

$k \in \mathbb{Z}$  field  
 subfield  $k \subset K \in \mathbb{Z}$  field

$$\dim_{\mathbb{Z}} K = n$$

$$\dim_{\mathbb{Z}} K \otimes_{\mathbb{Z}} \mathbb{Z} = n$$

$$\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z} = \mathbb{Z}$$

$$e \in K \xrightarrow{\cong} \mathbb{Z} \otimes_{\mathbb{Z}} K \ni e$$

$$\Rightarrow \dim_{\mathbb{Z}} K = \dim_{\mathbb{Z}} \mathbb{Z} \otimes_{\mathbb{Z}} K = n = \dim_{\mathbb{Z}} K$$