$$C_n = \sharp \text{ Klammerungen } a_1 \cdots a_n$$

$$C_0 = 0$$

$$C_1 = 1$$

$$C_2 = 1: \quad \underbrace{ab}$$

$$C_3 = 2: \quad a\underbrace{bc}: \quad \underbrace{ab}c$$

$$C_4 = 5: \quad a\overbrace{b\underbrace{cd}}: \quad a\overbrace{\underbrace{bc}d}: \quad \overbrace{ab}\overbrace{cd}: \quad \overbrace{a\underbrace{bc}}d: \quad \overbrace{\underbrace{ab}c}d$$

$$C_4 = C_1 C_3 + C_2 C_2 + C_3 C_1$$

quadratic Recursion

$$n \geqslant 2: \quad C_n = C_1 C_{n-1} + C_2 C_{n-2} + \cdot\cdot + C_{n-2} C_2 + C_{n-1} C_1$$

$$= \sum_{1 \leqslant m \leqslant n-1} C_m C_{n-m} = \sum_{m}^{0|n} C_m C_{n-m} = \sum_{i+j=n} C_i C_j$$

$$1 \leqslant m \leqslant n-1: \quad a_1 \cdots a_n = \overbrace{a_1 \cdots a_m}^{C_m \text{ Klammern}} \overbrace{a_{m+1} \cdot\cdot a_n}^{C_{n-m} \text{ Klammern}}$$

$$C(x) = \sum_{n \geqslant 0} C_n x^n = x + \sum_{n \geqslant 2} C_n x^n$$

$$C^2(x) - C(x) + x = 0$$

$$C^2(x) = C(x)\, C(x) = \sum_{i \geqslant 0} C_i x^i \sum_{j \geqslant 0} C_j x^j = \sum_{n \geqslant 0} x^n \sum_{i+j=n} C_i C_j$$

$$= \sum_{n \geqslant 2} x^n \sum_{i+j=n} C_i C_j = \sum_{n \geqslant 2} x^n C_n = C(x) - x$$

$$C\left(x\right)=\frac{1-\sqrt{1-4x}}{2}$$

$$C\left(x\right)=\frac{1}{2}\pm\sqrt{\frac{1}{4}-x}=\frac{1\pm\sqrt{1-4x}}{2}$$

$$C\left(0\right)=C_{0}=0\Rightarrow-\text{ sign}$$

$$C_{n}=\frac{1}{n}\begin{bmatrix}2n-2\\ n-1\end{bmatrix}$$

$$\frac{1}{2}\left(-1\right)^{n-1}4^{n}\,{}_{n}(1/2)=\frac{1}{2}\left(-1\right)^{n-1}4^{n}\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\cdots\left(\frac{1}{2}-n+1\right)$$

$$=4^{n-1}\left(1-\frac{1}{2}\right)\left(2-\frac{1}{2}\right)\cdots\left(n-\frac{3}{2}\right)=4^{n-1}\frac{1}{2}\frac{3}{2}\cdots\frac{2n-3}{2}=2^{n-1}1\cdot3\cdots\left(2n-3\right)=$$

$$=\frac{2\cdot4\cdot\left(2n-2\right)}{1\cdot2\cdot\cdots\left(n-1\right)}1\cdot3\cdots\left(2n-3\right)=\frac{\left(2n-2\right)!}{\left(n-1\right)!}$$

$$\text{binomial series } 2C\left(x\right)=1-\sqrt{1-4x}=1-\left(1-4x\right)^{1/2}=1-\sum_{n\geqslant0}\frac{{}_{n}(1/2)}{n!}\left(-4x\right)^{n}$$

$$=-\sum_{n\geqslant1}\frac{{}_{n}(1/2)}{n!}\left(-1\right)^{n}4^{n}x^{n}=\sum_{n\geqslant1}\frac{{}_{n}(1/2)}{n!}\left(-1\right)^{n-1}4^{n}x^{n}$$

$$\Rightarrow C_{n}=\frac{1}{2}\frac{{}_{n}(1/2)}{n!}\left(-1\right)^{n-1}4^{n}=\frac{\left(2n-2\right)!}{n!\left(n-1\right)!}=\frac{1}{n}\frac{\left(2n-2\right)!}{\left(n-1\right)!\left(n-1\right)!}=\text{ RHS}$$