

$$\overline{\partial \log h}^\alpha = \underline{\partial^\alpha h} h_\alpha$$

$$\overline{\partial \log h^j}^\alpha = \underline{\partial_{ik}^\alpha h}^{kj} h_\alpha$$

$${}^\beta h = \beta g_\alpha {}^\alpha h \beta g_\alpha^*$$

$$\overline{\partial \log h}^\beta = \underline{\partial^\beta g_\alpha} {}^\alpha g_\beta + \beta g_\alpha \overline{\partial \log h}^\alpha {}^\alpha g_\beta$$

$$\partial^\beta h = \underline{\partial^\beta g_\alpha} {}^\alpha h \beta g_\alpha^* + \beta g_\alpha \underline{\partial^\alpha h} \beta g_\alpha^* + \beta g_\alpha {}^\alpha h \overline{\partial^\beta g_\alpha^*} \stackrel{=0}{}$$

$$\text{LHS} = \underline{\partial^\beta h} h_\beta = \overline{\underline{\partial^\beta g_\alpha} {}^\alpha h \beta g_\alpha^* + \beta g_\alpha \underline{\partial^\alpha h} \beta g_\alpha^*} {}^\alpha g_\beta h_\alpha {}^\alpha g_\beta$$

$$= \underline{\partial^\beta g_\alpha} {}^\alpha h \underbrace{\beta g_\alpha^* {}^\alpha g_\beta}_{=1} h_\alpha {}^\alpha g_\beta + \beta g_\alpha \underline{\partial^\alpha h} \underbrace{\beta g_\alpha^* {}^\alpha g_\beta}_{=1} h_\alpha {}^\alpha g_\beta = \underline{\partial^\beta g_\alpha} \underbrace{{}^\alpha h h_\alpha}_{=1} {}^\alpha g_\beta + \beta g_\alpha \underline{\partial^\alpha h} h_\alpha {}^\alpha g_\beta = \text{RHS}$$