Covariant Symbolic Calculi on Real Symmetric Domains

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We introduce the concept of "covariant symbolic calculus" on real and complex symmetric domains, prove a general product formula for the link transform (generalized Berezin transform) between two such calculi, and describe a basic example (Toeplitz calculus) in more detail.

1. Introduction

The complex Hermitian spaces of non-compact type, realized as bounded symmetric domains $D \subset \mathbb{C}^n$, are a fundamental class of non-compact Kähler manifolds whose quantization, e.g. by the well-known Berezin-Toeplitz operators, has been studied intensively [BLU], [UU]. Writing $D = G/K$ for a semi-simple Lie group G of Hermitian type and its maximal compact subgroup K, the quantization map

$$\begin{array}{rcl} \mathcal{A} : \mathcal{C}^\infty(D) & \to & \mathcal{L}(H) \\ f & \mapsto & \mathcal{A}_f \end{array} \tag{1.1}$$

realized by (possibly unbounded) operators on a complex Hilbert space H should satisfy the covariance condition

$$\mathcal{A}_{f \circ g^{-1}} = U(g)\, \mathcal{A}_f\, U(g^{-1})$$

for all $g \in G$, where U denotes an irreducible (projective) representation of G acting on H. In [AU1] a general theory concerning such "covariant quantizations" on complex symmetric domains (including the flat case $D = \mathbb{C}^n$) is developed for the weighted Bergman spaces $H = H^2_\nu(D)$ of holomorphic functions on D, belonging to the scalar holomorphic discrete series of G. The Toeplitz calculus and also the Weyl calculus are natural examples of covariant quantizations, and the main result of [AU1] gives a "product formula" for the link transform $f \mapsto \mathcal{A}^*(\mathcal{B}_f)$ on $\mathcal{C}^\infty(D)$ in terms of the spherical Fourier transform of certain K-invariant "characteristic" functions associated with the covariant quantizations $\mathcal{A}$ and $\mathcal{B}$.

While the setting of complex symmetric domains is still an active research area (e.g., concerning vector-valued representations or the Weyl calculus for domains of higher dimension), a promising new direction is to apply ideas and methods from quantization theory to *real* symmetric domains $G_\mathbb{R}/K_\mathbb{R}$, for a semi-simple or reductive Lie group $G_{\mathbb{R}}$ not necessarily of Hermitian type. Besides dealing with a much wider geometric framework compared to the complex case, a major application of this more general approach is to harmonic analysis of $G_{\mathbb{R}}$, yielding an explicit decomposition of certain (reducible) $G_{\mathbb{R}}$-representations into irreducible components. The complex case described above corresponds to the special case of a tensor product representation.

2. Real Symmetric Domains

Let
$$D = G/K$$
be a Hermitian symmetric space of non-compact type, realized as the (spectral) unit ball
$$D = \{z \in Z : \ \|z\| < 1\}$$
of a complex JB^*-triple Z of finite dimension. Then
$$G = \mathrm{Aut}(D)^\circ$$
is the identity component of the holomorphic Automorphism group of D, and $K = \mathrm{Aut}(Z)^\circ$. Let
$$z \mapsto \overline{z}$$
be a conjugation of Z preserving the triple product, and define
$$\begin{aligned}
Z_{\mathbb{R}} &:= \{z \in Z : \ \overline{z} = z\}, \\
D_{\mathbb{R}} &:= D \cap Z_{\mathbb{R}} = \{z \in Z_{\mathbb{R}} : \ \|z\| < 1\}, \\
G_{\mathbb{R}} &:= \{g \in G : \ \overline{g(z)} = g(\overline{z}) \ \forall \ z \in Z\} = \{g \in G : \ g(D_{\mathbb{R}}) = D_{\mathbb{R}}\}, \\
K_{\mathbb{R}} &:= K \cap G_{\mathbb{R}}.
\end{aligned}$$
Then $Z_{\mathbb{R}}$ is a real JB^*-triple and
$$D_{\mathbb{R}} = G_{\mathbb{R}}/K_{\mathbb{R}}$$
is a real Riemannian symmetric space of non-compact type, called a *real bounded symmetric domain.* Up to a few low dimensional exceptions, all Riemannian symmetric spaces of non-compact type can be realized in this way. In the following we assume that $D_{\mathbb{R}}$ is *irreducible.* The domain D is called the *complexification* of $D_{\mathbb{R}}$. It is not necessarily irreducible.

Example 2.1. Let $Z_{\mathbb{R}}$ be a complex (irreducible) JB^*-triple, considered as a real JB^*-triple. Then its unit ball $D_{\mathbb{R}}$ is a complex bounded symmetric domain, considered as a real bounded symmetric domain. Put
$$Z := Z_{\mathbb{R}} \times \overline{Z}_{\mathbb{R}}, \quad D := D_{\mathbb{R}} \times \overline{D}_{\mathbb{R}}$$

endowed with the flip conjugation

$$\overline{(z_1, \overline{z}_2)} := (z_2, \overline{z}_1) \tag{2.1}$$

for all $z_1, z_2 \in Z_{\mathbb{R}}$. The corresponding real form is the diagonal

$$\begin{aligned} \{(z_1, \overline{z}_2) \in Z : \ z_1 = z_2\} &= \{(z, \overline{z}) : \ z \in Z_{\mathbb{R}}\} \equiv Z_{\mathbb{R}}, \\ \{(z_1, \overline{z}_2) \in D : \ z_1 = z_2\} &= \{(z, \overline{z}) : \ z \in D_{\mathbb{R}}\} \equiv D_{\mathbb{R}}. \end{aligned}$$

Thus the complex case gives rise to a product domain in the complexification. Putting

$$(g_1, \overline{g}_2)(z_1, \overline{z}_2) := (g_1(z_1), \overline{g_2(z_2)})$$

for $z_1, z_2 \in D_{\mathbb{R}}$ and $g_1, g_2 \in G_{\mathbb{R}} := \mathrm{Aut}(D_{\mathbb{R}})^\circ$, we have

$$G := \mathrm{Aut}(D)^\circ = \{(g_1, \overline{g}_2) : \ g_1, g_2 \in G_{\mathbb{R}}\} \approx G_{\mathbb{R}} \times G_{\mathbb{R}}.$$

Furthermore, an Automorphism $(g_1, \overline{g}_2) \in G$ commutes with (2.1) if and only if $g_1 = g_2$, since we have for $z_1, z_2 \in D_{\mathbb{R}}$

$$(g_1, \overline{g}_2)\overline{(z_1, \overline{z}_2)} = (g_1, \overline{g}_2)(z_2, \overline{z}_1) = (g_1(z_2), \overline{g_2(z_1)})$$

and

$$\overline{(g_1, \overline{g}_2)(z_1, \overline{z}_2)} = \overline{(g_1(z_1), \overline{g_2(z_2)})} = (g_2(z_2), g_1(z_1)).$$

Example 2.2. The complex matrix ball

$$D = \{z \in \mathbb{C}^{p \times q} : \ z^* z < I\}$$

in $Z := \mathbb{C}^{p\times q}$, endowed with the usual conjugation $z \mapsto \overline{z}$, gives rise to the real matrix ball

$$D_{\mathbb{R}} = \{z \in \mathbb{R}^{p \times q} : \ z^* z < I\}$$

in $Z_{\mathbb{R}} = \mathbb{R}^{p\times q}$. In particular, the interval $D_{\mathbb{R}} = (-1, 1) \subset \mathbb{R}$ has the unit disk $D = \{z \in \mathbb{C} : \ |z| < 1\}$ as complexification.

Example 2.3. The complex matrix ball

$$D = \{z \in \mathbb{C}^{2 \times 2} : \ z^* z < I\}$$

in $Z = \mathbb{C}^{2\times 2}$, endowed with the conjugation

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-} := \begin{pmatrix} \overline{d} & -\overline{c} \\ -\overline{b} & \overline{a} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \overline{a} & \overline{b} \\ \overline{c} & \overline{d} \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

has the real form

$$D_{\mathbb{R}} = \{z \in \mathbb{H} : \ z\, z^* < I\}$$

in $Z_{\mathbb{R}} = \mathbb{H}$. Here $\mathbb{H}$ denotes the real division algebra of quaternions.

Example 2.4. Every (irreducible) symmetric cone Ω in a Euclidean Jordan algebra X is a real symmetric domain (realized as the unit ball of X via a Cayley transform) whose complexification can be realized as the *tube domain*

$$T(\Omega) := \{x + iy : \ x \in \Omega, \ y \in X\}$$

in $Z := X^{\mathbb{C}}$.

Generalizing Example 2.4 every irreducible real bounded symmetric domain can be realized as a *real Siegel domain* as follows:

Let Z be an irreducible complex JB^*-triple with conjugation $z \mapsto \overline{z}$, and consider the Peirce decomposition

$$Z = U \oplus V, \ \ U = Z_1(e), \ \ V = Z_{1/2}(e)$$

with respect to a maximal tripotent $e = \overline{e} \in Z$. Then the respective real forms satisfy

$$Z_{\mathbb{R}} = U_{\mathbb{R}} \oplus V_{\mathbb{R}} \tag{2.2}$$

and $U_{\mathbb{R}}$ is a semi-simple (not necessarily Euclidean) real Jordan algebra with a decomposition

$$U_{\mathbb{R}} = X \oplus Y, \tag{2.3}$$

where

$$X := \{x \in U_{\mathbb{R}} : \ x^* = x\}$$

is an irreducible Euclidean Jordan algebra, and we put

$$Y := \{y \in U_{\mathbb{R}} : \ y^* = -y\}.$$

Combining (2.2) and (2.3) we obtain

$$Z_{\mathbb{R}} = X \oplus Y \oplus V_{\mathbb{R}}. \tag{2.4}$$

The self-adjoint part $X \oplus iY$ of U is also a Euclidean Jordan algebra. According to [L2, 10.1] or [U, Section 21], the unit ball of Z is equivalent (via a Cayley transformation) to a complex Siegel domain

$$D = \left\{ u + v \in Z : \ \frac{u + u^*}{2} - \{e\, v^*\, v\} \in \Omega_{X \oplus iY} \right\}$$

where $\Omega_{X \oplus iY}$ is the positive cone of $X \oplus iY$. It follows that

$$D_{\mathbb{R}} = \{x + y + v \in Z_{\mathbb{R}} : \ x - \{e\, v^*\, v\} \in \Omega\}, \tag{2.5}$$

where $\Omega = X \cap \Omega_{X \oplus iY}$ is the positive cone of X. This is the unbounded realization of $D_{\mathbb{R}}$ as a real Siegel domain.

Summing up, real symmetric domains $D_{\mathbb{R}}$ are of the following types.

Type 1 Z is irreducible and $D_{\mathbb{R}}$ has an unbounded realization

$$D_{\mathbb{R}} \approx \{x+y+v \in Z_{\mathbb{R}} : \ x - \{e\, v^*\, v\} \in \Omega\}$$

as described above.

Type 2 $Z_{\mathbb{R}}$ is a complex irreducible JB^*-triple, with unit ball $D_{\mathbb{R}}$ being a complex Hermitian symmetric domain, and the complexification is given by

$$Z = Z_{\mathbb{R}} \times \overline{Z}_{\mathbb{R}} \quad \text{and} \quad D = D_{\mathbb{R}} \times \overline{D}_{\mathbb{R}}.$$

We also include the *flat* case:

Type 3 $D = Z = \mathbb{C}^n$, endowed with the usual conjugation, so that $D_{\mathbb{R}} = Z_{\mathbb{R}} = \mathbb{R}^n$.

3. Quantization Hilbert Spaces

For a real symmetric domain $D_{\mathbb{R}} = G_{\mathbb{R}}/K_{\mathbb{R}}$, with complexification $D = G/K$, let

$$H^2_\nu(D) \tag{3.1}$$

be the ν-th weighted Bergman (resp. Bargmann) space of holomorphic functions on D, endowed with the canonical irreducible (projective) representation

$$U_\nu : G \to U(H^2_\nu(D)). \tag{3.2}$$

More precisely, according to the three types of domains (cf. Section 1), (3.1) and (3.2) are defined as follows.

Case 1 Let Z be an irreducible JB^*-triple, with unit ball D of rank r, dimension n and genus p. Let $\Delta(z,w)$ denote the Jordan triple determinant and denote by $(z|w)$ the unique scalar product invariant under $K := \mathrm{Aut}(Z)^\circ$ normalized by the condition $(c|c) = 1$ for all minimal tripotents $c \in Z$. Let $dm(z)$ be the associated Lebesgue measure. Fix $\nu > p-1$ and consider the probability measure

$$d\mu_\nu(z) = \frac{\Gamma_\Omega(\nu)}{\pi^n\ \Gamma_\Omega(\nu - \frac{n}{r})}\ \Delta(z,z)^{\nu-p}\, dm(z) \tag{3.3}$$

where Γ_Ω denotes the so-called Koecher-Gindikin Γ-function of the positive cone Ω associated with Z [FK]. The *weighted Bergman space*

$$H^2_\nu(D) := \{h \in L^2(D, \mu_\nu) : h \ \text{holomorphic}\}$$

$$= \{h : D \to \mathbb{C} \text{ holomorphic } : \ \|h\|_\nu^2 := \int_D d\mu_\nu(z)\, |h(z)|^2 < +\infty\}$$

has the reproducing kernel

$$K(z,w) = \Delta(z,w)^{-\nu} \tag{3.4}$$

for all $z, w \in D$. The irreducible unitary (projective) representation U_ν of G on $H^2_\nu(D)$ is defined by

$$(U_\nu(g^{-1})h)(z) := j(g,z)\, h(g(z)) \tag{3.5}$$

for all $g \in G, \ \ h \in H^2_\nu(D)$ and $z \in D$, where

$$j(g,z) := [\mathrm{Det}_Z\, g'(z)]^{\nu/p}.$$

<u>Case 2</u> If $D_\mathbb{R} \subset Z_\mathbb{R}$ is an irreducible *complex* symmetric domain, we consider the Hilbert space tensor product

$$H^2_\nu(D) := H^2_\nu(D_\mathbb{R}) \otimes \overline{H^2_\nu(D_\mathbb{R})} \tag{3.6}$$

for the weighted Bergman space $H^2_\nu(D_\mathbb{R})$ over $D_\mathbb{R}$, as described in Case 1. Here $\nu > p_\mathbb{R} - 1$ where $p_\mathbb{R} = \text{genus}\,(D_\mathbb{R})$. $H^2_\nu(D)$ consists of sesqui-holomorphic functions $h(z_1, \overline{z}_2)$ on $D := D_\mathbb{R} \times \overline{D}_\mathbb{R} \subset Z := Z_\mathbb{R} \times \overline{Z}_\mathbb{R}$ which are square-integrable under the product measure

$$d\mu_\nu(z_1, z_2) := d\mu^\mathbb{R}_\nu(z_1)\, d\mu^\mathbb{R}_\nu(z_2). \tag{3.7}$$

An equivalent realization of (3.6) is via Hilbert-Schmidt integral operators

$$(h\phi)(z_1) = \int_{D_\mathbb{R}} d\mu^\mathbb{R}_\nu(z_2)\, h(z_1, \overline{z}_2)\, \phi(z_2) \tag{3.8}$$

on $H^2_\nu(D_\mathbb{R})$. The reproducing kernel of $H^2_\nu(D)$ is the product

$$K(z_1, \overline{z}_2, \ w_1, \overline{w}_2) := K^\mathbb{R}(z_1, w_1)\ K^\mathbb{R}(w_2, z_2), \tag{3.9}$$

where $K^\mathbb{R}(z_1, w_1)$ is the kernel function for $H^2_\nu(D_\mathbb{R})$. $H^2_\nu(D)$ carries an irreducible unitary (projective) representation U_ν of $G = G_\mathbb{R} \times G_\mathbb{R}$ defined by

$$U_\nu(g_1, \overline{g}_2)\, T = U^\mathbb{R}_\nu(g_1)\, T\, U^\mathbb{R}_\nu(g_2)^* \tag{3.10}$$

for all $g_1, g_2 \in G_\mathbb{R}$ and $T \in H^2_\nu(D)$, regarded as a Hilbert-Schmidt operator on $H^2_\nu(D_\mathbb{R})$. For the diagonal $(g, \overline{g}) \in G$, with $g \in G_\mathbb{R}$, we obtain the adjoint action

$$U_\nu(g, \overline{g})\, T = U^\mathbb{R}_\nu(g)\, T\, U^\mathbb{R}_\nu(g)^{-1}.$$

We put

$$j(g_1, \overline{g}_2; \ z_1, \overline{z}_2) := j^\mathbb{R}(g_1, z_1)\ \overline{j^\mathbb{R}(g_2, z_2)}$$

for all $g_1, g_2 \in G_{\mathbb{R}}$ and $z_1, z_2 \in D_{\mathbb{R}}$.

Case 3 For the *flat* case $D = \mathbb{C}^n$, let $(z|w)$ denote the inner product and let $dm(z)$ be the associated Lebesgue measure. Fix $\nu > 0$ and consider the probability measure

$$d\mu_\nu(z) := \left(\frac{\nu}{\pi}\right)^n e^{-\nu(z|z)} dm(z). \tag{3.11}$$

The *weighted Bargmann space*

$$\begin{aligned} H^2_\nu(\mathbb{C}^n) &:= \{h \in L^2(\mathbb{C}^n;\, d\mu_\nu) : \ h \text{ holomorphic}\} \\ &= \{h : \mathbb{C}^n \to \mathbb{C} \text{ holomorphic } : \ \|h\|^2_\nu := \int\limits_{\mathbb{C}^n} d\mu_\nu(z)\, |h(z)|^2 < \infty\} \end{aligned}$$

has the reproducing kernel

$$K(z, w) = e^{\nu(z|w)} \tag{3.12}$$

for all $z, w \in \mathbb{C}^n$. $H^2_\nu(\mathbb{C}^n)$ carries an irreducible unitary (projective) representation of the semi-direct product $G := U(n) \triangleleft \mathbb{C}^n$ via

$$(U_\nu(g)\, h)(z) := j(g^{-1}, z)\, h(g^{-1}(z)) \tag{3.13}$$

for all $g \in G$, $h \in H^2_\nu(\mathbb{C}^n)$ and $z \in \mathbb{C}^n$, where

$$j(g^{-1}, z) = \frac{K(z, g(0))}{K(g(0), g(0))} = \exp\left(\nu(z|g(0)) - \frac{\nu}{2}\,(g(0)|g(0))\right).$$

This completes the definition of $H^2_\nu(D)$. In all three cases we have

$$j(g, z)\ K(g(z), g(w))\ \overline{j(g, w)} = K(z, w) \tag{3.14}$$

for all $g \in G$ and $z, w \in D$. Put

$$K_w(z) := K(z, w). \tag{3.15}$$

Then $K_w \in H^2_\nu(D)$ for all $w \in D$, and (3.14) shows

$$U(g)\, K_w = \overline{j(g, w)}\ K_{g(w)}$$

for all $g \in G$ and $w \in D$ since

$$\begin{aligned} \overline{j(g, w)}\ K_{g(w)}(g(z)) &= \overline{j(g, w)}\ K(g(z), g(w)) \\ &= j(g, z)^{-1}\, K(z, w) = j(g^{-1}, g(z))\ K_w(z) \\ &= (U(g)\, K_w)(g(z)) \end{aligned}$$

for all $z \in D$. Let

$$(h|k)_\nu := \int\limits_D d\mu_\nu(z)\, \overline{h(z)}\, k(z)$$

denote the scalar product in $H^2_\nu(D)$. Then

$$h(z) = (K_z|h)_\nu \tag{3.16}$$

for all $h \in H^2_\nu(D)$ by the reproducing kernel property.

Let $z \mapsto \overline{z}$ denote the conjugation of D with real form $D_{\mathbb{R}}$. Since D is simply connected

$$I(z) := K(z, \overline{z})^{1/2} \tag{3.17}$$

defines a holomorphic function on D (not belonging to $H^2_\nu(D)$).

Example 3.1. In the product case (Case 2) we have

$$\begin{aligned} I(z_1, \overline{z}_2) &= K(z_1, \overline{z}_2, \overline{(z_1, \overline{z}_2)})^{1/2} = K(z_1, \overline{z}_2, z_2, \overline{z}_1)^{1/2} \\ &= [K^{\mathbb{R}}(z_1, z_2)\, K^{\mathbb{R}}(z_1, z_2)]^{1/2} = K^{\mathbb{R}}(z_1, z_2) \end{aligned} \tag{3.18}$$

as a holomorphic function on $D = D_{\mathbb{R}} \times \overline{D}_{\mathbb{R}}$. Since the reproducing kernel property implies

$$\phi(z_1) = \int\limits_{D_{\mathbb{R}}} d\mu^{\mathbb{R}}_\nu(z_2)\, K^{\mathbb{R}}(z_1, z_2)\, \phi(z_2)$$

for all $\phi \in H^2_\nu(D_{\mathbb{R}})$, it follows that I corresponds to the identity operator on $H^2_\nu(D_{\mathbb{R}})$. On the other hand, the kernel

$$\begin{aligned} K_{w_1, \overline{w}_2}(z_1, \overline{z}_2) &= K(z_1, \overline{z}_2, w_1, \overline{w}_2) \\ &= K^{\mathbb{R}}(z_1, w_1)\, K^{\mathbb{R}}(w_2, z_2) = K^{\mathbb{R}}_{w_1}(z_1)\, \overline{K^{\mathbb{R}}_{w_2}(z_2)} \end{aligned} \tag{3.19}$$

for $(w_1, \overline{w}_2) \in D$ corresponds to the rank 1 operator $K^{\mathbb{R}}_{w_1}(K^{\mathbb{R}}_{w_2})^*$ acting on $H^2_\nu(D_{\mathbb{R}})$ since for all $\phi \in H^2_\nu(D_{\mathbb{R}})$

$$\begin{aligned} & \int\limits_{D_{\mathbb{R}}} d\mu^{\mathbb{R}}_\nu(z_2)\ K_{w_1, w_2}(z_1, \overline{z}_2)\, \phi(z_2) \\ =\ & \int\limits_{D_{\mathbb{R}}} d\mu^{\mathbb{R}}_\nu(z_2)\ K^{\mathbb{R}}_{w_1}(z_1)\ K^{\mathbb{R}}(w_2, z_2)\, \phi(z_2) \\ =\ & K^{\mathbb{R}}_{w_1}(z_1) \int\limits_{D_{\mathbb{R}}} d\mu^{\mathbb{R}}_\nu(z_2)\ K^{\mathbb{R}}(w_2, z_2)\, \phi(z_2) \\ =\ & K^{\mathbb{R}}_{w_1}(z_1)\, \phi(z_2) = K^{\mathbb{R}}_{w_1}(K^{\mathbb{R}}_{w_2}|\phi)_\nu. \end{aligned}$$

Proposition 3.2. *I is the unique (up to a multiplicative constant) holomorphic function on D which is U_ν-invariant under $G_{\mathbb{R}} \subset G$.*

Proof. Let $g \in G_{\mathbb{R}}$. In order to show

$$I(z) = (U_\nu(g^{-1})\, I)(z) = j(g, z)\, I(g(z)) \tag{3.20}$$

for all $z \in D$, we may assume that $z = \zeta \in D_{\mathbb{R}}$ since both sides of (3.20) are holomorphic. Since $I(\zeta)$, $j(g,\zeta)$ and $I(g(\zeta))$ are positive, the assertion follows by taking squares

$$[j(g,\zeta)\, I(g(\zeta))]^2 = j(g,\zeta)\, K(g(\zeta),\, g(\zeta))\; j(g,\zeta) = K(\zeta,\zeta) = I(\zeta)^2.$$

□

As a consequence of (3.20) we have

$$I(\zeta) = I(g_\zeta(o)) = j(g_\zeta, o)^{-1}\, I(o) \tag{3.21}$$

for all $\zeta \in D_{\mathbb{R}}$, with $g_\zeta \in G_{\mathbb{R}}$ satisfying $g_\zeta(o) = \zeta$. Here o is the origin of $D_{\mathbb{R}}$.

Lemma 3.3. *For all $z \in D$ we have*

$$\int_{D_{\mathbb{R}}} d\mu_0(\zeta)\; K(z,\zeta)\; I(\zeta)^{-1} = I(z) \int_{D_{\mathbb{R}}} d\mu_0(\zeta)\; K(o,\zeta)\; I(\zeta)^{-1}. \tag{3.22}$$

Proof. Since both sides of (3.22) are holomorphic on D we may assume $z \in D_{\mathbb{R}}$. Write $z = g(o)$ for some $g \in G_{\mathbb{R}}$. Then $G_{\mathbb{R}}$-invariance of μ_0 implies

$$\begin{aligned}
& \int_{D_{\mathbb{R}}} d\mu_0(\zeta)\; K(z,\zeta)\; K(\zeta,\zeta)^{-1/2} \\
= & \int_{D_{\mathbb{R}}} d\mu_0(\zeta)\; K(z,g(\zeta))\; K(g(\zeta),\, g(\zeta))^{-1/2} \\
= & \int_{D_{\mathbb{R}}} d\mu_0(\zeta)\; j(g,o)^{-1}\; K(o,\zeta)\; j(g,\zeta)^{-1}\; j(g,\zeta)\; K(\zeta,\zeta)^{-1/2} \\
= &\; j(g,o)^{-1} \int_{D_{\mathbb{R}}} d\mu_0(\zeta)\; K(o,\zeta)\; K(\zeta,\zeta)^{-1/2}.
\end{aligned}$$

Since $K(z,\overline{z}) = K(g(o),g(o)) = j(g,o)^{-2}\, K(o,o) = j(g,o)^{-2}$ the assertion follows.

□

Notation. In view of Lemma 3.3 it is natural to normalize the $G_{\mathbb{R}}$-invariant measure μ_0 on $D_{\mathbb{R}}$ by the condition

$$\int_{D_{\mathbb{R}}} d\mu_0(\zeta)\; K(o,\zeta)\; I(\zeta)^{-1} = 1. \tag{3.23}$$

This normalization (depending on ν) will be chosen in the sequel. Then

$$I(z) = \int_{D_{\mathbb{R}}} d\mu_0(\zeta)\; K(z,\zeta)\; I(\zeta)^{-1} \tag{3.24}$$

for all $z \in D$. In the case of complex bounded symmetric domains $D_{\mathbb{R}}$ the normalization (3.23) amounts to

$$d\mu_0(\zeta) = d\mu_\nu^{\mathbb{R}}(\zeta)\ K^{\mathbb{R}}(\zeta,\zeta) \tag{3.25}$$

since $K^{\mathbb{R}}(o,\zeta) = 1$.

Proposition 3.4. *For holomorphic $h \in H_\nu^2(D)$ we have*

$$\int_D d\mu_\nu(w)\ \overline{h(w)}\ I(w) = \int_{D_{\mathbb{R}}} d\mu_0(\zeta)\ \overline{h(\zeta)}\ I(\zeta)^{-1}. \tag{3.26}$$

Proof. We may assume $h = K_z$ for some $z \in D$. Since $I(z) := K(z,\overline{z})^{1/2}$ is holomorphic, Lemma 3.3 implies

$$\begin{aligned}
&\int_{D_{\mathbb{R}}} d\mu_0(\zeta)\ \overline{K_z(\zeta)}\ I(\zeta)^{-1} = \int_{D_{\mathbb{R}}} d\mu_0(\zeta)\ K(z,\zeta)\ I(\zeta)^{-1} \\
&= I(z) = (K_z|I)_\nu = \int_D d\mu_\nu(w)\ \overline{K_z(w)}\ I(w).
\end{aligned}$$

□

4. Covariant Symbolic Calculi on Real Symmetric Domains

We will now introduce the new concept of covariant calculus for real symmetric domains, which arises as a natural generalization of the complex case [AU1] replacing operators by Hilbert space vectors.

Definition 4.1. A *covariant symbolic calculus* is given by a linear "symbol map"

$$\begin{aligned}
\sigma : H_\nu^2(D) &\ \to\ \{\text{functions on } D_{\mathbb{R}}\} \\
h &\ \mapsto\ \sigma h
\end{aligned} \tag{4.1}$$

satisfying the covariance condition

$$\sigma(U_\nu(g)\,h) = (\sigma h) \circ g^{-1} \tag{4.2}$$

for all $g \in G_{\mathbb{R}}$. More precisely, σ should have a $G_{\mathbb{R}}$-invariant domain $\mathrm{Dom}(\sigma)$ of holomorphic functions on D containing all reproducing kernel vectors $K_w,\ w \in D$. Then the covariance condition becomes

$$(\sigma K_w) \circ g^{-1} = \sigma(U_\nu(g)\,K_w) = \overline{j(g,w)}\ \sigma(K_{gw}) \tag{4.3}$$

for all $w \in D$ and $g \in G_{\mathbb{R}}$. In addition we require that the holomorphic function $I(z) := K(z, \overline{z})^{1/2}$ on D belongs to $\mathrm{Dom}(\sigma)$. Then σI is a constant function by (4.2) and Lemma 3.3, and we normalize σ by assuming

$$\sigma I = 1. \tag{4.4}$$

The function σh is also called the *passive* (or weak) *symbol* of h. The main examples of symbolic calculi are "real" in the sense that σK_o is a real-valued function on $D_{\mathbb{R}}$. This will be assumed in the sequel.

Proposition 4.2. *For every symbolic calculus σ we have*

$$(\sigma h)(\zeta) = (\overline{(\sigma K_{\sqcup})(\zeta)}|h)_\nu \tag{4.5}$$

for all $h \in \mathrm{Dom}(\sigma)$ and $\zeta \in D_{\mathbb{R}}$, where for fixed ζ the function

$$z \mapsto (\sigma K_{\sqcup})(\zeta)\,(z) := (\sigma K_z)(\zeta) \tag{4.6}$$

is anti-holomorphic on D.

Proof. We may assume $h = K_w$ for $w \in D$. Then

$$\overline{(\sigma K_w)(\zeta)} = \overline{(\sigma K_{\sqcup})(\zeta)}\,(w) = (K_w|\overline{(\sigma K_{\sqcup})(\zeta)})_\nu$$

and hence

$$(\sigma K_w)(\zeta) = \overline{(K_w|\overline{(\sigma K_{\sqcup})(\zeta)})_\nu} = (\overline{(\sigma K_{\sqcup})(\zeta)}|K_w)_\nu.$$

□

Definition 4.3. The *adjoint*

$$\sigma^* : \ \{\text{functions on } D_{\mathbb{R}}\} \to H^2_\nu(D) \tag{4.7}$$

of a covariant symbolic calculus σ is defined by assigning to a function f on $D_{\mathbb{R}}$ (belonging to $\mathrm{Dom}(\sigma^*)$) the holomorphic function

$$(\sigma^* f)(z) := \int_{D_{\mathbb{R}}} d\mu_0(\zeta)\ f(\zeta)\ \overline{(\sigma K_z)(\zeta)} \tag{4.8}$$

on D. Here the invariant measure μ_0 is normalized by the condition (3.23), depending on ν but not on σ. We call f the *active* (or strong) *symbol* of $\sigma^* f$. The adjoint σ^*, corresponding to the map $\mathcal{A}$ of (1.1), can of course also be taken as the starting point of the theory. The "dual" view point emphasizing the symbol map σ is closer to Berezin's original approach.

Proposition 4.4. *A covariant symbolic calculus σ and its adjoint σ^* are related by duality*

$$(\sigma^* f|h)_\nu = (f|\sigma h) := \int_{D_\mathbb{R}} d\mu_0(\zeta)\ \overline{f(\zeta)}\ (\sigma h)(\zeta) \tag{4.9}$$

for all $f \in \mathrm{Dom}(\sigma^*)$ and $h \in \mathrm{Dom}(\sigma)$.

Proof. We may assume $h = K_z$ for some $z \in D$. Then

$$(\sigma^* f|K_z)_\nu = \overline{(K_z|\sigma^* f)_\nu} = \overline{(\sigma^* f)(z)}$$

and (4.9) follows from (4.8) by taking conjugates. □

Since (4.9) involves the $G_\mathbb{R}$-invariant measure μ_0 and σ is $G_\mathbb{R}$-covariant, it follows that σ^* satisfies the covariance condition

$$\sigma^*\left(f \circ g^{-1}\right) = U_\nu(g)(\sigma^* f) \tag{4.10}$$

for all $g \in G_\mathbb{R}$ and $f \in \mathrm{Dom}(\sigma^*)$.

Proposition 4.5. *For any covariant symbolic calculus σ the adjoint σ^* satisfies*

$$\sigma^* 1 = I. \tag{4.11}$$

Proof. For $\zeta \in D_\mathbb{R}$ and $g_\zeta \in G_\mathbb{R}$ satisfying $\zeta = g_\zeta(o)$ the covariance of σ implies

$$(\sigma K_o)(g_\zeta^{-1}(o)) = \sigma(U_\nu(g_\zeta)K_o)(o) = j(g_\zeta, o)(\sigma K_\zeta)(o) = I(\zeta)^{-1}(\sigma K_\zeta)(o).$$

Since $G_\mathbb{R}$ is unimodular and $(\sigma K_\xi)(\eta)$ is real for $\xi, \eta \in D_\mathbb{R}$, Proposition 3.4 and Proposition 4.2 imply

$$\begin{aligned}(\sigma^* 1)(o) &= \int_{D_\mathbb{R}} d\mu_0(\zeta)(\sigma K_o)(\zeta) = \int_{D_\mathbb{R}} d\mu_0(\zeta)(\sigma K_o)(g_\zeta(o)) \\ &= \int_{D_\mathbb{R}} d\mu_0(\zeta)(\sigma K_o)(g_\zeta^{-1}(o)) = \int_{D_\mathbb{R}} d\mu_0(\zeta)\ I(\zeta)^{-1}(\sigma K_\zeta)(o) \\ &= \int_D d\mu_\nu(z)\ I(z)(\sigma K_z)(o) = (\overline{(\sigma K_\sqcup)(o)}\,|\,I)_\nu = (\sigma I)(o) = 1.\end{aligned}$$

Since $\sigma^* 1 = (\sigma^* 1)(o)\, I$ by covariance, the assertion follows. □

Definition 4.6. Let σ be a covariant symbolic calculus with adjoint σ^*. The composite map

$$f \mapsto (\sigma\sigma^*)\, f = \sigma(\sigma^* f)$$

acting on functions on $D_{\mathbb{R}}$ is called the *link transform* associated with σ. Note that $\sigma\sigma^*$ maps the active symbol to the passive symbol. By our assumptions, $\sigma\sigma^*$ has a $G_{\mathbb{R}}$-invariant domain and commutes with the $G_{\mathbb{R}}$-action

$$(\sigma\sigma^*)(f \circ g) = (\sigma\sigma^* f) \circ g \tag{4.12}$$

for all functions $f \in \mathrm{Dom}(\sigma\sigma^*)$ and $g \in G_{\mathbb{R}}$. More generally, one may consider transforms

$$f \mapsto (\sigma_1\sigma_2^*)\, f = \sigma_1(\sigma_2^* f)$$

linking two covariant symbolic calculi σ_1, σ_2 on $H^2_\nu(D)$, and the invariance property (4.12) still holds. In view of (4.4) and (4.11) these transforms are "stochastic" operators: $(\sigma_1\sigma_2^*)\, 1 = \sigma_1(\sigma_2^* 1) = \sigma_1 I = 1.$

Proposition 4.7. *Let σ_1, σ_2 be covariant symbolic calculi. Then the link transform $\sigma_1\sigma_2^*$ has the integral kernel*

$$(\overline{(\sigma_1 K_\sqcup)(\xi)} \,|\, \overline{(\sigma_2 K_\sqcup)(\eta)})_\nu = \int_D d\mu_\nu(z) (\sigma_1 K_z)(\xi)\, \overline{(\sigma_2 K_z)(\eta)} \tag{4.13}$$

with respect to μ_0 (normalized by (3.23)*).*

Proof. Using (4.5) and (4.8) we obtain

$$\begin{aligned}
(\sigma_1\sigma_2^* f)(\xi) &= (\overline{(\sigma_1 K_\sqcup)(\xi)} \,|\, \sigma_2^* f)_\nu \\
&= \int_D d\mu_\nu(z) (\sigma_1 K_z)(\xi) (\sigma_2^* f)(z) \\
&= \int_D d\mu_\nu(z) (\sigma_1 K_z)(\xi) \int_{D_{\mathbb{R}}} d\mu_0(\eta)\, f(\eta)\, \overline{(\sigma_2 K_z)(\eta)} \\
&= \int_{D_{\mathbb{R}}} d\mu_0(\eta)\, f(\eta) \int_D d\mu_\nu(z) (\sigma_1 K_z)(\xi)\, \overline{(\sigma_2 K_z)(\eta)}.
\end{aligned}$$

□

It is well known [H] that the L^2-space $L^2(D_{\mathbb{R}}, \mu_0)$ has a multiplicity-free decomposition

$$L^2(D_{\mathbb{R}}, \mu_0) \equiv \int_{\mathfrak{a}^\#} |c(\lambda)|^{-2} d\lambda\, \langle G_{\mathbb{R}} \rangle_\lambda \tag{4.14}$$

under the group $G_{\mathbb{R}}$, where $\langle G_{\mathbb{R}} \rangle_\lambda$ denotes the principal series representation with spectral parameter $\lambda \in \mathfrak{a}^\#$, $c(\lambda)$ is Harish-Chandra's c-function and we choose an Iwasawa decomposition

$$G_{\mathbb{R}} = K_{\mathbb{R}}\, A\, N \tag{4.15}$$

of $G_{\mathbb{R}}$, with $\mathfrak{a} = \mathrm{Lie}(A)$. By $G_{\mathbb{R}}$-invariance it follows that the link transform $\sigma_1\sigma_2^*$ is diagonalized under the $G_{\mathbb{R}}$-action. More precisely, let $\phi_\lambda \in \mathcal{C}^\infty(D_{\mathbb{R}})$ be the normalized spherical function associated with λ. Then

$$\sigma_1\sigma_2^*\phi_\lambda = \widetilde{\sigma_1\sigma_2^*}(\lambda)\,\phi_\lambda \tag{4.16}$$

for all $\phi_\lambda \in \mathrm{Dom}(\sigma_1\sigma_2^*)$, and the eigenvalues $\widetilde{\sigma_1\sigma_2^*}(\lambda)$ completely characterize the link transform. For every $\lambda \in \mathfrak{a}^{\#}$ the Iwasawa decomposition (4.15) defines "exponential functions" [H]

$$e_\lambda(\zeta) = \exp\langle A(g)|\lambda + \rho\rangle, \quad g \in G_{\mathbb{R}},\ g(o) = \zeta$$

on $D_{\mathbb{R}}$ satisfying $e_\lambda(o) = 1$ and

$$e_\lambda(g(\zeta)) = e_\lambda(g(o))\,e_\lambda(\zeta)$$

for all $g \in NA$, such that

$$\phi_\lambda(\zeta) = \int\limits_{K_{\mathbb{R}}} dk\ e_\lambda(k\zeta)$$

is the spherical function of type λ. In the flat case e_λ is expressed in terms of the exponential function [AU1]; in the curved setting an explicit description of e_λ can be given by realizing $D_{\mathbb{R}}$ as a real Siegel domain (cf. Section 4).

Proposition 4.8. *For every covariant symbolic calculus σ we have*

$$(\sigma^* e_\lambda)(z) = (\sigma^* e_\lambda)(o)\,e_\lambda(z)\ I(z) \tag{4.17}$$

where $e_\lambda(z)$ denotes the unique extension of e_λ to a holomorphic function on D.

Proof. Since both sides of (4.17) are holomorphic it suffices to let $z = \zeta \in D_{\mathbb{R}}$. Let $g_\zeta \in NA$ satisfy $g_\zeta(o) = \zeta$. Then

$$\begin{aligned}(\sigma^* e_\lambda)(\zeta) &= (\sigma^* e_\lambda)(g_\zeta(o)) = j(g_\zeta, o)^{-1}(U_\nu(g_\zeta^{-1})(\sigma^* e_\lambda))(o)\\ &= j(g_\zeta, o)^{-1}(\sigma^*(e_\lambda \circ g_\zeta))(o) = j(g_\zeta, o)^{-1} e_\lambda(\zeta)\,(\sigma^* e_\lambda)(o)\\ &= I(\zeta)\ e_\lambda(\zeta)\,(\sigma^* e_\lambda)(o).\end{aligned}$$

□

Our main result is a "product formula" for the link transform for any pair of covariant symbolic calculi.

Theorem 4.9. *Let σ_1, σ_2 be covariant symbolic calculi on an irreducible real symmetric domain $D_{\mathbb{R}}$. Then the $G_{\mathbb{R}}$-invariant link transform σ_1, σ_2^* has the eigenvalues*

$$\widetilde{\sigma_1\sigma_2^*}(\lambda) = \frac{\overline{(\sigma_1^* e_\lambda)(o)}\,(\sigma_2^* e_\lambda)(o)}{c_\lambda} \tag{4.18}$$

for all $\lambda \in \mathrm{Dom}(\widetilde{\sigma_1\sigma_2^*}) \subset \mathfrak{a}^\#$, *where* c_λ *is a positive constant independent of* σ_1, σ_2 *which will be computed explicitly in Section 5. In particular,*

$$\widetilde{\sigma\sigma^*}(\lambda) = \frac{1}{c_\lambda}\,|(\sigma^* e_\lambda)(o)|^2. \tag{4.19}$$

Proof. By Proposition 4.7, $\sigma_1\sigma_2^*$ is the adjoint of $\sigma_2\sigma_1^*$ with respect to μ_0. Using Proposition 4.8 and Proposition 4.2 it follows that

$$\begin{aligned}
&(\sigma_2^* e_\lambda)(o)\,\overline{(e_\lambda I|\overline{(\sigma_1 K_\sqcup)(o)})_\nu} \\
&= (\sigma_2^* e_\lambda)(o)\,(\overline{(\sigma_1 K_\sqcup)(o)}\,|\,e_\lambda I)_\nu = \overline{(\overline{(\sigma_1 K_\sqcup)(o)}\,|\,\sigma_2^* e_\lambda)_\nu} \\
&= \sigma_1(\sigma_2^* e_\lambda)(o) = \widetilde{\sigma_1\sigma_2^*}(\lambda) = \overline{\widetilde{\sigma_2\sigma_1^*}(\lambda)} \\
&= \overline{(\sigma_1^* e_\lambda)(o)}\,(e_\lambda I|\overline{(\sigma_2 K_\sqcup)(o)})_\nu.
\end{aligned} \tag{4.20}$$

Therefore

$$c_\lambda := \frac{(\sigma_2^* e_\lambda)(o)}{(e_\lambda I|\overline{(\sigma_2 K_\sqcup)(o)})_\nu} = \overline{\left[\frac{(\sigma_1^* e_\lambda)(o)}{(e_\lambda I|\overline{(\sigma_1 K_\sqcup)(o)})_\nu}\right]}.$$

Taking $\sigma_1 = \sigma_2$ it follows that c_λ is real. Therefore c_λ is independent of σ_1, σ_2 and (4.18) follows from (4.20). □

Remark 4.10. In terms of harmonic analysis

$$\begin{aligned}
(\sigma^* e_\lambda)(o) &= \int\limits_{D_{\mathbb{R}}} d\mu_0(\zeta)\, e_\lambda(\zeta)\ (\sigma K_o)(\zeta) \\
&= \int\limits_{D_{\mathbb{R}}} d\mu_0(\zeta)\, \phi_\lambda(\zeta)\ (\sigma K_o)(\zeta) = (\sigma K_o)^\sim(\lambda)
\end{aligned}$$

can be identified with the *spherical Fourier transform* [H] of the real $K_{\mathbb{R}}$-invariant function σK_o on $D_{\mathbb{R}}$. Viewed as a function of λ, we call

$$a_o(\lambda) := (\sigma^* e_\lambda)(o)$$

the *fundamental function* of σ.

5. The Toeplitz-Berezin Calculus

Up to now the discussion of covariant symbolic calculi was quite general. In this section we describe a basic example, the Toeplitz-Berezin calculus, and show that it determines the value of the constant c_λ in an explicit way. Let $D_{\mathbb{R}} = G_{\mathbb{R}}/K_{\mathbb{R}}$ be an irreducible real symmetric domain, with complexification D and quantization Hilbert space $H^2_\nu(D)$, as introduced in Section 2. Let $o \in D_{\mathbb{R}} \subset D$ be the origin.

Definition 5.1. The *Toeplitz-Berezin symbol*

$$\tau : H^2_\nu(D) \to \mathcal{C}^\infty(D_\mathbb{R})$$

is defined by

$$(\tau h)(\zeta) := K(\zeta,\zeta)^{-1/2} h(\zeta) = I(\zeta)^{-1}\, h(\zeta) \tag{5.1}$$

for all $h \in H^2_\nu(D)$ and $\zeta \in D_\mathbb{R}$. In particular

$$(\tau K_z)(\zeta) = I(\zeta)^{-1}\, K(\zeta, z) \tag{5.2}$$

for all $z \in D$ and $\zeta \in D_\mathbb{R}$.

Note that τ is well-defined (and injective) for *all* holomorphic functions h on D since $D_\mathbb{R} \subset D$ is a set of uniqueness. The normalization $\tau I = 1$ made in (4.4) is trivially satisfied.

By Definition 4.3 the adjoint τ^* of τ (called the *Toeplitz-Berezin calculus*) is defined by

$$(\tau^* f)(z) = \int\limits_{D_\mathbb{R}} d\mu_0(\zeta)\, f(\zeta)\, I(\zeta)^{-1}\, K(z,\zeta) \tag{5.3}$$

for all functions $f \in L^\infty(D_\mathbb{R})$, since $d\mu_0(\zeta)\, I(\zeta)^{-1}$ is a finite measure for ν large enough. Here μ_0 is normalized by (3.23) so that (3.24) implies $\tau^* 1 = I$. According to Proposition 4.7, the link transform $\tau\tau^*$ (called the *Berezin transform* in the complex case) has the integral kernel

$$\begin{aligned} & \int\limits_D d\mu_\nu(z) (\tau K_z)(\xi)\, \overline{(\tau K_z)(\eta)} \\ = & \int\limits_D d\mu_\nu(z)\, I(\xi)^{-1}\, K(\xi, z)\, I(\eta)^{-1}\, \overline{K(\eta, z)} \\ = & \; I(\xi)^{-1}\, I(\eta)^{-1}\, K(\xi,\eta) = \frac{K(\xi,\eta)}{K(\xi,\xi)^{1/2}\, K(\eta,\eta)^{1/2}} \end{aligned} \tag{5.4}$$

for all $\xi, \eta \in D_\mathbb{R}$.

Example 5.2. In case $D_\mathbb{R}$ is complex (Case 2) $H^2_\nu(D) = H^2_\nu(D_\mathbb{R}) \otimes \overline{H^2_\nu(D_\mathbb{R})}$ is the space of Hilbert-Schmidt operators via the identification (3.8), where h is a sesqui-holomorphic function on $D_\mathbb{R} \times D_\mathbb{R}$. For fixed $z \in D_\mathbb{R}$ we have

$$(K^\mathbb{R}_z \,|\, h K^\mathbb{R}_z)_\nu = (h K^\mathbb{R}_z)(z) = \int\limits_{D_\mathbb{R}} d\mu^\mathbb{R}_\nu(w)\, h(z, \overline{w})\, K^\mathbb{R}_z(w) = h(z, \overline{z})$$

since $w \mapsto \overline{h(z,\overline{w})}$ is holomorphic. Therefore (5.1) amounts to

$$\begin{aligned} (\tau h)(z, \overline{z}) &= I(z,\overline{z})^{-1}\, h(z,\overline{z}) = K^\mathbb{R}(z,z)^{-1}\, h(z,\overline{z}) \\ &= \frac{(K^\mathbb{R}_z | h K^\mathbb{R}_z)_\nu}{(K^\mathbb{R}_z | K^\mathbb{R}_z)_\nu} \end{aligned}$$

for all $z \in D_{\mathbb{R}}$. This is the classical Berezin symbol of the operator h.

On the other hand, the sesqui-holomorphic function

$$(\tau^* f)(z,\overline{w}) = \int_{D_{\mathbb{R}}} d\mu_0(\zeta)\ f(\zeta)\ I(\zeta,\overline{\zeta})^{-1}\, K(z,\overline{w},\zeta,\overline{\zeta})$$

$$= \int_{D_{\mathbb{R}}} d\mu_0(\zeta)\ f(\zeta)\ K^{\mathbb{R}}(\zeta,\zeta)^{-1}\ K^{\mathbb{R}}(z,\zeta)\ K^{\mathbb{R}}(\zeta,w)$$

$$= \int_{D_{\mathbb{R}}} d\mu_\nu^{\mathbb{R}}(\zeta)\ f(\zeta)\ K^{\mathbb{R}}(z,\zeta)\ K^{\mathbb{R}}(\zeta,w)$$

on $D_{\mathbb{R}} \times D_{\mathbb{R}}$, defined via (5.3), gives rise to the integral operator

$$((\tau^* f)\phi)(z) = \int_{D_{\mathbb{R}}} d\mu_\nu^{\mathbb{R}}(w)\,(\tau^* f)(z,\overline{w})\,\phi(w)$$

$$= \int_{D_{\mathbb{R}}} d\mu_\nu^{\mathbb{R}}(w) \int_{D_{\mathbb{R}}} d\mu_\nu^{\mathbb{R}}(\zeta)\ f(\zeta)\ K^{\mathbb{R}}(z,\zeta)\ K^{\mathbb{R}}(\zeta,w)\,\phi(w)$$

$$= \int_{D_{\mathbb{R}}} d\mu_\nu^{\mathbb{R}}(\zeta)\ f(\zeta)\ K^{\mathbb{R}}(z,\zeta) \int_{D_{\mathbb{R}}} d\mu_\nu^{\mathbb{R}}(w)\ K^{\mathbb{R}}(\zeta,w)\,\phi(w)$$

$$= \int_{D_{\mathbb{R}}} d\mu_\nu^{\mathbb{R}}(\zeta)\ f(\zeta)\ K^{\mathbb{R}}(z,\zeta)\,\phi(\zeta) = E(f\phi)(z)$$

where E denotes the orthogonal projection onto $H^2_\nu(D_{\mathbb{R}})$. Thus

$$(\tau^* f)\phi = E(f\phi) =: T_f\phi$$

gives the *Toeplitz operator* with symbol f acting on $H^2_\nu(D_{\mathbb{R}})$. By (5.4) the Berezin transform in the complex case has the kernel

$$\frac{K(\xi,\overline{\xi},\eta,\overline{\eta})}{K(\xi,\overline{\xi},\xi,\overline{\xi})^{1/2}\,K(\eta,\overline{\eta},\eta,\overline{\eta})^{1/2}} = \frac{K^{\mathbb{R}}(\xi,\eta)\,K^{\mathbb{R}}(\eta,\xi)}{K^{\mathbb{R}}(\xi,\xi)\,K^{\mathbb{R}}(\eta,\eta)} = \frac{|K^{\mathbb{R}}(\xi,\eta)|^2}{K^{\mathbb{R}}(\xi,\xi)\,K^{\mathbb{R}}(\eta,\eta)}$$

for all $\xi,\eta \in D_{\mathbb{R}}$.

Proposition 5.3. *For all $\lambda \in \mathfrak{a}^\#$ we have*

$$\widetilde{\tau\tau^*}(\lambda) = (\tau e_\lambda)(o) = c_\lambda. \tag{5.5}$$

Proof. Since

$$\overline{(\tau K_z)(o)} = \overline{K(o,o)^{-1/2}\,K_z(o)} = K_o(z)$$

for all $z \in D$, Proposition 4.2 implies

$$\widetilde{\tau\tau^*}(\lambda) = \tau(\tau^* e_\lambda)(o) = (\overline{(\tau K_\sqcup)(o)} \,|\, \tau^* e_\lambda)_\nu = (K_o|\tau^* e_\lambda)_\nu = (\tau^* e_\lambda)(o).$$

This shows that $(\tau^* e_\lambda)(o)$ is real, and the product formula (4.19) implies

$$(\tau^* e_\lambda)(o) = \widetilde{\tau\tau^*}(\lambda) = \frac{|(\tau^* e_\lambda)(o)|^2}{c_\lambda} = \frac{(\tau^* e_\lambda)(o)^2}{c_\lambda}.$$

□

Combining Proposition 5.3 with Theorem 4.9 and Remark 4.10 we obtain

Corollary 5.4. *For covariant symbolic calculi σ_1, σ_2 on $H^2_\nu(D)$ the link transform $\sigma_1\sigma_2^*$ has the eigenvalues*

$$\widetilde{\sigma_1\sigma_2^*}(\lambda) = \frac{(\sigma_2^* e_\lambda)(o)\ \overline{(\sigma_1^* e_\lambda)(o)}}{(\tau^* e_\lambda)(o)} = \frac{(\sigma_2 K_o)^\sim(\lambda)\ \overline{(\sigma_1 K_o)^\sim(\lambda)}}{(\tau K_o)^\sim(\lambda)}$$

where τ is the Toeplitz-Berezin calculus and $\sim$ denotes spherical Fourier transforms.

In view of Corollary 5.4 it is important to compute the integral

$$\begin{aligned}(\tau^* e_\lambda)(o) &= \int\limits_{D_\mathbb{R}} d\mu_0(\zeta)\ e_\lambda(\zeta)\ (\tau K_o)(\zeta) \\ &= \int\limits_{D_\mathbb{R}} d\mu_0(\zeta)\ e_\lambda(\zeta)\ I(\zeta)^{-1}\ K(o,\zeta) \\ &= \int\limits_{D_\mathbb{R}} d\mu_0(\zeta)\ e_\lambda(\zeta)\ K(\zeta,\zeta)^{-1/2}\ K(o,\zeta)\end{aligned}$$

explicitly. In the complex case this has been carried out in [UU], verifying a long-standing conjecture of Berezin. For details on the complex case and also the flat case we refer to [AU1]. In the sequel we determine $(\tau^* e_\lambda)(o)$ for real symmetric domains. The Toeplitz-Berezin calculus has also been studied in [DP], [N], [Z]; we avoid a case-by-case separation according to the various root systems of $D_\mathbb{R}$, thus giving a more uniform treatment.

As in Section 2 consider the unbounded realization

$$D_\mathbb{R} = \{x + y + v \in Z_\mathbb{R} :\ x - \{ev^*v\} \in \Omega\}$$

as a real Siegel domain. For the Jordan theoretic concepts used in the sequel we refer to [FK], [L2], [U]. Let $e_1, \ldots, e_r \in X$ be a frame with $e = e_1 + \cdots + e_r$. Then we have Peirce decompositions

$$X = \sum_{1\le i\le j\le r} X_{ij}, \quad Y = \sum_{1\le i\le j\le r} Y_{ij}, \quad V_\mathbb{R} = \sum_{1\le j\le r} V_j^\mathbb{R}$$

and we put

$$\begin{array}{rcll} 1 & = & \dim X_{jj} & (1 \le j \le r) \\ a & := & \dim X_{ij} & (1 \le i < j \le r). \end{array}$$

In case $Z_{\mathbb{R}} \neq X$ we also put

$$\begin{array}{rcll} b & := & \dim V_j^{\mathbb{R}} & (1 \le j \le r) \\ c & := & \dim Y_{jj} & (1 \le j \le r). \end{array}$$

It is known that

$$\dim Y_{ij} = a \quad (1 \le i < j \le r)$$

up to one exception (root system D_2) which we omit in the sequel. The dimensions of the respective subspaces are then given by

$$\begin{array}{rcccl} n_X & := & \dim_{\mathbb{R}} X & = & r + r(r-1)\,a/2 \\ n_Y & := & \dim_{\mathbb{R}} Y & = & cr + r(r-1)\,a/2 \\ n_V & := & \dim_{\mathbb{C}} V & = & \dim_{\mathbb{R}} V_{\mathbb{R}} = rb. \end{array}$$

Put $n := \dim_{\mathbb{R}} Z_{\mathbb{R}} = n_X + n_Y + n_V$. Let P denote the quadratic representation of X and let $\Delta(x)$ be the Jordan determinant of X.

Lemma 5.5. *For $x \in \Omega$ we have*

$$\mathrm{Det}_X\ P_x^{1/2} = \Delta(x)^{n_X/r}.$$

Proof. Without loss of generality (due to the spectral decomposition and transitivity of $K_{\mathbb{R}}$ on frames) we may assume $x = \sum_j x_j e_j$ diagonal. Putting $t' = P_x^{1/2} t$ for $t \in X$, the respective Peirce components satisfy $t'_{ij} = x_i^{1/2}\, t_{ij}\, x_j^{1/2}$. It follows that

$$\begin{aligned} \mathrm{Det}_X\, P_x^{1/2} &= \prod_{1 \le j \le r} x_j \prod_{1 \le i < j \le r} (x_i x_j)^{a/2} \\ &= (x_1 \cdots x_r)^{1+(r-1)a/2} = \Delta(x)^{1+(r-1)a/2} = \Delta(x)^{n_X/r}. \end{aligned}$$

□

In the sequel we use the *conical functions*

$$\Delta^{\underline{\alpha}}(x) = \Delta_1(x)^{\alpha_1 - \alpha_2} \Delta_2(x)^{\alpha_2 - \alpha_3} \cdots \Delta_r(x)^{\alpha_r} \tag{5.6}$$

on Ω, associated with $\underline{\alpha} = (\alpha_1, \ldots, \alpha_r)$, where $\Delta_1, \ldots, \Delta_r$ are the Jordan theoretic minors. Let $\Delta_*^{\underline{\alpha}}$ denote the corresponding function using the minors in reverse order $e_r, \ldots, e_1$. For the following result, cf. [FK, Section VII.1].

Lemma 5.6. *Let $x \in \Omega$ and $\mathrm{Re}(\alpha_j) > -1-(r-j)\,\frac{a}{2}$. Then*

$$\int_\Omega dt\ \Delta^{\underline{\alpha}}(t)\, e^{-(x|t)} = \Gamma_\Omega\left(\underline{\alpha}+\frac{n_X}{r}\right)\ \Delta_*^{-\underline{\alpha}^*-n_X/r}(x)$$

where

$$\Gamma_\Omega\left(\underline{\alpha}+\frac{n_X}{r}\right) = \int_\Omega dt\ \Delta^{\underline{\alpha}}(t)\, e^{-(e|t)}$$

is the (multi-variable) Koecher-Gindikin Γ-function and $\underline{\alpha}^* := (\alpha_r,\ldots,\alpha_1)$.

Lemma 5.7. *Let $\mathrm{Re}(\beta) > 1+\frac{a}{2}\,(2r-j-1)+\mathrm{Re}(\alpha_j) > \frac{a}{2}\ (r-1)$. Then*

$$\Gamma_\Omega(\beta)\int_\Omega dt\ \Delta^{\underline{\alpha}}(t)\,\Delta(e+t)^{-\beta} = \Gamma_\Omega\left(\underline{\alpha}+\frac{n_X}{r}\right)\ \Gamma_\Omega\left(\beta-\underline{\alpha}+\underline{\delta}-\frac{2n_X}{r}\right)$$

where $\underline{\delta} = (\delta_1,\ldots,\delta_r)$ is defined by

$$\delta_j := 1+(j-1)a. \tag{5.7}$$

P r o o f. Applying Lemma 5.6 twice we obtain

$$\begin{aligned}
&\Gamma_\Omega(\beta)\int_\Omega dt\ \Delta^{\underline{\alpha}}(t)\ \Delta(e+t)^{-\beta}\\
=\ &\int_\Omega dt\ \Delta^{\underline{\alpha}}(t)\int_\Omega ds\ \Delta(s)^{\beta-n_X/r}\, e^{-(s|e+t)}\\
=\ &\int_\Omega ds\ \Delta(s)^{\beta-n_X/r}\, e^{-(s|e)}\int_\Omega dt\ \Delta^{\underline{\alpha}}(t)\, e^{-(s|t)}\\
=\ &\Gamma_\Omega\left(\underline{\alpha}+\frac{n_X}{r}\right)\ \int_\Omega ds\ \Delta_*^{\beta-\underline{\alpha}^*-2n_X/r}(s)\, e^{-(s|e)}\\
=\ &\Gamma_\Omega\left(\underline{\alpha}+\frac{n_X}{r}\right)\ \Gamma_\Omega\left(\beta-\underline{\alpha}^*-\frac{n_X}{r}\right).
\end{aligned}$$

Now the assertion follows from the well-known identity

$$\Gamma_\Omega(\underline{\alpha}^*) = \Gamma_\Omega\left(\underline{\alpha}+\underline{\delta}-\frac{n_X}{r}\right). \tag{5.8}$$

□

Lemma 5.8. *[FK, p. 142, Exercise 5] Let $2\gamma > 1+(r-1)\,a$. Then*

$$\Gamma_\Omega(\gamma)\int_X dx\ \Delta(e+x^2)^{-\gamma} = \pi^{n_X}\ 4^{n_X-r\gamma}\ \frac{\Gamma_\Omega(2\gamma-n_X/r)}{\Gamma_\Omega(\gamma)}.$$

Proof. Let $\Delta^{\mathbb{C}}$ be the holomorphic extension of Δ to the complexified Jordan algebra $X^{\mathbb{C}}$. By analytic continuation, Lemma 5.6 implies for every $x \in X$

$$\Gamma_\Omega(\gamma)\,\Delta^{\mathbb{C}}(e+ix)^{-\gamma} = \int_\Omega ds\ \Delta(s)^{\gamma-n_X/r}\, e^{-(e+ix|s)}$$
$$= \int_\Omega ds\ \Delta(s)^{\gamma-n_X/r}\, e^{-(e|s)}\, e^{-i(x|s)}.$$

It follows that $\Gamma_\Omega(\gamma)\,\Delta^{\mathbb{C}}(e+ix)^{-\gamma}$ is the Fourier transform of the function

$$f(s) := \chi_\Omega(s)\ \Delta(s)^{\gamma-n_X/r}\, e^{-(e|s)}$$

on X, where χ_Ω is the characteristic function of $\Omega \subset X$. Therefore Parseval's formula implies

$$\Gamma_\Omega(\gamma)^2 \int_X dx\ |\Delta^{\mathbb{C}}(e+ix)|^{-2\gamma} = (2\pi)^{n_X} \int_X ds\ |f(s)|^2$$
$$= (2\pi)^{n_X} \int ds\ \Delta(s)^{2\gamma-2n_X/r}\, e^{-2(e|s)}$$
$$= (2\pi)^{n_X}\,\Gamma_\Omega\left(2\gamma - \frac{n_X}{r}\right)\ \Delta(2e)^{-2\gamma+n_X/r}$$
$$= \pi^{n_X}\, 4^{n_X-r\gamma}\,\Gamma_\Omega\left(2\gamma - \frac{n_X}{r}\right).$$

Now the assertion follows from

$$|\Delta^{\mathbb{C}}(e+ix)|^2 = \Delta^{\mathbb{C}}(e+ix)\,\Delta^{\mathbb{C}}(e-ix) = \Delta^{\mathbb{C}}((e+ix)(e-ix)) = \Delta(e+x^2).$$

$\square$

From now on we assume $Y \neq \{0\}$. For $t \in X$ the Jordan multiplication operator M_t leaves X and Y invariant.

Lemma 5.9. *For $t \in \Omega$ we have*

$$\Delta(t)^{n_X/r}\,\mathrm{Det}_Y\, M_t = \Delta(t)^{n_Y/r}\,\mathrm{Det}_X\, M_t.$$

Proof. Assuming $t = \sum_j t_j\, e_j$ diagonal and putting $x' = M_t\, x,\ \ y' = M_t\, y$ for $x \in X$ and $y \in Y$, the respective Peirce components satisfy $x'_{ij} = x_{ij}(t_i+t_j)/2,\ \ y'_{ij} = y_{ij}(t_i+t_j)/2$ for all $1 \le i \le j \le r$. It follows that

$$\mathrm{Det}_X\, M_t = \prod_{1\le j\le r} t_j \cdot \prod_{1\le i<j\le r} \left(\frac{t_i+t_j}{2}\right)^a,$$
$$\mathrm{Det}_Y\, M_t = \prod_{1\le j\le r} t_j^c \cdot \prod_{1\le i<j\le r} \left(\frac{t_i+t_j}{2}\right)^a.$$

Since $\Delta(t) = t_1 \cdots t_r$ and $c - 1 = \frac{n_Y - n_X}{r}$, the assertion follows. $\square$

Let r_U and Δ_U be the rank and Jordan determinant of U. We define

$$\nu_{\mathbb{R}} := \frac{\nu r_U}{2r}, \quad p_{\mathbb{R}} := \frac{p r_U}{2r} = \frac{n_X + n_Y + n_V/2}{r}. \tag{5.9}$$

Lemma 5.10. *Let $2\nu_{\mathbb{R}} > (r-1)\,a + c$. Then*

$$2^{\nu r_U} \int\limits_Y dy\ \Delta_U(e-y)^{-\nu} = \pi^{(n_X+n_Y)/2}\, 2^{n_X+n_Y}\ \frac{\Gamma_\Omega\left(2\nu_{\mathbb{R}} - \frac{n_Y}{r}\right)}{\Gamma_\Omega(\nu_{\mathbb{R}})\ \Gamma_\Omega\left(\nu_{\mathbb{R}} + \frac{n_X - n_Y}{2r}\right)}.$$

Proof. For $t \in \Omega$, $x \in X$ and $y \in Y$ we have

$$\begin{aligned}(x^2|t) &= (\{xx^*e\}|t) = (x|\{xe^*t\}) = (x|M_t x),\\ -(y^2|t) &= (\{yy^*e\}|t) = (y|\{ye^*t\}) = (y|M_t y).\end{aligned}$$

In view of Lemma 5.9 this implies

$$\begin{aligned}\pi^{-n_Y/2} \int\limits_Y dy\ e^{(y^2|t)} &= \pi^{-n_Y/2} \int\limits_Y dy\ e^{-(y|M_t\, y)}\\ &= \mathrm{Det}_Y^{-1/2}\ M_t = \Delta(t)^{(n_X-n_Y)/2r}\ \mathrm{Det}_X^{-1/2}\ M_t\\ &= \Delta(t)^{(n_X-n_Y)/2r}\ \pi^{-n_X/2} \int\limits_X dx\ e^{-(x|M_t\, x)}\\ &= \Delta(t)^{(n_X-n_Y)/2r}\ \pi^{-n_X/2} \int\limits_X dx\ e^{-(x^2|t)}.\end{aligned}$$

Since $\Delta_U(e-y)$ is real for all $y \in Y$, we have

$$\Delta_U(e-y) = \overline{\Delta_U(e-y)} = \Delta_U((e-y)^*) = \Delta_U(e+y)$$

and therefore

$$\Delta_U(e-y)^2 = \Delta_U(e-y)\ \Delta_U(e+y) = \Delta_U((e-y)(e+y)) = \Delta_U(e-y^2) = \Delta(e-y^2)^{r_U/r}.$$

Applying Lemma 5.6 and Lemma 5.8 it follows that

$$\begin{aligned}&\pi^{-n_Y/2}\ \Gamma_\Omega(\nu_{\mathbb{R}}) \int\limits_Y dy\ \Delta_U(e-y)^{-\nu}\\ =\ &\pi^{-n_Y/2}\ \Gamma_\Omega(\nu_{\mathbb{R}}) \int\limits_Y dy\ \Delta(e-y^2)^{-\nu_{\mathbb{R}}}\\ =\ &\pi^{-n_Y/2} \int\limits_Y dy \int\limits_\Omega dt\ \Delta(t)^{\nu_{\mathbb{R}} - n_X/r}\ e^{-(e-y^2|t)}\end{aligned}$$

$$
\begin{aligned}
&= \pi^{-n_Y/2} \int\limits_\Omega dt\ \Delta(t)^{\nu_{\mathbb{R}}-n_X/r}\ e^{-(e|t)} \int\limits_Y dy\ e^{(y^2|t)} \\
&= \pi^{-n_X/2} \int\limits_\Omega dt\ \Delta(t)^{\nu_{\mathbb{R}}-(n_X+n_Y)/2r}\ e^{-(e|t)} \int\limits_X dx\ e^{-(x^2|t)} \\
&= \pi^{-n_X/2} \int\limits_X dx \int\limits_\Omega dt\ \Delta(t)^{\nu_{\mathbb{R}}-(n_X+n_Y)/2r}\ e^{-(e+x^2|t)} \\
&= \pi^{-n_X/2}\ \Gamma_\Omega\left(\nu_{\mathbb{R}} + \frac{n_X - n_Y}{2r}\right) \int\limits_X dx\ \Delta(e+x^2)^{(n_Y-n_X)/2r-\nu_{\mathbb{R}}} \\
&= \pi^{-n_X/2}\ \pi^{n_X}\ 4^{n_X+(n_Y-n_X-\nu r_U)/2}\ \frac{\Gamma_\Omega\left(2\nu_{\mathbb{R}} - \frac{n_Y}{r}\right)}{\Gamma_\Omega\left(\nu_{\mathbb{R}} + \frac{n_X-n_Y}{2r}\right)}.
\end{aligned}
$$

□

For $x \in X$ the quadratic representation P_x acts also on Y.

Lemma 5.11. *For $x \in \Omega$ we have*

$$\mathrm{Det}_Y\ P_x^{1/2} = \Delta(x)^{n_Y/r}.$$

Proof. Assuming $x = \sum_j x_j\, e_j$ diagonal and putting $y' = P_x^{1/2}\, y$ for $y \in Y$, the respective Peirce components satisfy $y'_{ij} = x_i^{1/2}\, y_{ij}\, x_j^{1/2}$. It follows that

$$
\begin{aligned}
&\mathrm{Det}_Y\ P_x^{1/2} = \prod_{1\le j\le r} x_j^c \prod_{1\le i<j\le r} (x_i\, x_j)^{a/2} \\
&= (x_1 \cdots x_r)^{c+(r-1)\,a/2} = \Delta(x)^{c+(r-1)\,a/2} = \Delta(x)^{n_Y/r}.
\end{aligned}
$$

□

Lemma 5.12. *Let $x \in \Omega$ and $2\nu_{\mathbb{R}} > (r-1)\,a + c$. Then*

$$
\begin{aligned}
&2^{\nu r_U} \int\limits_Y dy\ \Delta_U(x-y)^{-\nu} \\
= \quad &\Delta(x)^{-2\nu_{\mathbb{R}}+n_Y/r}\ \pi^{(n_X+n_Y)/2}\ 2^{n_X+n_Y}\ \frac{\Gamma_\Omega\left(2\nu_{\mathbb{R}} - \frac{n_Y}{r}\right)}{\Gamma_\Omega\left(\nu_{\mathbb{R}}\right)\ \Gamma_\Omega\left(\nu_{\mathbb{R}} + \frac{n_X-n_Y}{2r}\right)}.
\end{aligned}
$$

Proof. Putting $y' = P_x^{-1/2}\, y$ for $y \in Y$, we have

$$dy = \Delta(x)^{n_Y/r}\, dy'$$

by Lemma 5.8. Since

$$\Delta_U(x-y) = \Delta_U(P_x^{1/2}(e-y')) = \Delta_U(x)\ \Delta_U(e-y') = \Delta(x)^{r_U/r}\, \Delta_U(e-y')$$

it follows that

$$\int\limits_Y dy\ \Delta_U(x-y)^{-\nu} = \Delta(x)^{-2\nu_{\mathbb{R}}+n_Y/r} \int\limits_Y dy'\ \Delta_U(e-y')^{-\nu}.$$

Now the assertion follows from Lemma 5.10. □

Let R denote the canonical Jordan representation of X on $V_{\mathbb{R}}$ [FK], [L1].

Lemma 5.13. *For $x \in \Omega$ we have*

$$\mathrm{Det}_{V_{\mathbb{R}}}\ R_x^{1/2} = \Delta(x)^{n_V/2r}.$$

Proof. Assuming $x = \sum_j x_j\, e_j$ diagonal and putting $v' = R_x^{1/2}\, v$ for $v \in V_{\mathbb{R}}$, the respective Peirce components satisfy $v'_j = x_j^{1/2}\, v_j$. It follows that

$$\mathrm{Det}_{V_{\mathbb{R}}}\ R_x^{1/2} = \Big(\prod_{1 \le j \le r} x_j^{1/2}\Big)^b = (x_1 \cdots x_r)^{b/2} = \Delta(x)^{b/2} = \Delta(x)^{n_V/2r}.$$

□

For $v \in V_{\mathbb{R}}$ we have $(e|\{ev^*v\}) = (\{ee^*v\}|v) = (v|v)/2$ and therefore

$$\int\limits_{V_{\mathbb{R}}} dv\ e^{-(e|\{ev^*v\})} = \int\limits_{V_{\mathbb{R}}} dv\ e^{-(v|v)/2} = (2\pi)^{n_V/2}. \tag{5.10}$$

Lemma 5.14. *Let $x \in \Omega$. Then*

$$\int\limits_{V_{\mathbb{R}}} dv\ e^{-(x|\{ev^*v\})} = (2\pi)^{n_V/2}\ \Delta(x)^{-n_V/2r}.$$

Proof. Putting $v' = R_x^{1/2}\, v$ for $v \in V_{\mathbb{R}}$, we have

$$dv = \Delta(x)^{-n_V/2r}\, dv'$$

by Lemma 5.13. Moreover

$$P_x^{1/2}\,\{ev^*v\} = \{e(R_x^{1/2}\, v)^*\,(R_x^{1/2}\, v)\} = \{e\overset{*}{v}{}'v'\}$$

and hence

$$(x|\{ev^*v\}) = (P_x^{1/2}\, e|\{ev^*v\}) = (e|P_x^{1/2}\,\{ev^*v\}) = (e|\{e\overset{*}{v}{}'v'\}).$$

Now the assertion follows from (5.10). □

Lemma 5.15. *Let $\beta > (r-1)\,\frac{a}{2} + \frac{b}{2}$. Then*

$$\Gamma_\Omega(\beta) \int\limits_{V_{\mathbb{R}}} dv\ \Delta(e + \{ev^*v\})^{-\beta} = (2\pi)^{n_V/2}\ \Gamma_\Omega\Big(\beta - \frac{n_V}{2r}\Big).$$

Proof. Applying Lemma 5.6 and Lemma 5.14 we obtain

$$\begin{aligned}
&\Gamma_\Omega(\beta)\int\limits_{V_{\mathbb{R}}} dv\ \Delta(e+\{ev^*v\})^{-\beta}\\
=&\int\limits_{V_{\mathbb{R}}} dv \int\limits_{\Omega} dx\ \Delta(x)^{\beta-n_X/r}\, e^{-(x|e+\{ev^*v\})} =\\
&\int\limits_{\Omega} dx\ \Delta(x)^{\beta-n_X/r}\, e^{-(x|e)} \int\limits_{V_{\mathbb{R}}} dv\ e^{-(x|\{ev^*v\})}\\
=&\ (2\pi)^{n_V/2}\int\limits_{\Omega} dx\ \Delta(x)^{\beta-n_X/r-n_V/2r}\ e^{-(x|e)}\\
=&\ (2\pi)^{n_V/2}\ \Gamma_\Omega\left(\beta-\frac{n_V}{2r}\right).
\end{aligned}$$

□

Lemma 5.16. *Let $x\in\Omega$ and $\beta>(r-1)\,\frac{a}{2}+\frac{b}{2}$. Then*

$$\Gamma_\Omega(\beta)\int\limits_{V_{\mathbb{R}}} dv\ \Delta(x+\{ev^*v\})^{-\beta}=(2\pi)^{n_V/2}\ \Gamma_\Omega\left(\beta-\frac{n_V}{2r}\right)\ \Delta(x)^{n_V/2r-\beta}.$$

Proof. Putting $v'=R_x^{-1/2}\,v$ for $v\in V_{\mathbb{R}}$, we have

$$dv=\Delta(x)^{n_V/2r}\,dv'$$

by Lemma 5.13. Moreover

$$P_x^{-1/2}\,\{ev^*v\}=\{e(R_x^{-1/2}\,v)^*\,(R_x^{-1/2}\,v)\}=\{e\overset{*}{v}{}'v'\}$$

and hence

$$\Delta(x+\{ev^*v\})=\Delta(P_x^{1/2}\,(e+P_x^{-1/2}\,\{ev^*v\}))=\Delta(x)\ \Delta(e+\{e\overset{*}{v}{}'v'\}).$$

Therefore

$$\int\limits_{V_{\mathbb{R}}} dv\ \Delta(x+\{ev^*v\})^{-\beta}=\Delta(x)^{n_V/2r-\beta}\int\limits_{V_{\mathbb{R}}} dv'\ \Delta(e+\{e\overset{*}{v}{}'v'\})^{-\beta}$$

and Lemma 5.15 implies the assertion. □

Theorem 5.17. *The link transform $\tau\tau^*$ of the Toeplitz calculus with "Wallach" parameter ν has the eigenvalues*

$$\widetilde{\tau\tau^*}(\lambda)=\frac{\Gamma_\Omega\left(\underline{\lambda}+\underline{\rho}+\nu_{\mathbb{R}}-\frac{n_Y+n_V/2}{r}\right)\ \Gamma_\Omega\left(-\underline{\lambda}+\underline{\rho}+\nu_{\mathbb{R}}-\frac{n_Y+n_V/2}{r}\right)}{\Gamma_\Omega\left(\nu_{\mathbb{R}}-\frac{n_Y+n_V/2}{r}\right)\ \Gamma_\Omega(\nu_{\mathbb{R}})}.$$

Here $\underline{\lambda} = (\lambda_1, \ldots, \lambda_r)$ *is the spectral parameter in* $\mathfrak{a}^\#$ *and* $\underline{\rho} = (\rho_1, \ldots, \rho_r)$ *is the half-sum of positive restricted roots [L2], [Z] given by*

$$2\rho_j = 1 + (j-1)a + \frac{n_Y - n_X + n_V/2}{r}. \tag{5.11}$$

Proof. Writing $\zeta = x + y + v \in X \oplus Y \oplus V_{\mathbb{R}}$ according to (2.4), the domain $D_{\mathbb{R}}$ is defined by the condition

$$t := x - \{ev^*v\} \in \Omega. \tag{5.12}$$

Using (3.4) and applying Lemma 5.12, Lemma 5.16 and Lemma 5.7 we obtain in case $Z_{\mathbb{R}} \neq X$

$$\begin{aligned}
& I_\alpha := \int dx \int dv \int dy\ \Delta_U(x - \{ev^*v\})^{-p/2}\, \Delta^{\underline{\alpha}}(x - \{ev^*v\}) \cdot \\
& \cdot \Delta_U\left(\frac{e + x - y}{2}\right)^{-\nu}\ \Delta_U(x - \{ev^*v\})^{\nu/2} \\
= \quad & 2^{\nu r_U} \int dx \int dv \int dy\ \Delta^{\underline{\alpha} + \nu_{\mathbb{R}} - p_{\mathbb{R}}}(x - \{ev^*v\}) \cdot \Delta_U(e + x - y)^{-\nu} \\
= \quad & 2^{\nu r_U} \int_\Omega dt\ \Delta^{\underline{\alpha} + \nu_{\mathbb{R}} - p_{\mathbb{R}}}(t) \int_{V_{\mathbb{R}}} dv \int_Y dy \cdot \Delta_U(e + t + \{ev^*v\} - y)^{-\nu} \\
= \quad & 2^{n_X + n_Y}\ \pi^{(n_X + n_Y)/2}\ \frac{\Gamma_\Omega\left(2\nu_{\mathbb{R}} - \frac{n_Y}{r}\right)}{\Gamma_\Omega(\nu_{\mathbb{R}})\ \Gamma_\Omega\left(\nu_{\mathbb{R}} + \frac{n_X - n_Y}{2r}\right)} \cdot \\
& \cdot \int_\Omega dt\ \Delta^{\underline{\alpha} + \nu_{\mathbb{R}} - p_{\mathbb{R}}}(t) \int_{V_{\mathbb{R}}} dv\ \Delta(e + t + \{ev^*v\})^{-2\nu_{\mathbb{R}} + n_Y/r} \\
= \quad & 2^{n_X + n_Y}\ \pi^{(n_X + n_Y)/2}\ \frac{(2\pi)^{n_V/2}\ \Gamma_\Omega\left(2\nu_{\mathbb{R}} - \frac{n_Y + n_V/2}{r}\right)}{\Gamma_\Omega(\nu_{\mathbb{R}})\ \Gamma_\Omega\left(\nu_{\mathbb{R}} + \frac{n_X - n_Y}{2r}\right)} \cdot \\
& \cdot \int_\Omega dt\ \Delta^{\underline{\alpha} + \nu_{\mathbb{R}} - p_{\mathbb{R}}}(t)\, \Delta(e + t)^{-2\nu_{\mathbb{R}} + n_Y/r + n_V/2r} = 2^{n_X + n_Y + n_V/2}\, \pi^{n/2} \cdot \\
& \cdot \frac{\Gamma_\Omega\left(\underline{\alpha} + \nu_{\mathbb{R}} - p_{\mathbb{R}} + \frac{n_X}{r}\right)\ \Gamma_\Omega\left(2\nu_{\mathbb{R}} - \frac{n_Y + n_V/2}{r} - \underline{\alpha} - \nu_{\mathbb{R}} + p_{\mathbb{R}} + \underline{\delta} - \frac{2n_X}{r}\right)}{\Gamma_\Omega(\nu_{\mathbb{R}})\ \Gamma_\Omega\left(\nu_{\mathbb{R}} + \frac{n_X - n_Y}{2r}\right)} \\
= \quad & 2^{n_X + n_Y + n_V/2}\, \pi^{n/2}\ \frac{\Gamma_\Omega\left(\underline{\alpha} + \nu_{\mathbb{R}} - \frac{n_Y + n_V/2}{r}\right)\ \Gamma_\Omega\left(\nu_{\mathbb{R}} - \underline{\alpha} + \underline{\delta} - \frac{n_X}{r}\right)}{\Gamma_\Omega(\nu_{\mathbb{R}})\ \Gamma_\Omega\left(\nu_{\mathbb{R}} + \frac{n_X - n_Y}{2r}\right)}
\end{aligned}$$

since $rp_{\mathbb{R}} = n_X + n_Y + n_V/2$. In the remaining case $Z_{\mathbb{R}} = X$, Lemma 5.7 yields

$$I_\alpha \ := \ \int_\Omega dx\ \Delta(x)^{-n_X/r}\ \Delta^{\underline{\alpha}}(x)\ \Delta\left(\frac{e + x}{2}\right)^{-\nu}\ \Delta(x)^{\nu/2}$$

$$= \quad 2^{r\nu} \int_\Omega dx\ \Delta^{\underline{\alpha}-n_X/r+\nu/2}(x)\ \Delta(e+x)^{-\nu}$$

$$= \quad 2^{r\nu}\ \frac{\Gamma_\Omega\left(\frac{\nu}{2}+\underline{\alpha}\right)\ \Gamma_\Omega\left(\frac{\nu}{2}-\underline{\alpha}+\underline{\delta}-\frac{n_X}{r}\right)}{\Gamma_\Omega(\nu)}.$$

In general [Z, Lemma 2.3] we have

$$e_\lambda(x+y+v) = \Delta^{\underline{\lambda}+\underline{\rho}}(x-\{ev^*v\}).$$

Since $\widetilde{\tau\tau^*}(-\rho) = (\tau\tau^*1)(e) = 1$, it follows that

$$\widetilde{\tau\tau^*}(\lambda) = I_{\lambda+\rho}/I_0 = \frac{\Gamma_\Omega\left(\underline{\lambda}+\underline{\rho}+\nu_{\mathbb{R}}-\frac{n_Y+n_V/2}{r}\right)\ \Gamma_\Omega\left(\nu_{\mathbb{R}}-\underline{\lambda}-\underline{\rho}+\underline{\delta}-\frac{n_X}{r}\right)}{\Gamma_\Omega\left(\nu_{\mathbb{R}}-\frac{n_Y+n_V/2}{r}\right)\ \Gamma_\Omega\left(\nu_{\mathbb{R}}+\underline{\delta}-\frac{n_X}{r}\right)}$$

in both cases (since $\nu_{\mathbb{R}} = \nu/2$ if $Z_{\mathbb{R}} = X$). Since

$$2\underline{\rho} = \underline{\delta} + \frac{n_Y - n_X + n_V/2}{r}$$

by (5.7) and (5.11) and

$$\Gamma_\Omega\left(\nu_{\mathbb{R}}+\underline{\delta}-\frac{n_X}{r}\right) = \Gamma_\Omega\left(\nu_{\mathbb{R}}\right)$$

by (5.8), the assertion follows. □

Remark 5.18. In case $D_{\mathbb{R}}$ is complex, we have $\nu_{\mathbb{R}} = \nu$ and $d := \dim_{\mathbb{C}} D_{\mathbb{R}} = n_Y + n_V/2$. Therefore (5.10) simplifies to

$$\widetilde{\tau\tau^*}(\lambda) = \frac{\Gamma_\Omega(\underline{\rho}+\nu-\frac{d}{r}+\underline{\lambda})\ \Gamma_\Omega(\underline{\rho}+\nu-\frac{d}{r}-\underline{\lambda})}{\Gamma_\Omega(\nu-\frac{d}{r})\ \Gamma_\Omega(\nu)}$$

(cf. [AU1]).

Remark 5.19. While the Toeplitz-Berezin calculus is certainly fundamental, it is important to study other covariant symboli calculi such as the *Wick calculus* and the *Weyl calculus* and the relationship between them. In [AU1] a detailed investigation is carried out in the complex case, in particular for the Bargmann spaces over $\mathbb{C}^n$. In the curved setting the Weyl calculus, which involves the *symmetries* in a crucial way, poses many open problems, but in [AU2] the eigenvalues for the link transform of the Weyl calculus are determined for all (real and complex) symmetric domains of rank 1. The surprising new feature is the deep role played by hypergeometric functions in this context.

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