$$\begin{array}{c} \mathbb{1}^\sharp\llcorner{}^{m}\mathbb{1} \\ \Big\downarrow d \\ \mathbb{1}^\sharp\llcorner{}^{m+1}\mathbb{1} \end{array}$$

$$\underbrace{\mathbb{1}^\sharp\llcorner\mathbb{1}}\underset{\mathbb{1}}{\otimes}\cdots\underset{\mathbb{1}}{\otimes}\underbrace{\mathbb{1}^\sharp\llcorner\mathbb{1}} \ni \phi = \varphi^0\otimes\cdots\otimes\varphi^m$$

$$d\overbrace{\varphi^0\otimes\cdots\otimes\varphi^m} = \sum_{0\leqslant j\leqslant m}\overset{j}{-1}\,\varphi^0\otimes\cdots\otimes\varphi^{j-1}\otimes\underbrace{d\varphi^j}\otimes\varphi^{j+1}\otimes\cdots\otimes\varphi^m = \sum_{0\leqslant j\leqslant m}\overset{j}{-1}\,\overbrace{\bigotimes_i^{j}\varphi^i}\otimes\underbrace{d\varphi^j}\otimes\overbrace{\bigotimes_k^{m\llcorner j}\varphi^{k+1}}$$

$$\begin{array}{ccc} & & {}^{m}\otimes\underbrace{\mathbb{1}^\sharp\llcorner\mathbb{1}} \\ & \swarrow d & \Big\downarrow \\ {}^{m+1}\otimes\underbrace{\mathbb{1}^\sharp\llcorner\mathbb{1}} & \xleftarrow{\ d\ } & {}^{m}\otimes\underbrace{\mathbb{1}^\sharp\llcorner\mathbb{1}} \end{array}$$

$$\overbrace{d\underbrace{\phi\phi}}\otimes\phi' - \underbrace{\phi\phi}\otimes\underbrace{d\phi'} - \underbrace{d\phi}\otimes\underbrace{\phi\phi'} + \phi\otimes\overbrace{d\underbrace{\phi\phi'}}$$

$$= \overbrace{\underbrace{d\phi}\phi - \phi\otimes\underbrace{d\phi}}\otimes\phi' - \underbrace{\phi\phi}\otimes\underbrace{d\phi'} - \underbrace{d\phi}\otimes\underbrace{\phi\phi'} + \phi\otimes\overbrace{\underbrace{d\phi}\otimes\phi' + \phi\underbrace{d\phi'}}$$

$$= \overbrace{\underbrace{d\phi}\phi}\otimes\phi' - \phi\otimes\underbrace{d\phi}\otimes\phi' - \underbrace{\phi\phi}\otimes\underbrace{d\phi'} - \underbrace{d\phi}\otimes\underbrace{\phi\phi'} + \phi\otimes\underbrace{d\phi}\otimes\phi' + \phi\otimes\overbrace{\phi\underbrace{d\phi'}} = 0$$

$$d\overbrace{\varphi^0\otimes\cdots\otimes\varphi^M} = \sum_{0\leqslant m\leqslant M}\overset{m}{-1}\,\varphi^0\otimes\cdots\otimes\varphi^{m-1}\otimes\underbrace{d\varphi^m}\otimes\varphi^{m+1}\otimes\cdots\otimes\varphi^M = \sum_{0\leqslant m\leqslant M}\overset{m}{-1}\,\underbrace{\bigotimes_\mu^{m}\varphi^\mu}\otimes\underbrace{d\varphi^m}\otimes\underbrace{\bigotimes_\mu^{M\llcorner m}\varphi^{\mu+1}}$$

$$d\underbrace{\bigotimes_m^{M}\varphi^m}\otimes\underbrace{\overbrace{\phi\phi}\otimes\phi' - \phi\otimes\overbrace{\phi\phi'}}\otimes\underbrace{\bigotimes_n^{N}\varphi^n}$$

$$= \sum_{0\leqslant m\leqslant M}\overset{m}{-1}\,\underbrace{\bigotimes_\mu^{m}\varphi^\mu}\otimes\underbrace{d\varphi^m}\otimes\underbrace{\bigotimes_\mu^{M\llcorner m}\varphi^{\mu+1}}\otimes\underbrace{\overbrace{\phi\phi}\otimes\phi' - \phi\otimes\overbrace{\phi\phi'}}_{=0}\otimes\underbrace{\bigotimes_n^{N}\varphi^n}$$

$$+\underbrace{\bigotimes_m^{M}\varphi^m}\otimes\underbrace{d\underbrace{\phi\phi}\otimes\phi' - \overbrace{\phi\phi}\otimes\overbrace{d\phi'} - \overbrace{d\phi}\otimes\overbrace{\phi\phi'} + \phi\otimes d\underbrace{\phi\phi'}}_{=0}\otimes\underbrace{\bigotimes_n^{N}\varphi^n}$$

$$+\left(\bigotimes_{m}^{M}\omega^{m}\right)\otimes\underbrace{\left(\omega\omega\otimes\omega'-\omega\otimes\omega\omega'\right)}_{=0}\otimes\sum_{0\leqslant n\leqslant N}(-1)^{n}\left(\bigotimes_{\nu}^{n}\omega^{\nu}\right)\otimes d\omega^{n}\otimes\left(\bigotimes_{\nu}^{N-n}\omega^{\nu+1}\right)$$

$$d\left(\omega^{0}\otimes\cdots\otimes\omega^{m}\right)=\sum_{0\leqslant j\leqslant m}(-1)^{j}\,\omega^{0}\otimes\cdots\otimes\omega^{j-1}\otimes d\omega^{j}\otimes\omega^{j+1}\otimes\cdots\otimes\omega^{m}=\sum_{0\leqslant j\leqslant m}(-1)^{j}\left(\bigotimes_{i\in j}\omega^{i}\right)\otimes d\omega^{j}\otimes\left(\bigotimes_{k\in m-j}\omega^{k+1}\right)$$

$$d(\omega\otimes\omega')=d\omega\otimes\omega'+(-1)^{k}\otimes d\omega'$$

$$d\left(\bigotimes_{m}^{M}\omega^{m}\otimes\bigotimes_{n}^{N}\omega^{n}\right)=\sum_{0\leqslant m\leqslant M}(-1)^{m}\left(\bigotimes_{\mu}^{m}\omega^{\mu}\right)\otimes d\omega^{m}\otimes\left(\bigotimes_{\mu}^{M-m}\omega^{\mu+1}\right)\otimes\left(\bigotimes_{n}^{N}\omega^{n}\right)$$

$$+(-1)^{m}\left(\bigotimes_{m}^{M}\omega^{m}\right)\otimes\sum_{0\leqslant n\leqslant N}(-1)^{n}\left(\bigotimes_{\nu}^{n}\omega^{\nu}\right)\otimes d\omega^{n}\otimes\left(\bigotimes_{\nu}^{N-n}\omega^{\nu+1}\right)=d\left(\bigotimes_{m}^{M}\omega^{m}\right)\otimes\left(\bigotimes_{n}^{N}\omega^{n}\right)+(-1)^{m}\left(\bigotimes_{m}^{M}\omega^{m}\right)\otimes d\left(\bigotimes_{n}^{N}\omega^{n}\right)$$