel{2}} \\ 2C_0 - C_{\cancel{5}} \end{cases} \;=\; 56 \;\; =< \; E_7 \; >$$

$$\begin{cases} D=5 \\ k=6 \end{cases} \quad \begin{bmatrix}6\\1\end{bmatrix} \begin{cases} C_{\cancel{1}} \\ 2C_0 - C_{\cancel{5}} \end{cases} + \begin{bmatrix}6\\2\end{bmatrix} \underbrace{C_0 - C_{\cancel{2}}} \;=\; 27 \;\; =< \; E_6 \; >$$

$$\begin{cases} D=6 \\ k=5 \end{cases} \quad \begin{bmatrix}5\\1\end{bmatrix} \underbrace{C_{\cancel{1}}} + \begin{bmatrix}5\\2\end{bmatrix} \underbrace{C_0 - C_{\cancel{2}}} + \begin{bmatrix}5\\0\end{bmatrix} \underbrace{2C_0 - C_{\cancel{5}}} \;=\; 16 \;\; =< SO\,(10) >= \; \text{Segre}$$

$$\begin{cases} D=7 \\ k=4 \end{cases} \quad \begin{bmatrix}4\\1\end{bmatrix} \underbrace{C_{\cancel{1}}} + \begin{bmatrix}4\\2\end{bmatrix} \underbrace{C_0 - C_{\cancel{2}}} \;=\; 10 \;\; =< SU\,(5) >$$

$$\begin{cases} D=8 \\ k=3 \end{cases} \quad \begin{bmatrix}3\\1\end{bmatrix} \begin{cases} C_{\cancel{1}} \\ C_0 - C_{\cancel{2}} \end{cases} \;= 6\, A_2\, A_1$$

$$\begin{cases} D=9 \\ k=2 \end{cases} \quad \begin{bmatrix}2\\1\end{bmatrix} \underbrace{C_{\cancel{1}}} + \begin{bmatrix}2\\0\end{bmatrix} \underbrace{C_0 - C_{\cancel{2}}} = 3$$

$$\begin{cases} D=10 \\ k=1 \end{cases} \quad \begin{bmatrix}2\\0\end{bmatrix} \underbrace{C_{\cancel{1}}} = 1 \text{ Hirz}$$