

$$\text{free } \mathbb{1} = \underbrace{\mathbb{1}^0 \dots \mathbb{1}^n}_{\mathbb{1}} \supset \mathbb{F} \Rightarrow \mathbb{F} = \underbrace{\mathbb{F}^0 \dots \mathbb{F}^m}_{\mathbb{1}} \text{ free } m \leq n$$

$$\mathbb{F} \cap \underbrace{\mathbb{1}^0 \dots \mathbb{1}^{k-1}}_{\mathbb{1}} = \underbrace{\mathbb{F}^0 \dots \mathbb{F}^j}_{\mathbb{1}} \text{ free } j \leq k$$

$$\mathbb{F} = \frac{a \in \mathbb{1}}{\underbrace{\mathbb{1}^k a + \mathbb{1}^0 \dots \mathbb{1}^{k-1}}_{\mathbb{1}} \cap \mathbb{F} \neq \emptyset} \triangleleft \mathbb{1}$$

$$a \in \mathbb{F} \Rightarrow \mathbb{1}^k a + \sum_{i \leq k} \mathbb{1}^i \mathbb{1} \in \mathbb{F}$$

$$\Rightarrow \mathbb{1}^k \underbrace{a + a}_{\mathbb{1}} + \sum_{i \leq k} \mathbb{1}^i \underbrace{\mathbb{1} + \mathbb{1}}_{\mathbb{1}} = \overbrace{\mathbb{1}^k a + \sum_{i \leq k} \mathbb{1}^i \mathbb{1}}^{\in \mathbb{F}} + \overbrace{\mathbb{1}^k a + \sum_{i \leq k} \mathbb{1}^i \mathbb{1}}^{\in \mathbb{F}} \in \mathbb{F} \Rightarrow a + a \in \mathbb{F}$$

$$b \in \mathbb{1} \Rightarrow \mathbb{1}^k ab + \sum_{i \leq k} \mathbb{1}^i \mathbb{1} b = \overbrace{\mathbb{1}^k a + \sum_{i \leq k} \mathbb{1}^i \mathbb{1}}^{\in \mathbb{F}} b \in \mathbb{F} \Rightarrow ab \in \mathbb{F}$$

$$\Rightarrow \mathbb{F} = \mathbb{F}^k \mathbb{1} \Rightarrow \mathbb{F}^k = \sum_{i \leq k} \mathbb{1}^i \mathbb{F}^k + \mathbb{1}^k \mathbb{F}^k \in \mathbb{F}$$

$$\mathbb{F} \cap \underbrace{\mathbb{1}^0 \dots \mathbb{1}^k}_{\mathbb{1}} = \mathbb{F} \cap \underbrace{\mathbb{1}^0 \dots \mathbb{1}^{k-1}}_{\mathbb{1}} + \mathbb{F}^k \mathbb{1}$$

$$\supset: \mathbb{1}^k \mathbb{F}^k + \sum_{i \leq k} \mathbb{1}^i \mathbb{F}^k \in \mathbb{F} \cap \underbrace{\mathbb{1}^0 \dots \mathbb{1}^k}_{\mathbb{1}}$$

$$\subset: \mathbb{F} \cap \underbrace{\mathbb{1}^0 \dots \mathbb{1}^k}_{\mathbb{1}} \ni \mathbb{1} = \sum_{i \leq k} \mathbb{1}^i \mathbb{1} = \sum_{i \leq k} \mathbb{1}^i \mathbb{1} + \mathbb{1}^k \mathbb{1} \in \mathbb{F} \Rightarrow \mathbb{1} \in \mathbb{F} \Rightarrow \mathbb{1} = \mathbb{F}^k b$$

$$\Rightarrow \mathbb{1} - \mathbb{F}^k b = \sum_{i \leq k} \mathbb{1}^i \mathbb{1} + \mathbb{1}^k \mathbb{1} - \mathbb{1}^k \mathbb{F}^k b - \sum_{i \leq k} \mathbb{1}^i \mathbb{F}^k b$$

$$= \sum_{i \leq k} \mathbb{1}^i \underbrace{\mathbb{1} - \mathbb{F}^k b}_{\mathbb{1}} + \mathbb{1}^k \underbrace{\mathbb{1} - \mathbb{F}^k b}_{=0} \in \mathbb{F} \cap \underbrace{\mathbb{1}^0 \dots \mathbb{1}^{k-1}}_{\mathbb{1}}$$

$$\mathbb{F} \cap \underbrace{\mathbb{1}^0 \dots \mathbb{1}^{k-1}}_{\mathbb{1}} \cap \mathbb{F}^k \mathbb{1} = \mathbb{F}^k \begin{cases} b \in \mathbb{1} \\ \mathbb{F}^k b = 0 \end{cases}$$

$$\mathbb{F}^k b = \mathbb{1}^k \mathbb{F}^k b + \sum_{i \leq k} \mathbb{1}^i \mathbb{F}^k b \in \mathbb{F} \cap \underbrace{\mathbb{1}^0 \dots \mathbb{1}^{k-1}}_{\mathbb{1}} \Leftrightarrow \mathbb{F}^k b = 0$$

$$\mathbb{F} \cap \underbrace{\mathbb{1}^0 \dots \mathbb{1}^k}_{\mathbb{1}} = \underbrace{\mathbb{F}^0 \dots \mathbb{F}^j \mathbb{F}^k}_{\mathbb{1}} \text{ free } j + 1 \leq k + 1$$