$$\pi = \pi_1 \cdots \pi_n \in \mathsf{C}|n: \quad \pi_i = \pi(i)$$

$$i{:}\pi \in \underbrace{1|n-} \times \underbrace{\mathsf{C}|n} \to 0|1 \ni i\sharp\pi = \begin{cases} 1 & \pi_i > \pi_{i+} \\ 0 & \pi_i \leqslant \pi_{i+} \end{cases}$$

$$\text{descent } D(\pi) = \frac{1 \leqslant i < n}{\pi_i > \pi_{i+}}$$

$$S = \{0 < s_1 < s_2 << s_k < n\} \subset 1|n-$$

$$\underbrace{\mathsf{C}|n}_S^{\subset} = \frac{\pi \in \mathsf{C}|n}{D(\pi) \subset S} \text{ descents hochstens in S}$$

$$\underbrace{\mathsf{C}|n}_S^{=} = \frac{\pi \in \mathsf{C}|n}{D(\pi) = S} \text{ descents genau in S}$$

$$\sharp \underbrace{\mathsf{C}|n}_{S}^{\subset} = \begin{bmatrix} n \\ s_1{:}s_2 - s_1{:}s_3 - s_2{::}n - s_k \end{bmatrix} \text{ multi-nomial}$$

$$\frac{1}{n!}\sharp \underbrace{\mathsf{C}|n}_{S}^{\subset} = \frac{1}{s_1!\,(s_2 - s_1)!\,(s_3 - s_2)! \cdot (n - s_k)!}$$

$$\text{no descent } 0 < i < s_1 \Rightarrow \{\pi_1 << \pi_{s_1}\} \subset 1|n \Rightarrow \begin{bmatrix} n \\ s_1 \end{bmatrix} \text{ choices}$$

$$\text{no descent } s_1 < i < s_2 \Rightarrow \{\pi_{s_1+} << \pi_{s_2}\} \subset 1|n \llcorner \{\pi_1 << \pi_{s_1}\} \Rightarrow \begin{bmatrix} n - s_1 \\ s_2 - s_1 \end{bmatrix} \text{ choices}$$

$$\text{no descent } s_2 < i < s_3 \Rightarrow \{\pi_{s_2+} << \pi_{s_3}\} \subset 1|n \llcorner \{\pi_1 << \pi_{s_1}\} \underset{\text{disj}}{\dot{\cup}} \{\pi_{s_1+} << \pi_{s_2}\}$$

$$= 1|n \llcorner \{\pi_1 \cdots \pi_{s_2}\} \Rightarrow \begin{bmatrix} n - s_2 \\ s_3 - s_2 \end{bmatrix} \text{ choices}$$

$$\text{no descent } s_{k-} < i < s_k \Rightarrow \{\pi_{s_{k-}+} << \pi_{s_k}\} \subset 1|n \llcorner \{\pi_1 \cdots \pi_{s_{k-}}\} \Rightarrow \begin{bmatrix} n - s_{k-} \\ s_k - s_{k-} \end{bmatrix} \text{ choices}$$

$$\text{no descent } s_k < i < n \Rightarrow \{\pi_{s_k+} << \pi_n\} \subset 1|n \llcorner \{\pi_1 \cdots \pi_{s_k}\} \Rightarrow \begin{bmatrix} n - s_k \\ n - s_k \end{bmatrix} = 1 \text{ choice}$$

$$\Rightarrow \text{ LHS } = \begin{bmatrix} n \\ s_1 \end{bmatrix} \begin{bmatrix} n - s_1 \\ s_2 - s_1 \end{bmatrix} \begin{bmatrix} n - s_2 \\ s_3 - s_2 \end{bmatrix} \cdot\cdot \begin{bmatrix} n - s_{k-} \\ s_k - s_{k-} \end{bmatrix} \begin{bmatrix} n - s_k \\ n - s_k \end{bmatrix}$$

$$= \frac{n!}{s_1!\,(n - s_1)!} \frac{(n - s_1)!}{(s_2 - s_1)!\,(n - s_2)!} \frac{(n - s_2)!}{(s_3 - s_2)!\,(n - s_3)!} \cdots \frac{\left(n - s_{k-}\right)!}{\left(s_k - s_{k-}\right)!\,(n - s_k)!}$$

$$= \frac{n!}{s_1!\,(s_2 - s_1)!\,(s_3 - s_2)!\,\left(s_k - s_{k-}\right)!\,(n - s_k)!} = \text{ RHS}$$

$$\sharp \underbrace{\mathsf{C}|n}_{T}^{\subset} = \sum_{S \subset T} \sharp \underbrace{\mathsf{C}|n}_{S}^{=}$$

$$D(\pi) \subset T \Rightarrow \bigvee_{\text{ex}} S \subset T\text{:} \quad D(\pi) = S \Rightarrow \underbrace{\mathsf{C}|n}_{T}^{\subset} = \dot{\bigcup_{S \subset T}} \underbrace{\mathsf{C}|n}_{S}^{=}$$

$$T = \{\, t_0 = 0 << t_1 < t_2 << t_\ell < n = t_{\ell+} \}$$

$$\frac{1}{n!} \sharp \underbrace{\mathsf{C}|n}_{T}^{=} = \det_{i \leqslant j}^{0|\ell} \frac{1}{\left(t_{j+} - t_i\right)!}$$

$$T \supset S = \{1 \leqslant t_{i_1} < t_{i_2} << t_{i_k} < n\}$$

$$\frac{1}{n!} \sharp \underbrace{\mathsf{C}|n}_{T}^{=} \underset{\text{Moe}}{=} \frac{1}{n!} \sum_{S \subset T} (-1)^{T-S} \sharp \underbrace{\mathsf{C}|n}_{S}^{\subset} = \frac{1}{n!} \sum_{i_1 << i_k}^{1|\ell} (-1)^{\ell - k} \begin{bmatrix} n \\ t_{i_1} {:} t_{i_2} - t_{i_1} {:} t_{i_3} - t_{i_2} {::} n - t_{i_k} \end{bmatrix}$$

$$= \sum_{k}^{0|\ell} (-1)^{\ell - k} \sum_{i_1 << i_k}^{1|\ell} \frac{1}{t_{i_1}! \left(t_{i_2} - t_{i_1}\right)! \left(t_{i_3} - t_{i_2}\right)! \cdot\cdot \left(n - t_{i_k}\right)!} = \det_{i \leqslant j}^{0|\ell} \frac{1}{\left(t_{j+} - t_i\right)!}$$

$$\sharp \underbrace{\mathsf{C}|n}_{T}^{=} = \det_{i \leqslant j}^{0|\ell} \begin{bmatrix} n - t_i \\ t_{j+} - t_i \end{bmatrix}$$