$$\text{IntR } Z \subset Q = \frac{Z}{\overset{\times}{Z}}$$

$$0 \neq I \blacktriangleleft Q \Leftrightarrow \begin{cases} ZI \subset I \\ \overset{\times}{\underset{a}{\bigvee}} aI \subset Z \end{cases}$$

$$I \blacktriangleleft Q \blacktriangleright J \Rightarrow \begin{cases} I + J \\ I \cap J \quad \blacktriangleleft Q \\ IJ \end{cases}$$

$$\underset{\substack{\in \overset{\times}{Z}}}{\underline{a\acute{a}}} \underbrace{I + \acute{I}} = \underset{\subset Z}{\underline{aI}} \, \acute{a} + a \underset{\subset Z}{\underline{\acute{I}\acute{a}}} \subset Z\acute{a} + aZ \subset Z$$

$$\dot{a} \underline{I \cap \acute{I}} \subset \dot{a} \, \dot{I} \subset Z$$

$$0 \neq \dot{b} \in \dot{I} \Rightarrow 0 \neq \dot{a} \, \dot{b} \in Z\dot{b} \cap \dot{a} \, \dot{I} \subset \dot{I} \cap Z \Rightarrow 0 \neq ab\acute{a} \, \acute{b} \in I \cap \acute{I}$$

$$0 \neq b\acute{b} \in I \acute{I}$$

$$\underset{\substack{\in \overset{\times}{Z}}}{\underline{a\acute{a}}} \, \underline{I\acute{I}} = \underset{\subset Z}{\underline{aI}} \, \underset{\subset Z}{\underline{\acute{a}\acute{I}}} \subset Z$$

$$\bar{I} = \begin{cases} r \in Q \\ rI \subset Z \end{cases} \quad \blacktriangleleft Q$$

$$\underline{Z\bar{I}}I = Z\underline{\bar{I}I} \subset ZZ = Z \Rightarrow Z\bar{I} \subset \bar{I}$$

$$a \in \overset{\times}{Z} : aI \subset Z \Rightarrow a \in \bar{I} \neq 0$$

$$0 \neq b \in I \Rightarrow 0 \neq ab \in I \cap Z \Rightarrow ab \in \overset{\times}{Z}$$

$$\underline{ab} \, \bar{I} = a \underset{\subset Z}{\underline{b\bar{I}}} \subset aZ \subset Z$$

$$I\,\bar{I} \triangleleft Z$$

$$Z\,\underline{I\,\bar{I}} \subset I\,\bar{I} \subset Z$$

$$I \text{ inv } \Leftrightarrow I\,\bar{I} = Z \Leftrightarrow I\,\bar{I} \supset Z$$

$$I \prec J \Leftrightarrow \bigvee \mathfrak{a} \triangleleft Z: \quad J = I\mathfrak{a} \subset IZ = I$$

$$I \underset{\text{inv}}{\blacktriangleleft} Q \underset{\text{inv}}{\blacktriangleright} J \Rightarrow \begin{cases} IJ \underset{\text{inv}}{\blacktriangleleft} Q \\[4pt] \overline{IJ} = \bar{I}\,\bar{J} \end{cases}$$

$$\underline{\bar{I}\,\bar{J}}\,\underline{IJ} = \underline{I\,\bar{I}}\,\underline{J\,\bar{J}} = ZZ = Z \Rightarrow \bar{I}\,\bar{J} \subset \overline{IJ}$$

$$Z \supset \overline{IJ}\,IJ \supset \underline{\bar{I}\,\bar{J}}\,\underline{IJ} = \underline{I\,\bar{I}}\,\underline{J\,\bar{J}} = Z$$

$$\Rightarrow IJ \text{ inv} : \overline{IJ}\,IJ = Z = IJ\,\bar{I}\,\bar{J} \Rightarrow \overline{IJ} = \overline{IJ}\,Z = \overline{IJ}\,IJ\,\bar{I}\,\bar{J} = Z\,\bar{I}\,\bar{J} = \bar{I}\,\bar{J}$$

$$\text{inv } I \supset J \Rightarrow I \prec J$$

$$Z = \bar{I}\,I \supset \bar{I}\,J \triangleleft Z$$

$$J = ZJ = \underline{I\,\bar{I}}\,J = I\,\underline{\bar{I}\,J}$$

$$\text{inv } I = JK \Rightarrow J \text{ inv } K$$

$$Z = \bar{I}\,I = \bar{I}\,\underline{JK} = \underline{\bar{I}\,J}\,K \Rightarrow \bar{I}\,J \subset \bar{K} \Rightarrow Z = \underline{\bar{I}\,J}\,K \subset \bar{K}\,K$$