$$
{}_{z}^{\tau}\vartheta = \sum_{n}^{\mathbb{Z}} \exp\left(\pi i n^2\tau + 2\pi i z n\right)
$$

$$
{}_{z}^{\tau}\vartheta_{\varkappa}^{k} = {}_{z+\varkappa\tau/\ell+k/\ell}^{\tau}\vartheta \; \mathfrak{e}^{\,\pi i\varkappa\tau\varkappa/\ell^2 + 2\pi iz\varkappa/\ell + 2\pi ik\varkappa/\ell^2}
$$

$$
{}_{z}\underbrace{S_{k/\ell}T_{\varkappa/\ell}\vartheta}_{} = {}_{z+k/\ell}\underbrace{T_{\varkappa/\ell}\vartheta}_{} = \exp\left(\pi i\left(\frac{\varkappa}{\ell}\right)^2\tau + 2\pi i\left(z + \frac{k}{\ell}\right)\frac{\varkappa}{\ell}\right){}_{z+k/\ell+\varkappa\tau/\ell}\vartheta
$$

$$
= \exp\left(\pi i\left(\frac{\varkappa}{\ell}\right)^2\tau + 2\pi i\left(z + \frac{k}{\ell}\right)\frac{\varkappa}{\ell}\right){}_{z+k/\ell+\varkappa\tau/\ell}\vartheta
$$

$$
= \exp\left(\pi i\left(\frac{\varkappa}{\ell}\right)^2\tau + 2\pi i\left(z + \frac{k}{\ell}\right)\frac{\varkappa}{\ell}\right)\sum_{n}^{\mathbb{Z}} \exp\left(\pi i n^2\tau + 2\pi i\left(z + \frac{k}{\ell} + \frac{\varkappa}{\ell}\tau\right)n\right)
$$

$$
= \sum_{n}^{\mathbb{Z}} \exp\left(\pi i\left(n + \frac{\varkappa}{\ell}\right)^2\tau + 2\pi i\left(z + \frac{k}{\ell}\right)\left(n + \frac{\varkappa}{\ell}\right)\right) = {}_{z}\vartheta_{k:\varkappa}^{\ell}
$$

$$
{}_{z}\underbrace{T_{\xi/\ell}S_{x/\ell}\vartheta_{k:\varkappa}^{\ell}}_{} = \exp\left(\pi i\left(\frac{\xi}{\ell}\right)^2\tau + 2\pi iz\frac{\xi}{\ell}\right){}_{z+\xi\tau/\ell}\underbrace{S_{x/\ell}\vartheta_{k:\varkappa}^{\ell}}_{}
$$

$$
= \exp\left(\pi i\left(\frac{\xi}{\ell}\right)^2\tau + 2\pi iz\frac{\xi}{\ell}\right){}_{z+\xi\tau/\ell+x/\ell}\vartheta_{k:\varkappa}^{\ell}
$$

$$
= \exp\left(\pi i\left(\frac{\xi}{\ell}\right)^2\tau + 2\pi iz\frac{\xi}{\ell}\right)\sum_{n}^{\mathbb{Z}} \exp\left(\pi i\left(n + \frac{\varkappa}{\ell}\right)^2\tau + 2\pi i\left(z + \frac{\xi}{\ell}\tau + \frac{x}{\ell} + \frac{k}{\ell}\right)\left(n + \frac{\varkappa}{\ell}\right)\right)
$$

$$
= \exp\left(-2\pi i\frac{(x+k)\,\xi}{\ell^2}\right)\sum_{n}^{\mathbb{Z}} \exp\left(\pi i\left(n + \frac{\xi+\varkappa}{\ell}\right)^2\tau + 2\pi i\left(z + \frac{\xi+\varkappa}{\ell}\tau + \frac{x+k}{\ell}\right)\left(n + \frac{\xi+\varkappa}{\ell}\right)\right)
$$

$$
= \exp\left(-2\pi i\frac{(x+k)\,\xi}{\ell^2}\right){}_{z}\vartheta_{x+k:\xi+\varkappa}^{\ell}
$$

$$
{}_{\ell z}\underbrace{S_{k/\ell}T_{\varkappa/\ell}\vartheta}_{} = {}_{\ell z+k/\ell}\underbrace{T_{\varkappa/\ell}\vartheta}_{} = \exp\left(\pi i\left(\frac{\varkappa}{\ell}\right)^2\tau + 2\pi i\left(\ell z + \frac{k}{\ell}\right)\frac{\varkappa}{\ell}\right){}_{\ell z+k/\ell+\varkappa\tau/\ell}\vartheta
$$

$$
= \exp\left(\pi i\left(\frac{\varkappa}{\ell}\right)^2\tau + 2\pi i\left(\ell z + \frac{k}{\ell}\right)\frac{\varkappa}{\ell}\right){}_{\ell z+k/\ell+\varkappa\tau/\ell}\vartheta
$$

$$
= \exp\left(\pi i\left(\frac{\varkappa}{\ell}\right)^2\tau + 2\pi i\left(\ell z + \frac{k}{\ell}\right)\frac{\varkappa}{\ell}\right)\sum_{n}^{\mathbb{Z}} \exp\left(\pi i n^2\tau + 2\pi i\left(\ell z + \frac{k}{\ell} + \frac{\varkappa}{\ell}\tau\right)n\right)
$$

$$= \sum_{n}^{\mathbb{Z}} \exp\left(\pi i \left(n + \frac{\varkappa}{\ell}\right)^2 \tau + 2\pi i \left(\ell z + \frac{k}{\ell}\right)\left(n + \frac{\varkappa}{\ell}\right)\right) = {}_z\vartheta^{\ell}_{k:\varkappa}$$