String Theory in Physics and Mathematics

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Elementary particle physics: matter fields

Space-time coordinates

$$x = (x^0, x^1, x^2, x^3) = (x^\mu),\ x^0 = \text{ time}$$

Minkowski metric

$$dx^\mu\ dx_\mu = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2$$

matter fields = spinor fields $\psi(x)$ (fermions, half-integral spin)
3 colors plus no color

0	electron	charge -1
$1, 2, 3$	down quarks	charge $-\frac{1}{3}$
$\underline{0}$	neutrino	charge 0
$\underline{1}, \underline{2}, \underline{3}$	up quarks	charge $\frac{2}{3}$

plus **antiparticles** (opposite charge)
These sixteen matter particles occur in **3 almost identical generations**

Elementary particle physics: force fields

force fields = differential 1-forms $A(x)$ (bosons, integral spin)

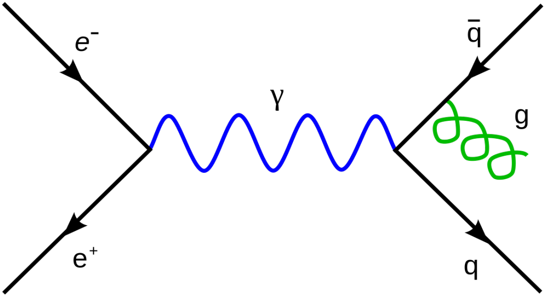
$$A_\mu(x)dx^\mu = A_0(x)dx^0 + A_1(x)dx^1 + A_2(x)dx^2 + A_3(x)dx^3$$

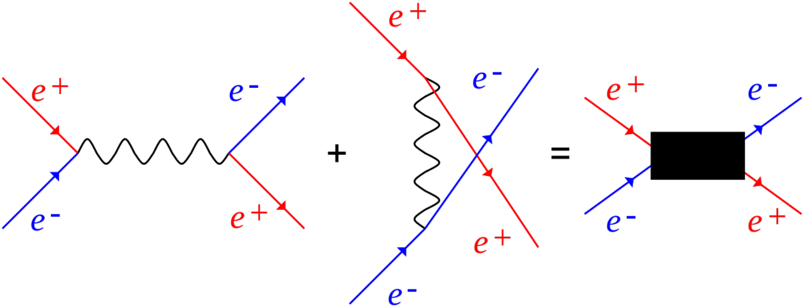
with **matrix-valued** coefficient functions $A_\mu(x)$.
$U(n) = \{u \in \mathbf{C}^{n\times n} : \ uu^* = I\}$ unitary group
$U(1) = \{u \in \mathbf{C} : \ |u| = 1\} = \mathbf{T}$ unit circle, commutative
3 values plus no value:

$A^1 = A^1_\mu(x)dx^\mu$	electro-magnetism, photon (light)	$U(1)$
$A^2 = A^2_\mu(x)dx^\mu$	weak nuclear force, radioactive decay	$U(2)$
$A^3 = A^3_\mu(x)dx^\mu$	strong nuclear force, gluons (binding quarks)	$U(3)$
$A^0 = A^0_\mu(x)dx^\mu$	gravitation	no value

no generations

Feynman diagrams, scattering





no loop   one loop

Principle of Least Action

Scalar field: real valued function $\phi(x^\mu)$ on space-time
Lagrange action functional

$$\mathcal{L}(\phi) := \int_{\mathbf{R}^4} dx(\partial_\mu\phi\ \partial^\mu\phi - m^2\phi^2)$$

classical fields minimize action (**field equation**)

$$\frac{\partial\mathcal{L}}{\partial\phi}(\dot{\phi}) = \frac{d}{d\epsilon}\Big|_{\epsilon=0}\mathcal{L}(\phi+\epsilon\dot{\phi}) = 0$$

free fields: **wave equation**

$$(\partial_\mu\partial^\mu + m^2)\phi = 0$$

$$\partial_\mu\partial^\mu = \partial_0^2 - \partial_1^2 - \partial_2^2 - \partial_3^2$$

Interacting fields: non-linear PDE

Standard model (theory of almost everything)

Lagrangian with several hundred summands and 26 free parameters

$$\mathcal{L} = -\frac{1}{4}B_{\mu\nu}B^{\mu\nu} - \frac{1}{8}tr\ W_{\mu\nu}W^{\mu\nu} - \frac{1}{2}tr\ G_{\mu\nu}G^{\mu\nu}$$

$$+(\overline{\nu}_L, \overline{e}_L)\tilde{\sigma}^\mu iD_\mu \begin{pmatrix}\nu_L \\ e_L\end{pmatrix} + \overline{e}_R\sigma^\mu iD_\mu e_R + \overline{\nu}_R\sigma^\mu iD_\mu \nu_R + h.c.$$

$$+(\overline{u}_L, \overline{d}_L)\tilde{\sigma}^\mu iD_\mu \begin{pmatrix}u_L \\ d_L\end{pmatrix} + \overline{u}_R\sigma^\mu iD_\mu u_R + \overline{d}_R\sigma^\mu iD_\mu d_R + h.c.$$

$$+\overline{D_\mu\phi}D_\mu\phi - \frac{m_h^2}{2v^2}\Big(\overline{\phi}\phi - \frac{v^2}{2}\Big)^2 \quad \text{Higgs field}$$

$$-\frac{\sqrt{2}}{v}\Big((\overline{\nu}_L : \overline{e}_L)\phi M^e e_R + \overline{e}_R\overline{M}^e\overline{\phi}\begin{pmatrix}\nu_L \\ e_L\end{pmatrix} + (-\overline{e}_L, \overline{\nu}_L)\phi^* M^\nu \nu_R + \overline{\nu}_R\overline{M}^\nu\phi^+\begin{pmatrix}-e_L \\ \nu_L\end{pmatrix}\Big)$$

$$-\frac{\sqrt{2}}{v}\Big((\overline{u}_L, \overline{d}_L)\phi M^d d_R + \overline{d}_R\overline{M}^d\overline{\phi}\begin{pmatrix}u_L \\ d_L\end{pmatrix} + (-\overline{d}_L, \overline{u}_L)\phi^* M^u u_R + \overline{u}_R\overline{M}^u\phi^+\begin{pmatrix}-d_L \\ u_L\end{pmatrix}\Big)$$

1+3 Quantum fields: Feynman integrals

h=Planck's constant (very small)
time-ordered positions $x_1, \dots, x_n \in \mathbf{R}^4$, correlation functions

$$< x_1, \dots, x_n > := \int \mathcal{D}\phi \ \exp\big(\frac{2\pi i}{h}\mathcal{L}(\phi)\big) \prod_{j=1}^{n} \phi(x_j)$$

LSZ-scattering formula: momenta $p_1, \dots, p_n$, incoming/outgoing

$$< p_1, \dots, p_n > = \int \mathcal{D}\phi \ \exp\big(\frac{2\pi i}{h}\mathcal{L}(\phi)\big) \prod_{j=1}^{n} \big((\partial_\mu \partial^\mu + m^2)\phi\big)^\wedge(\pm p_j)$$

1+0 fields, paths $X(t)$, Path integrals $t_1 < \dots t_n$

$$< t_1, \dots, t_n > := \int \mathcal{D}X \ \exp\big(-\mathcal{L}(X)\big) \prod_{j=1}^{n} X(t_j)$$

Wiener measure, stochastic processes
Stationary phase method: asymptotic expansion (zero radius of convergence)

$$I(h) \approx \sum_g h^g A_g.$$

Standard model agrees with experiment up to 12 decimal places
On the other hand:

- gravity not included in quantization
- three generations of matter fields not explained
- too many free parameters, little predictive power
- needs summation over large number of Feynman graphs (ugly)
- pointlike singularities of Feynman graphs create infinities in quantum theory

String theory (theory of everything)

Replace **0-dimensional particles** by **1-dimensional strings**, depending on 1 spatial coordinate $s,\ 0 \leq s \leq \pi$. Two types of strings

- **open** strings $\partial_s X^\mu = 0$ at endpoints
- **closed** strings $X(0,t) = X(\pi,t)$.

- all matter particles correspond to different vibrating modes of **open** strings
- gravity corresponds to **closed** strings

- **World line** of particle $t \mapsto X^\mu(t)$
- $\mathcal{L}(X) = \int_0^T dt \, \| \frac{dX}{dt} \|$ **arc length**
- Classical world lines ($\mathcal{L}'(X) = 0$): **geodesics**

$$\frac{d^2 X^\mu}{dt^2} = \Gamma^\mu_{\nu\rho}(X) \frac{dX^\nu}{dt} \frac{dX^\rho}{dt}$$

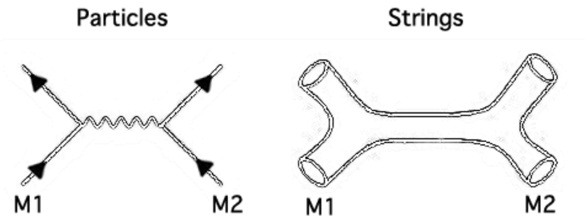
- **World sheet** of string $(s, t) \mapsto X^\mu(s, t)$
- $\mathcal{L}(X) = \int_0^\pi ds \int_0^T dt \, |\det(\partial_i X^\mu \, \partial_j X_\mu)|^{1/2}$ **surface area**
- Classical strings: **minimal area surfaces**

$$(\partial_s^2 - \partial_t^2) X^\mu = 0 \text{ wave equation}$$

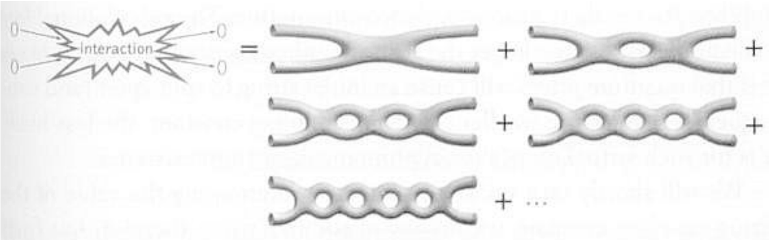
Classical string solutions have **Fourier expansion**

$$X^\mu(s, t) = x^\mu + i c_0^\mu + i \sum_{n \neq 0} \frac{c_n^\mu}{n} \cos(ns) e^{int},$$

Interacting strings: **Riemann surfaces** instead of Feynman graphs



Summation over Feynman graphs replaced by integration
over 'moduli space' $\mathcal{M}_{g,n}$ of Riemann surfaces of genus g with n punctures



Bosonic strings in 26 spacetime dimensions

$$\mathbf{R} \subset \mathbf{C} \subset \mathbf{H} \subset \mathbf{O} \text{ division algebras}$$

$$\dim \mathbf{K} = 2^a, \ a = 0, 1, 2, 3$$

8-dim **Cayley numbers** $\mathbf{O}$, non-associative, automorphism group

$$G_2 = Aut(\mathbf{O})$$

Jordan algebra of self-adjoint 3×3-matrices with octonion entries

$$\mathcal{H}_3(\mathbf{O}) = \left(\begin{array}{c|c|c} \mathbf{R} & \mathbf{O} & \mathbf{O} \\ \hline * & \mathbf{R} & \mathbf{O} \\ \hline * & * & \mathbf{R} \end{array}\right)$$

anti-commutator product $x \circ y = \frac{1}{2}(xy + yx)$, automorphism group

$$F_4 = Aut \ \mathcal{H}_3(\mathbf{O})$$

Superstrings in 10 spacetime dimensions

Supersymmetry (SUSY) is a one-one correspondence between matter fields (fermions) and force fields (bosons)

$$\mathbf{K}^0 = \{x \in \mathbf{K} : \ Re(x) = (1|x) = 0\}, \ \dim = 2^a - 1$$

$$\mathbf{K} = \mathbf{R} \cdot 1 \oplus \mathbf{K}^0 = \text{ spinors of } \mathbf{K}^0, \quad \dim = 2^a$$

In particular, 1 is a spinor for $\mathbf{K}^0$, $\quad Aut(\mathbf{K}) = \{g \in SO(\mathbf{K}^0) : \ g1 = 1\}$.

$$\mathcal{H}_2(\mathbf{K}) = \left(\begin{array}{c|c} \mathbf{R} & \mathbf{K} \\ \hline * & \mathbf{R} \end{array}\right) = \mathbf{R} \cdot 1 \oplus \mathcal{H}_2^0(\mathbf{K})$$

$$\mathcal{H}_2^0(\mathbf{K}) = \{x = x^* \in \mathcal{H}_2(\mathbf{K}) : \ tr(x) = 0\}, \ \dim = 2^a + 1$$

$$\begin{pmatrix} \mathbf{K} \\ \mathbf{K} \end{pmatrix} = \text{ spinors for } \mathcal{H}_2^0(\mathbf{K}), \quad \dim = 2^{a+1}$$

$$\begin{array}{c|c||c} \mathbf{R} & \mathbf{O} & \mathbf{O} \\ \hline * & \mathbf{R} & \mathbf{O} \\ \hline\hline * & * & \mathbf{R} \end{array}$$

Hidden microscopic dimensions

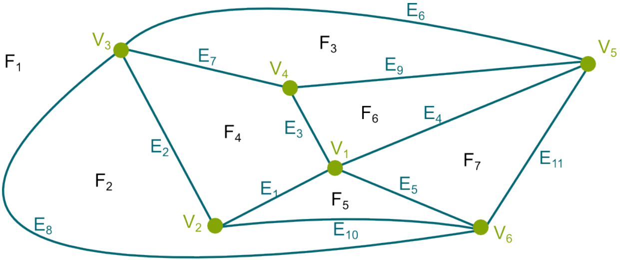
- Macroscopic spacetime is 4-dimensional

$$\text{10-dim spacetime } = \mathbf{R}^4 \times M,$$

where M is 'microscopic' spacetime, a compact 6-dimensional **Calabi-Yau manifold**, too small to be detected.

- Calabi-Yau manifolds generalize Riemann surfaces of genus 1 (elliptic curves, complex tori).
- Every Calabi-Yau manifold describes possible universe where strings can be quantized
- no experimental sign of supersymmetry
...desperately seeking SUSY

Euler characteristic



F - E + V = 7 - 11 + 6 = **2**

Number of generations = half of **Euler characteristic** of Calabi-Yau manifold M

$$\chi_M = \#\text{even-dim holes} - \#\text{odd-dimensional holes}$$

Standard model: $\chi_M = -6$

Bosonic strings and the Monster

Finite simple groups are classified:

- **Alternating** groups A_n (even permutations)
- finite groups of **Lie type** (matrix groups over finite fields, including exceptional groups)
- 26 **sporadic** groups.

The **Monster** $\mathbb{M}$ is the largest sporadic group, of order

$$2^{46} \cdot 3^{20} \cdot 5^{9} \cdot 7^{6} \cdot 11^{2} \cdot 13^{3} \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71$$

$$= 808017424794512875886459904961710757005754368000000000$$

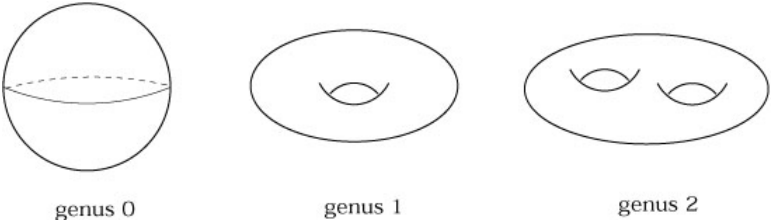
Total debt of Greece 403860000000

For a prime number p consider the 'congruence subgroup'

$$\Gamma_0(p) := \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbf{Z}),\ c \in p\mathbf{Z}\}$$

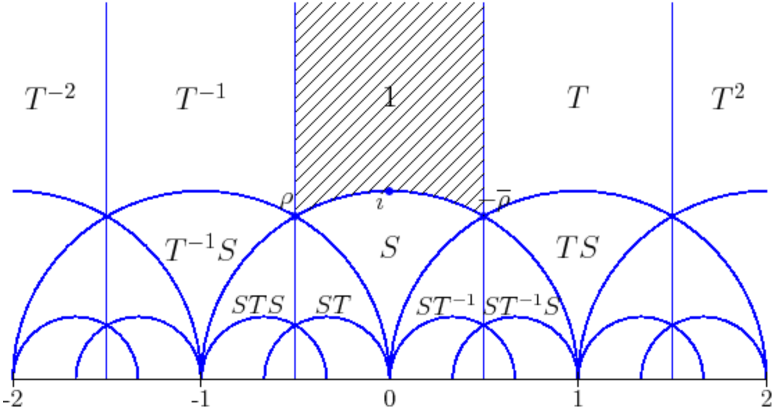
acting on the upper half-plane $H = \{\tau \in \mathbf{C} :\ Im(\tau) > 0\}$ via **Moebius transformations**

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}(\tau) = \frac{a\tau + b}{c\tau + d}.$$



The **modular curve** $H/\Gamma_0(p)$ has genus $g = 0$ if and only if

$$p = \mathbf{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31; 41, 47, 59, 71}$$



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$ST^{-1}S$
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-1
0
1
2

Vertex operator algebras

creation/annihilation operators $m, n \in \mathbf{Z},\ 0 \le \mu, \nu < 26$.

$$[C_m^\mu, C_n^\nu] = m\delta_{m,-n}\delta^{\mu,\nu}$$

$$C(z) = \sum_{n \in \mathbf{Z}} C_n z^n,\ \ z = e^{it}$$

$$\int \frac{dz}{z} C(z) = C_0 \log(z) + \sum_{n \neq 0} \frac{C_n}{n} z^n + q = \sum_{n>0} \frac{C_n}{n} z^n + (q + C_0 \log(z)) - \sum_{n>0} \frac{C_n^*}{n} z^{-n}$$

Vertex operator for emission of string of momentum $p = (p_\mu)$

$$V_p(z) =: \exp\Big(\int \frac{dz}{z} p \cdot C(z) \Big)$$

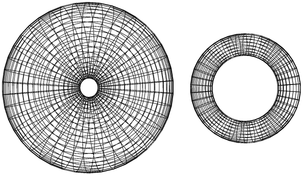
$$:= \exp\Big(p \cdot \sum_{n>0} \frac{C_n}{n} z^n \Big) \exp\Big(p \cdot (q + C_0 \log(z)) \Big) \exp\Big(-p \cdot \sum_{n>0} \frac{C_n^*}{n} z^{-n} \Big)$$

$\mathbf{M}$ is the **automorphisms group** of vertex operator algebra, compactified on the **Leech lattice** $\Lambda \subset \mathbf{R}^{24}$.

T-duality and mirror symmetry

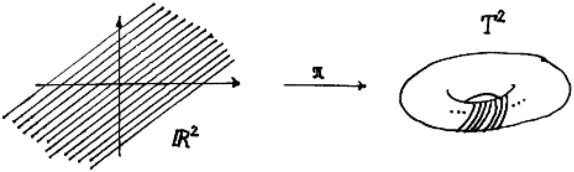
long strings (length ℓ) in Calabi-Yau manifold M equivalent to **short** strings (length $\frac{1}{\ell}$) in 'mirror' CY-manifold $\check{M}$.

$M = \mathbf{C}/\Lambda$ torus $\qquad \check{M} = \mathbf{C}/\Lambda^{-1}$ dual torus



'symplectic' category of Lagrangian subspaces in M=lines with rational slope $\theta = \frac{p}{q}$
'holomorphic' category of coherent analytic sheaves in $\check{M}$
Holomorphic vector bundles of degree p and rank q.

What if θ is **irrational**?
Symplectic side: **Kronecker foliation**



Irrational rotation algebras A_θ, generated by two Hilbert space unitaries u, v satisfying

$$uv = e^{2\pi i\theta} vu.$$

Holomorphic side completely unknown.

S-duality and Langlands program

S-duality: **weakly coupled** strings in M equivalent to **strongly coupled** strings in dual $\tilde{M}$.

$$A = A_0(x)dx^0 + \ldots + A_3(x)dx^3 \quad 4-\text{dim Yang-Mills Theory}$$

connection 1-form in Lie group G.

$$F_A = dA + k[A \wedge A] \text{ field strength, coupling constant } k$$

$$F^a_{\mu\nu} = \partial_\mu \ A^a_\nu - \partial_\nu \ A^a_\mu + k f^a_{bc} A^b_\mu A^c_\nu$$

$$F_A = dA + \frac{1}{k}[A \wedge A] \text{ dual field strength, coupling constant } \frac{1}{k}$$

weakly coupled YM-theory in Lie group G equivalent to **strongly coupled** YM-theory in **Langlands dual group** G^L.

formal series $\sum_{i \in \mathbf{Z}} a_i p^i, \quad 0 \le a_i < p$

real numbers $\mathbf{R} = \mathbf{Q}_\infty : \sum_{-\infty}^{k} a_i p^i$, p-adic numbers $\mathbf{Q}_p : \sum_{i=k}^{\infty} a_i p^i$

- commutative case (class field theory)

$$\text{characters} \quad Gal(\overline{\mathbf{Q}}/\mathbf{Q}) \xrightarrow{\chi} \mathbf{C}^\times \Longleftrightarrow \prod_{p \le \infty} \mathbf{Q}_p^\times \quad \text{ideles}$$

- non-commutative case: finite-dimensional representations (number theory)

$$\rho : Gal(\overline{\mathbf{Q}}/\mathbf{Q}) \to GL_n(\mathbf{C})$$

  infinite-dimensional representations of adelic Lie groups (harmonic analysis)

$$\pi : GL_n(\mathbf{Q}_p) \to U(H)$$

Chern-Simons TQFT and higher categories

$\dim M = 3$, topological QFT, independent of choice of metric

$$A = A_0(x)dx^0 + .. + A_2(x)dx^2,$$

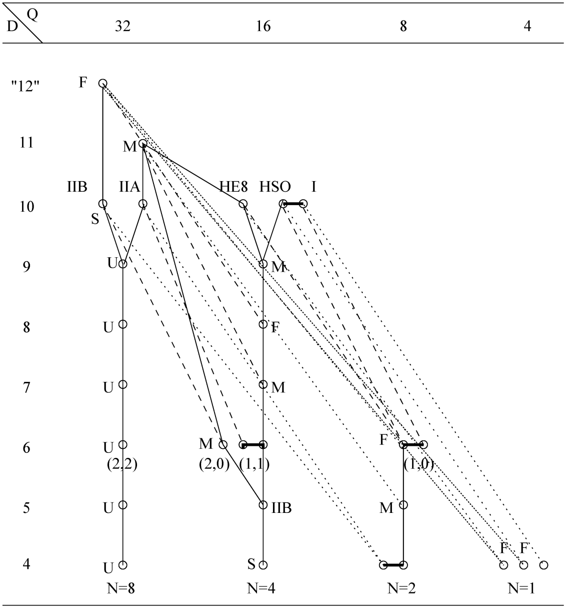
$$\mathcal{L}(A) = \int_M dA \wedge A + \frac{2}{3} A \wedge A \wedge A$$

classical solutions: affine space $H^1(M, G)$ of **flat connexions** (zero curvature)

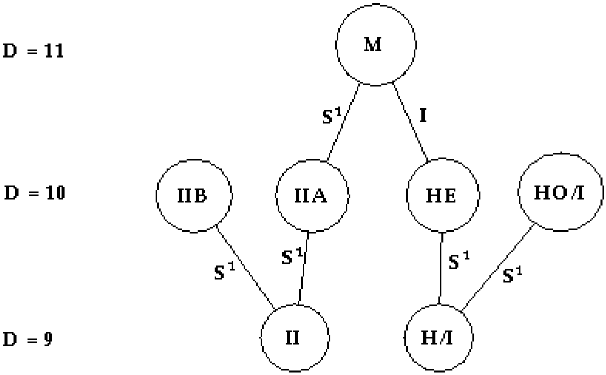
$$F_A = dA + A \wedge A = 0$$

Feynman integrals yield knot polynomials (V. Jones)
Open problem: What is $H^2(M, G)$? Not a set (member of a category) but member of 'higher category'

- Topology: higher homotopy theory
- Mathematical logic: Russell's type theory
- Theoretical computer science: programming language using proof assistant Coq (Voevodsky)



10^{500} choices of CY-manifolds (string landscape)


D = 11
M
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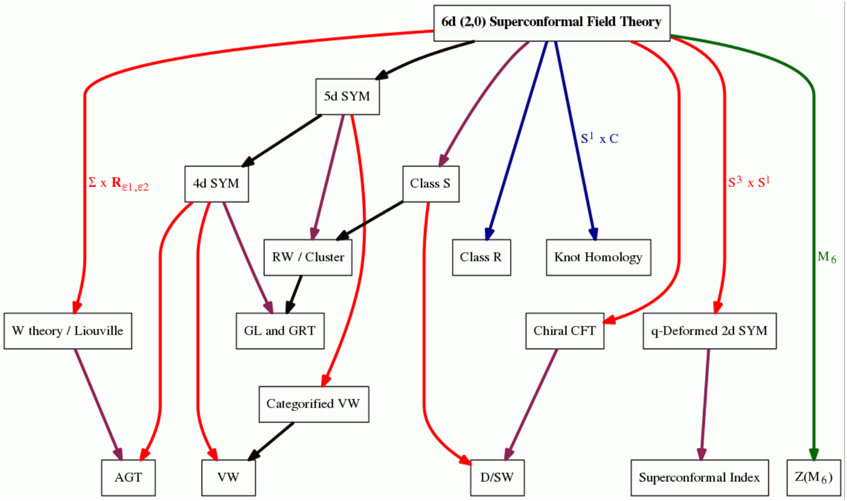
6d (2,0) Superconformal Field Theory
5d SYM
4d SYM
Class S
Σ x $\mathbf{R}_{\varepsilon1,\varepsilon2}$
S^1 x C
S^3 x S^1
M_6
RW / Cluster
Class R
Knot Homology
W theory / Liouville
GL and GRT
Chiral CFT
q-Deformed 2d SYM
Categorified VW
AGT
VW
D/SW
Superconformal Index
$Z(M_6)$