$$d_{\nabla}\nabla = d\nabla - \nabla \wedge \nabla$$

$$^{\alpha}\widehat{d_{\nabla}\nabla} = d\,^{\alpha}\nabla - {}^{\alpha}\nabla \wedge {}^{\alpha}\nabla$$

$$^{\beta}\widehat{d_{\nabla}\nabla} = {}^{\beta}g_{\alpha}\,{}^{\alpha}\widehat{d_{\nabla}\nabla}\,{}^{\alpha}g_{\beta}$$

$$-d\,{}^{\alpha}g_{\beta} = {}^{\alpha}g_{\beta}\underbrace{d^{\beta}g_{\alpha}}\,{}^{\alpha}g_{\beta}$$

$$^{\beta}\widehat{d_{\nabla}\nabla} = d\,^{\beta}\nabla - {}^{\beta}\nabla \wedge {}^{\beta}\nabla = d\,\widehat{\underbrace{d^{\beta}g_{\alpha}}\,{}^{\alpha}g_{\beta} + {}^{\beta}g_{\alpha}\,{}^{\alpha}\nabla{}^{\alpha}g_{\beta}} - \widehat{\underbrace{d^{\beta}g_{\alpha}}\,{}^{\alpha}g_{\beta} + {}^{\beta}g_{\alpha}\,{}^{\alpha}\nabla{}^{\alpha}g_{\beta}} \wedge \widehat{\underbrace{d^{\beta}g_{\alpha}}\,{}^{\alpha}g_{\beta} + {}^{\beta}g_{\alpha}\,{}^{\alpha}\nabla{}^{\alpha}g_{\beta}}$$

$$= \widehat{\overbrace{d\,\underbrace{d^{\beta}g_{\alpha}}}^{=0}\,{}^{\alpha}g_{\beta} - \underbrace{d^{\beta}g_{\alpha}} \wedge \underbrace{d^{\alpha}g_{\beta}} + \underbrace{d^{\beta}g_{\alpha}} \wedge {}^{\alpha}\nabla\,{}^{\alpha}g_{\beta} + {}^{\beta}g_{\alpha}\underbrace{d^{\alpha}\nabla}\,{}^{\alpha}g_{\beta} - {}^{\beta}g_{\alpha}\,{}^{\alpha}\nabla \wedge \underbrace{d^{\alpha}g_{\beta}}}$$

$$- \underbrace{d^{\beta}g_{\alpha}}\,{}^{\alpha}g_{\beta} \wedge \underbrace{d^{\beta}g_{\alpha}}\,{}^{\alpha}g_{\beta} - \underbrace{d^{\beta}g_{\alpha}}\,{}^{\alpha}g_{\beta} \wedge {}^{\beta}g_{\alpha}\,{}^{\alpha}\nabla\,{}^{\alpha}g_{\beta} - {}^{\beta}g_{\alpha}\,{}^{\alpha}\nabla\,{}^{\alpha}g_{\beta} \wedge \underbrace{d^{\beta}g_{\alpha}}\,{}^{\alpha}g_{\beta} - {}^{\beta}g_{\alpha}\,{}^{\alpha}\nabla\,{}^{\alpha}g_{\beta} \wedge {}^{\beta}g_{\alpha}\,{}^{\alpha}\nabla\,{}^{\alpha}g_{\beta}$$

$$= \underbrace{d^{\beta}g_{\alpha}} \overset{*}{\wedge}\, {}^{\alpha}g_{\beta}\underbrace{d^{\beta}g_{\alpha}}\,{}^{\alpha}g_{\beta} + \underbrace{d^{\beta}g_{\alpha}} \overset{\sharp}{\wedge}\, {}^{\alpha}\nabla\,{}^{\alpha}g_{\beta} + {}^{\beta}g_{\alpha}\underbrace{d^{\alpha}\nabla}\,{}^{\alpha}g_{\beta} + {}^{\beta}g_{\alpha}\,{}^{\alpha}\nabla \underset{*}{\wedge} {}^{\alpha}g_{\beta}\underbrace{d^{\beta}g_{\alpha}}\,{}^{\alpha}g_{\beta}$$

$$- \underbrace{d^{\beta}g_{\alpha}}\,{}^{\alpha}g_{\beta} \overset{*}{\wedge}\, \underbrace{d^{\beta}g_{\alpha}}\,{}^{\alpha}g_{\beta} - \underbrace{d^{\beta}g_{\alpha}} \overset{\sharp}{\wedge}\, {}^{\alpha}\nabla\,{}^{\alpha}g_{\beta} - {}^{\beta}g_{\alpha}\,{}^{\alpha}\nabla\,{}^{\alpha}g_{\beta} \underset{*}{\wedge} \underbrace{d^{\beta}g_{\alpha}}\,{}^{\alpha}g_{\beta} - {}^{\beta}g_{\alpha}\,{}^{\alpha}\nabla \wedge {}^{\alpha}\nabla\,{}^{\alpha}g_{\beta}$$

$$= {}^{\beta}g_{\alpha}\underbrace{d^{\alpha}\nabla - {}^{\alpha}\nabla \wedge {}^{\alpha}\nabla}\,{}^{\alpha}g_{\beta} = {}^{\beta}g_{\alpha}\,{}^{\alpha}\widehat{d_{\nabla}\nabla}\,{}^{\alpha}g_{\beta}$$