$$1{\times}\mathbb{Z}\ell{\times}\mathbb{Z}\ell \ni 1{:}\ell\varkappa{:}\ell k$$

$$_z\overbrace{1{:}\ell\varkappa{:}\ell k}\ltimes\lrcorner = \exp\left(\pi i\ell^2\varkappa\lrcorner\varkappa + 2\pi iz\ell k\right)\ {}_{z+\ell\varkappa\lrcorner+\ell k}\lrcorner$$

$$_z\lrcorner = \sum_{\varkappa}^{\ell^2} c_\varkappa \sum_k^{\mathbb{Z}} \exp\left(\pi i\left(\varkappa/\ell+\ell k\right)\lrcorner\left(\varkappa/\ell+\ell k\right)+2\pi iz\left(\varkappa/\ell+\ell k\right)\right)$$

$$= \sum_{\varkappa}^{\ell^2} c_\varkappa \sum_k^{\mathbb{Z}} \exp\left(\pi i\left(\varkappa/\ell+\ell k\right)\lrcorner\left(\varkappa/\ell+\ell k\right)\right)\exp\left(2\pi iz\left(\varkappa/\ell+\ell k\right)\right)$$

$$ {}^{\lrcorner}_z\vartheta^k_\varkappa = \exp\left(\pi i\varkappa\lrcorner\varkappa/\ell^2+2\pi i\left(z+k/\ell\right)\varkappa/\ell\right)\ {}^{\lrcorner}_{z+\varkappa\lrcorner/\ell+k/\ell}\vartheta$$

$$= \exp\left(\pi i\varkappa\lrcorner\varkappa/\ell^2\right)\exp\left(2\pi i\left(z+k/\ell\right)\varkappa/\ell\right)\ {}^{\lrcorner}_{z+\varkappa\lrcorner/\ell+k/\ell}\vartheta$$

$$= \sum_n^{\mathbb{Z}} \exp\left(\pi i\left(n+\varkappa/\ell\right)\lrcorner\left(n+\varkappa/\ell\right)\right)\exp\left(2\pi i\left(z+k/\ell\right)\left(n+\varkappa/\ell\right)\right)$$

$$ {}^{\lrcorner}\vartheta^{k+\ell q}_{\varkappa+\ell p} = \exp\left(2\pi i\varkappa q\right)\ {}^{\lrcorner}\vartheta^k_\varkappa$$

$$ {}^{\lrcorner}_z\vartheta^{k+\ell n}_{\varkappa+\ell\nu} = \exp\left(\pi i\left(\varkappa+\ell\nu\right)\lrcorner\left(\varkappa+\ell\nu\right)/\ell^2\right)\exp\left(2\pi i\left(z+\left(k+\ell n\right)/\ell\right)\left(\varkappa+\ell\nu\right)/\ell\right)\ {}^{\lrcorner}_{z+\left(\varkappa+\ell\nu\right)\lrcorner/\ell+\left(k+\ell n\right)/\ell}\vartheta$$

$$= \exp\left(\pi i\varkappa\lrcorner\varkappa/\ell^2+2\pi i\varkappa\lrcorner\nu/\ell+\pi i\nu\lrcorner\nu\right)$$

$$\exp\left(2\pi i\left(z+k/\ell+n\right)\left(\varkappa/\ell+\nu\right)\right)\ {}^{\lrcorner}_{z+\left(\varkappa/\ell+\nu\right)\lrcorner+k/\ell+n}\vartheta$$

$$= \exp\left(\pi i\varkappa\lrcorner\varkappa/\ell^2+2\pi i\varkappa\lrcorner\nu/\ell+\pi i\nu\lrcorner\nu\right)\exp\left(2\pi i\left(z+k/\ell+n\right)\left(\varkappa/\ell+\nu\right)\right)$$

$$\exp\left(-\pi i\nu\lrcorner\nu-2\pi i\left(z+\varkappa\lrcorner/\ell+k/\ell\right)\nu\right)\ {}^{\lrcorner}_{z+\varkappa\lrcorner/\ell+k/\ell}\vartheta$$

$$= \exp\left(\pi i\varkappa\lrcorner\varkappa/\ell^2+2\pi i\varkappa\lrcorner\nu/\ell+\pi i\nu\lrcorner\nu+2\pi i\left(k/\ell+n\right)\left(\varkappa/\ell+\nu\right)-\pi i\nu\lrcorner\nu-2\pi i\left(\varkappa\lrcorner/\ell+k/\ell\right)\nu\right)$$

$$\exp\left(2\pi iz\left(\varkappa/\ell+\nu\right)-2\pi iz\nu\right)\ {}^{\lrcorner}_{z+\varkappa\lrcorner/\ell+k/\ell}\vartheta$$

$$= \exp\left(2\pi i\varkappa\lrcorner\nu/\ell+\pi i\nu\lrcorner\nu+2\pi i\left(k/\ell+n\right)\left(\varkappa/\ell+\nu\right)-\pi i\nu\lrcorner\nu-2\pi i\left(\varkappa\lrcorner/\ell+k/\ell\right)\nu\right)$$

$$\exp\left(\pi i\varkappa\lrcorner\varkappa/\ell^2+2\pi i\sqrt{2}k/\ell\right)\ {}^{\lrcorner}_{z+\varkappa\lrcorner/\ell+k/\ell}\vartheta$$