Weyl Calculus for Complex and Real Symmetric Domains

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Abstract

We define the Weyl functional calculus for real and complex symmetric domains, and compute the associated Weyl transform in the rank 1 case.

0 Introduction

In the theory of pseudo-differential operators the Weyl calculus (a quantization method for the cotangent bundle $T^{\#}(\mathbb{R}^n)$) is of basic importance since it allows the full symplectic group $Sp(2n, \mathbb{R})$ as covariance group and the relationship between operators and symbols has optimal continuity properties. Unterberger [10, 11] has introduced an analogous Weyl calculus for (curved) hermitian symmetric spaces of non-compact type and computed the Weyl transform in the simplest case of the unit disk. The higher dimensional case is more difficult. In this paper we define the Weyl calculus for real symmetric domains and then determine the Weyl transform for all symmetric spaces of rank 1. The new feature is the appearance of a hypergeometric function in the spectral decomposition, indicating that the harmonic analysis underlying the Weyl calculus involves (multi-variable) special functions in a significant way.

1 Real symmetric domains and quantization Hilbert spaces

Real bounded symmetric domains, as defined in [7], are those Riemannian symmetric spaces $D = G/K$ of non-compact type which are real forms of the well-known complex hermitian bounded symmetric domains $D_{\mathbb{C}} = G_{\mathbb{C}}/K_{\mathbb{C}}$, where $G_{\mathbb{C}} = Aut(D_{\mathbb{C}})^o$ is a real semisimple Lie group of hermitian type and $K_{\mathbb{C}}$ is a maximal compact subgroup.

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The well-known Harish-Chandra embedding, in its Jordan theoretic form, realizes $D_{\mathbb{C}}$ as the open unit ball

$$D_{\mathbb{C}} = \{z \in Z_{\mathbb{C}} : \ \|z\| < 1\}$$

of a complex vector space $Z_{\mathbb{C}} \approx \mathbb{C}^n$ endowed with a Jordan triple product $\{uv^*w\}$ [7, 12]. Now let $z \mapsto \overline{z}$ be a conjugation on $Z_{\mathbb{C}}$ preserving the triple product and define

$$\begin{aligned} Z &:= \{z \in Z_{\mathbb{C}} : \ \overline{z} = z\}, \quad D := \{z \in D_{\mathbb{C}} : \ \overline{z} = z\} = Z \cap D_{\mathbb{C}}, \\ G &:= \{g \in G_{\mathbb{C}} : \ \overline{g(z)} = g(\overline{z}) \ \forall \ z \in D_{\mathbb{C}}\} = \{g \in G_{\mathbb{C}} : \ g(D) = D\}, \quad K := K_{\mathbb{C}} \cap G. \end{aligned}$$

Then $D = G/K$ is called a *real bounded symmetric domain* which is a Riemannian symmetric space under the reductive Lie group G. Up to a few low dimensional exceptions all irreducible Riemannian symmetric spaces of non-compact type can be realized this way. Assuming that Z is irreducible one can show that the following cases occur:

<u>Case 1</u> $D_{\mathbb{C}}$ is an irreducible hermitian domain with real form D.

We may also include the *flat* case

<u>Case 2</u> $D_{\mathbb{C}} = Z_{\mathbb{C}} \approx \mathbb{C}^n$, endowed with the usual conjugation, so that $D = Z \approx \mathbb{R}^n$. In this case we obtain (non-reductive) semi-direct products $G_{\mathbb{C}} = U(n) \triangleleft \mathbb{C}^n, \quad G = O(n) \triangleleft \mathbb{R}^n$.

<u>Case 3</u> D is itself a complex hermitian domain with complexification $D_{\mathbb{C}} = D \times \overline{D}$ endowed with the flip conjugation $\overline{(z_1, \overline{z}_2)} := (z_2, \overline{z}_1)$ for all $z_1, z_2 \in D$. In this case

$$G_{\mathbb{C}} = \{(g_1, \overline{g}_2) : \ g_1, g_2 \in G\} \approx G \times G$$

where $(g_1, \overline{g}_2)(z_1, \overline{z}_2) := (g_1(z_1), \ \overline{g_2(z_2)})$.

For every real symmetric domain D as above there exists a scale of "quantization Hilbert spaces" H_ν of holomorphic functions on the complexification $D_{\mathbb{C}}$ of D. These Hilbert spaces constitute the "scalar holomorphic discrete series" of $G_{\mathbb{C}} = Aut(D_{\mathbb{C}})^\circ$ via irreducible unitary (projective) representations $U_\nu : G_{\mathbb{C}} \to U(H_\nu)$ of the form

$$(U_\nu(g^{-1})h)(z) = j(g, z)\, h(g(z))$$

for all $g \in G_{\mathbb{C}}, \ \ h \in H_\nu$ and $z \in D_{\mathbb{C}}$. Here $j(g, z)$ is a suitable automorphy factor.

For each irreducible complex bounded symmetric domain B of dimension n, define the *weighted Bergman spaces*

$$H^2_\beta(B) := \{h \in L^2(B, d\mu_\beta) : \ h \text{ holomorphic}\}.$$

Here $\beta > p-1$ is a scalar parameter, where p is the genus of B. The probability measure

$$d\mu_\beta(z) := \frac{\Gamma_{\Omega_B}(\beta)}{\pi^n\, \Gamma_{\Omega_B}(\beta - n/r_B)}\ \Delta(z,z)^{\beta-p} dm(z) \tag{1.1}$$

involves the so-called Jordan triple determinant $\Delta(z,w)$ and the Gindikin Γ-function of the positive cone Ω_B associated with B. Moreover, r_B is the rank of B and $dm(z)$ is Lebesgue measure. The reproducing kernel of $H^2_\beta(B)$ has the form

$$K(z,w) = \Delta(z,w)^{-\beta}$$

for all $z,w \in B$. Returning to the real symmetric domain D with complexification $D_{\mathbb{C}}$, we consider the different cases:

<u>Case 1</u> If $D_{\mathbb{C}}$ is an irreducible complex symmetric domain with real form D, we define

$$H_\nu := H^2_{\nu_{\mathbb{C}}}(D_{\mathbb{C}}), \quad \nu_{\mathbb{C}} := 2r\nu/r_{\mathbb{C}} \tag{1.2}$$

where $r_{\mathbb{C}} \geq r$ is the rank of $D_{\mathbb{C}}$. We have $j(g,z) = (Det\, g'(z))^{\nu_{\mathbb{C}}/p_{\mathbb{C}}}$ in this case, where $p_{\mathbb{C}}$ is the genus of $D_{\mathbb{C}}$.

<u>Case 2</u> In the flat case $Z_{\mathbb{C}} = \mathbb{C}^n$, with real form $Z = \mathbb{R}^n$, the quantization Hilbert spaces are the *Bargmann spaces*

$$H_\nu := H^2_\nu(\mathbb{C}^n) = \{h \in L^2(\mathbb{C}^n, \mu_\nu) :\ h \text{ holomorphic}\}$$

with respect to the probability measure

$$d\mu_\nu(z) = \left(\frac{\nu}{\pi}\right)^n\ e^{-\nu(z|z)}\, dm(z)$$

where $(z|w)$ is the scalar product on $\mathbb{C}^n$ and $dm(z)$ is the associated Lebesgue measure. The reproducing kernel is

$$K(z,w) = e^{\nu(z|w)}$$

for all $z,w \in \mathbb{C}^n$ and U_ν is the Schrödinger representation of $U(n) \triangleleft \mathbb{C}^n$ in the "complex wave" realization, with its well-known multiplier $j(g,z)$ [1].

<u>Case 3</u> If D is itself a complex hermitian domain, with measure $d\mu_\nu$ as defined in (1.1), we consider the product probability measure $d\mu(z_1,z_2) := d\mu_\nu(z_1)\ d\mu_\nu(z_2)$ on $D_{\mathbb{C}} = D \times \overline{D}$ and put

$$H_\nu := \{h \in L^2(D_{\mathbb{C}}, d\mu) :\ h \text{ sesqui-holomorphic}\} = H^2_\nu(D) \otimes \overline{H^2_\nu(D)}$$

realized via *Hilbert-Schmidt operators*

$$(h\phi)(z) = \int_D d\mu_\nu(w)\ h(z,\overline{w})\ \phi(w)$$

for $\phi \in H^2_\nu(D)$ and $z \in D$. This Hilbert space has the reproducing kernel

$$K(z_1,\overline{z}_2;w_1,\overline{w}_2) = K(z_1,w_1)\ K(w_2,z_2), \tag{1.3}$$

with K the kernel function of $H^2_\nu(D)$, and

$$U_\nu(g_1,\overline{g}_2)\,h = U_\nu(g_1)\,h\,U_\nu(g_2)^*$$

is the corresponding irreducible unitary (projective) representation of $G_{\mathbb{C}} = G \times G$ on H_ν, realized as Hilbert-Schmidt operators. We put $j(g_1,\overline{g}_2;\ z_1,\overline{z}_2) := j(g_1,z_1)\,\overline{j(g_2,z_2)}$ in this case.

In all cases the reproducing kernel $K(z,w)$ and the (projective) multiplier $j(g,z)$ are related by

$$j(g,z)\ K(g(z),\ g(w))\ \overline{j(g,w)} = K(z,w) \tag{1.4}$$

for all $z,w \in D_{\mathbb{C}}$ and $g \in G_{\mathbb{C}}$. This implies

$$U_\nu(g)\,K_z = \overline{j(g,z)}\ K_{g(z)}. \tag{1.5}$$

2 The Weyl calculus and its basic properties

In [1, 2] a general concept of "covariant symbolic calculus" of symmetric domains has been developed. In the (more general) *real* version [2] one considers a linear "symbol" map $\sigma : H_\nu \to \{\text{functions on } D\}$ satisfying the covariance condition

$$\sigma(U_\nu(g)\,h) = (\sigma h) \circ g^{-1} \tag{2.1}$$

for all $g \in G \subset G_{\mathbb{C}}$ and $h \in H_\nu$. More precisely, the domain $Dom(\sigma)$ should contain all the kernel vectors

$$K_w(z) := K(z,w)$$

for $w \in D_{\mathbb{C}}$, and the condition (2.1) becomes

$$\overline{j(g,w)}\ \sigma(K_{g(w)}) = (\sigma\,K_w) \circ g^{-1} \tag{2.2}$$

for all $g \in G$ and $w \in D_{\mathbb{C}}$. In addition we assume that the holomorphic function

$$I(z) := K(z,\overline{z})^{1/2}$$

on $D_{\mathbb{C}}$, defined via the conjugation $z \mapsto \overline{z}$ and the holomorphic square-root on the (simply-connected) domain $D_{\mathbb{C}}$, belongs to $Dom(\sigma)$ and satisfies

$$\sigma I = 1. \tag{2.3}$$

Since $U_\nu(g)\, I = I$ for all $g \in G$, σI is a constant function according to (2.1) so that (2.3) is just a normalization.

In [2], the so-called *Toeplitz-Berezin calculus* has been studied in detail (cf. also [3, 9, 15]). We now consider another covariant symbolic calculus, the *Weyl calculus* introduced in the complex setting in [10]. For $\zeta \in D$ the *symmetry* $s_\zeta \in G$ is characterized by the conditions

$$s_\zeta^2 = id, \;\; s_\zeta(\zeta) = \zeta, \;\; s'_\zeta(\zeta) = -Id.$$

Lemma 2.1 *$j(s_\zeta, \zeta) = 1$ for all $\zeta \in D$.*

Proof: Since $s_\zeta^2 = id$ we have $1 = j(s_\zeta^2, \zeta) = j(s_\zeta, s_\zeta(\zeta))\, j(s_\zeta, \zeta) = j(s_\zeta, \zeta)^2$ and hence $j(s_\zeta, \zeta) \in \{\pm 1\}$. Since D is connected it follows that $j(s_\zeta, \zeta) = j(s_o, o) = 1$. ∎

Lemma 2.2 *For $\zeta \in D$ and $z \in D_{\mathbb{C}}$ we have*

$$j(s_\zeta, z) = \frac{K(z,\zeta)}{K(s_\zeta z, \zeta)}.$$

Proof: By Lemma 2.1 , we have $j(s_\zeta, z)\; K(s_\zeta z, \zeta) = j(s_\zeta, z)\; K(s_\zeta z, s_\zeta \zeta)\; j(s_\zeta, \zeta) =$
$= j(s_\zeta, z)\; K(s_\zeta z, \zeta) = K(z, \zeta)$. ∎

As a special case of Lemma 2.2 , we obtain

$$j(s, z) = \frac{K(z, o)}{K(sz, o)}$$

for the origin $\zeta = o \in D$ and its symmetry $s = s_o$.

Definition 2.1 The *Weyl symbol* map $\omega_\nu :\; \mathrm{span}\{K_z; z \in D_{\mathbb{C}}\} \to \mathcal{C}^\infty(D)$ is defined by

$$(\omega_\nu\, K_z)(\zeta) = c_\nu^{-1}\; \frac{K(\zeta, z)^{1/2}}{K(\zeta, s_\zeta z)^{1/2}}\; K(\overline{z}, s_\zeta z)^{1/2} \tag{2.4}$$

for all $z \in D_{\mathbb{C}}$ and $\zeta \in D$. Here $\overline{z}$ is the conjugate of z (so that (2.4) is anti-holomorphic in z). The normalization constant c_ν determined by the condition

$$\omega_\nu\, I = 1 \tag{2.5}$$

will be computed below. Note that Lemma 2.2 implies

$$\overline{(\omega_\nu K_z)(\zeta)} = c_\nu^{-1}\ j(s_\zeta, z)^{1/2}\ K(s_\zeta z, \overline{z})^{1/2} \tag{2.6}$$

as a holomorphic function in $z \in D_{\mathbb{C}}$. Together with (1.5), (2.6) implies

$$\begin{aligned} c_\nu(\omega_\nu K_z)(\zeta) &= (\overline{j(s_\zeta, z)}\ K(\overline{z}, s_\zeta z))^{1/2} = (\overline{j(s_\zeta, z)}\ K_{s_\zeta z}(\overline{z}))^{1/2} \\ &= (U_\nu(s_\zeta)\, K_z)(\overline{z})^{1/2} = (K_{\overline{z}}\,|\, U_\nu(s_\zeta)\, K_z)^{1/2}. \end{aligned} \tag{2.7}$$

Example 2.1 In the product case $D_{\mathbb{C}} = D \times \overline{D}$, with D complex hermitian, H_ν can be identified with the space of Hilbert-Schmidt operators acting on $H^2_\nu(D)$, and under this identification

$$K_{z_1, \overline{z}_2} = K_{z_1}\, K^*_{z_2} \qquad (z_1, z_2 \in D)$$

becomes a rank 1 operator [2, Example 3.1]. Therefore (2.7) yields

$$\begin{aligned} c_\nu(\omega_\nu K_{z_1,\overline{z}_2})(\zeta) &= (K_{z_2,\overline{z}_1}\,|\, U_\nu(s_\zeta)\, K_{z_1,\overline{z}_2})^{1/2} = (K_{z_2}\, K^*_{z_1}\,|\, U_\nu(s_\zeta)(K_{z_1}\, K^*_{z_2}))^{1/2} = \\ &(K_{z_2}\, K^*_{z_1}\,|\, U_\nu(s_\zeta) K_{z_1}\, K^*_{z_2}\ U_\nu(s_\zeta)^*)^{1/2} = (K_{z_2}\, K^*_{z_1}\,|\, (U_\nu(s_\zeta)\, K_{z_1})(U_\nu(s_\zeta)\, K_{z_2})^*)^{1/2} = \\ &[(K_{z_2}\,|\, U_\nu(s_\zeta)\, K_{z_1})(U_\nu(s_\zeta)\, K_{z_2}\,|\, K_{z_1})]^{1/2} = (K_{z_2}\,|\, U_\nu(s_\zeta)\, K_{z_1}) = tr(U_\nu(s_\zeta)\ K_{z_1}\, K^*_{z_2}). \end{aligned}$$

Hence we have

$$c_\nu(\omega_\nu T)(\zeta) = tr(U_\nu(s_\zeta)\, T)$$

for all (trace-class) operators T acting on $H^2_\nu(D)$. This coincides with the "Weyl symbol" of T as defined in [10].

Proposition 2.1 *The Weyl symbol (2.4) is covariant under G.*

Proof: Let $\zeta \in D$, $g \in G$ and $z \in D_{\mathbb{C}}$. Then $s_{g(\zeta)} = g\, s_\zeta\, g^{-1}$ and (1.2) and (2.6) imply

$$\begin{aligned} c_\nu \left[j(g,z)\ \overline{(\omega_\nu K_{g(z)})(g(\zeta))}\right]^2 &= j(g,z)^2\ j(s_{g(\zeta)}, g(z))\ K(s_{g(\zeta)}(g(z)),\ \overline{g(z)}) = \\ &j(g,z)\ j(g\, s_\zeta\, g^{-1}, g(z))\ K(g(s_\zeta(z)), g(\overline{z}))\ \overline{j(g,\overline{z})} = \\ &j(s_\zeta, z)\ j(g, s_\zeta(z))\ K(g(s_\zeta(z)), g(\overline{z}))\ \overline{j(g,\overline{z})} = j(s_\zeta, z)\ K(s_\zeta(z), \overline{z}) = c_\nu\ \overline{(\omega_\nu K_z)(\zeta)}^2 \end{aligned}$$

since $j(g,z)\ j(g\, s_\zeta\, g^{-1}, g(z)) = j(g\, s_\zeta, z) = j(s_\zeta, z)\ j(g, s_\zeta(z))$. Taking holomorphic square-roots and conjugates it follows that $\overline{j(g,z)}\ (\omega_\nu K_{g(z)})(g(\zeta)) = (\omega_\nu K_z)(\zeta)$ which yields covariance in view of (2.2). ∎

Given a covariant symbolic calculus σ one defines its *adjoint* $\sigma^* :$ {functions on D} $\to$ H_ν by assigning to a function $f \in Dom\,(\sigma^*)$ the holomorphic function

$$(\sigma^* f)(z) := \int_D d\mu_0(\zeta)\ f(\zeta)\ \overline{(\sigma\, K_z)(\zeta)}$$

on $D_{\mathbb{C}}$. Here $d\mu_0$ is the G-invariant measure on D normalized by the condition

$$\int_D d\mu_0(\zeta)\ I(\zeta)^{-1} = 1. \tag{2.8}$$

By [2, Proposition 4.4] σ^* is the adjoint of σ with respect to $L^2(D, d\mu_0)$. According to Definition 2.1 the adjoint $f \mapsto \omega_\nu^* f$ of the Weyl symbol map is given by

$$(\omega_\nu^* f)(z) = \int_D d\mu_0(\zeta)\ f(\zeta)\ \overline{(\omega_\nu\, K_z)(\zeta)} = c_\nu^{-1} \int_D d\mu_0(\zeta)\ f(\zeta)\ \frac{K(z,\zeta)^{1/2}}{K(s_\zeta z, \zeta)^{1/2}}\ K(s_\zeta z, \overline{z})^{1/2}$$

$$= c_\nu^{-1} \int_D d\mu_0(\zeta)\ f(\zeta)\ j(s_\zeta, z)^{1/2}\ K(s_\zeta z, \overline{z})^{1/2} = c_\nu^{-1} \int_D d\mu_0(\zeta)\ f(\zeta)\ \overline{(U_\nu(s_\zeta)\, K_z)(\overline{z})}^{1/2}$$

as a holomorphic function in $z \in D_{\mathbb{C}}$. Whereas the Toeplitz map $f \mapsto \tau_\nu^* f$ is well-defined for $f \in L^\infty(D)$, it is more difficult to find conditions on $f \in \mathcal{C}^\infty(D)$ such that $\omega_\nu^* f$ is well-behaved [11, 13].

Example 2.2 In the flat case $D = \mathbb{R}^n$ and $D_{\mathbb{C}} = \mathbb{C}^n$, the ν-th Bargmann space $H_\nu^2(\mathbb{C}^n)$ has the reproducing kernel $K(z, w) = exp\ \nu(z|w)$ and

$$s_\zeta z = 2\zeta - z$$

is the symmetry. Hence the Weyl calculus $\omega_\nu^* : L^2(\mathbb{R}^n) \to H_\nu^2(\mathbb{C}^n)$ has the form

$$c_\nu\,(\omega_\nu^* f)(z) = \int_{\mathbb{R}^n} d\zeta\ f(\zeta) \cdot exp\ \frac{\nu}{2}\,((z|\zeta) + (s_\zeta z|\overline{z}) - (s_\zeta z|\zeta)) =$$

$$\int_{\mathbb{R}^n} d\zeta\ f(\zeta)\ exp\ \frac{\nu}{2}\,((z|\zeta) + (2\zeta - z\,|\,\overline{z} - \zeta)) = \int_{\mathbb{R}^n} d\zeta\ f(\zeta)\ exp\,(2\nu(z|\zeta) - \frac{\nu}{2}\,(z|\overline{z}) - \nu(\zeta|\zeta)).$$

Since $c_\nu = 2^{-n/4}$ in this case, we obtain the *Bargmann transform* [4, p. 40].

The link *transform* of a covariant symbolic calculus σ is defined as the map $f \mapsto (\sigma\,\sigma^*)\, f := \sigma(\sigma^* f)$ acting on functions on D. By [2, Proposition 4.7] $\sigma\,\sigma^*$ is an integral operator in $L^2(D, d\mu_0)$ with kernel

$$\int_{D_{\mathbb{C}}} d\mu_\nu(z)(\sigma\, K_z)(\xi)\overline{(\sigma\, K_z)(\eta)}$$

for $\xi, \eta \in D$. For the *Weyl transform* $\omega_\nu \omega_\nu^*$ we obtain the integral kernel

$$c_\nu^{-2} \int_{D_{\mathbb{C}}} d\mu_\nu(z)\ \left(\frac{K(\xi, z)\, K(\overline{z}, s_\xi z)\, K(z, \eta)\, K(s_\eta z, \overline{z})}{K(\xi, s_\xi z)\, K(s_\eta z, \eta)} \right)^{1/2}$$

for $\xi, \eta \in D$. In comparison, the integral kernel of the *Toeplitz transform* $\tau_\nu \tau_\nu^*$ has the much simpler expression [2, (5.4)]

$$\frac{K(\xi,\eta)}{K(\xi,\xi)^{1/2}\ K(\eta,\eta)^{1/2}}.$$

Since $D = G/K$ is a Riemannian symmetric space there is an explicit *Plancherel decomposition* [6]

$$L^2(D,\mu_0) = \int\limits_{\mathfrak{a}^\#} d\lambda\, |c(\lambda)|^{-2}\ \langle G\rangle_\lambda$$

in terms of the Hilbert spaces $\langle G\rangle_\lambda$ of the principal series of G with parameter $\lambda \in \mathfrak{a}^\#$, where $c(\lambda)$ is Harish-Chandra's c-function and we use an Iwasawa decomposition $G = N\,A\,K$ with $\mathfrak{a} := Lie\,(A)$. By covariance every covariant symbolic calculus σ yields a multiplicity-free decomposition

$$H_\nu \approx \int d\sigma_0(\lambda)\,\langle G\rangle_\lambda$$

of H_ν, under the restricted action of $G \subset G_{\mathbb{C}}$. The defining measure $d\sigma_0(\lambda)$ depends on the choice of calculus, more precisely on the *eigenvalues*

$$\widetilde{\sigma\sigma^*}(\lambda) := (\sigma\sigma^*\phi_\lambda)(o)$$

of the G-invariant link transform $\sigma\sigma^*$, computed on the spherical function ϕ_λ of type $\lambda \in \mathfrak{a}^\#$. Here $o \in D$ is the origin. It is technically easier to use the NA-covariant "exponential functions" e_λ, where

$$\phi_\lambda(\zeta) = \int\limits_K dk\ e_\lambda(k\zeta)$$

for all $\zeta \in D$. For the Toeplitz-Berezin calculus τ_ν the eigenvalues of $\tau_\nu\tau_\nu^*$ are given by the integral

$$\widetilde{\tau_\nu\tau_\nu^*}(\lambda) = (\tau_\nu^* e_\lambda)(o) = \int\limits_D d\mu_0(\zeta)\, e_\lambda(\zeta)\,(\tau_\nu\, K_o)(\zeta) = \int\limits_D d\mu_0(\zeta)\, e_\lambda(\zeta)\, K(o,\zeta)\, K(\zeta,\zeta)^{-1/2}$$

which can be computed using the structure theory of Jordan triples ([2, 15, 3, 9]) yielding a (complicated) product of classical Γ-functions. For arbitrary covariant symbolic calculi σ_1, σ_2 on H_ν there is a "product formula" [2, Theorem 4.9]

$$\widetilde{\sigma_1\sigma_2^*}(\lambda) = \frac{\overline{(\sigma_1^* e_\lambda)(o)}(\sigma_2^* e_\lambda)(o)}{(\tau_\nu^* e_\lambda)(o)}.$$

Thus the integral

$$(\omega_\nu^* e_\lambda)(o) = \int_D d\mu_0(\zeta)\, e_\lambda(\zeta)\, (\omega_\nu K_o)(\zeta) = c_\nu^{-1} \int_D d\mu_0(\zeta)\, e_\lambda(\zeta)\, \frac{K(o,\zeta)^{1/2}}{K(\zeta, s_\zeta(o))^{1/2}}\, K(o, s_\zeta(o))^{1/2}$$

is needed for the computation of the eigenvalues of $\widetilde{\omega_\nu \omega_\nu^*}(\lambda)$ of the Weyl transform.

3 Polar coordinates and root decomposition

From now on we only consider the non-flat case. For a deeper study of the Weyl calculus (and other covariant symbolic calculi) on real symmetric domains it is necessary to recall the basic structure theory of symmetric spaces G/K (of non-compact type) related to the *root decomposition* [6]

$$\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{m} \oplus \int_{\alpha\in\Sigma} \mathfrak{g}_\alpha$$

induced by a Cartan subspace $\mathfrak{a} \subset \mathfrak{g}$. Here $\mathfrak{m}$ is the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$ and $\mathfrak{g}_\alpha$ denotes the root space associated with $\alpha \in \mathfrak{a}^\#$. Put $\Sigma := \{\alpha \in \mathfrak{a}^\# \setminus \{0\} : \ \mathfrak{g}_\alpha \neq \{0\}\}$. According to [7] every (irreducible) real symmetric domain $D = G/K$ has an unbounded realization as a *real Siegel domain*

$$D \approx \{x + y + v \in X \oplus Y \oplus V : \ x - \{e\, v^* v\} \in \Omega\}.$$

Here X is a euclidean Jordan algebra of rank r, with unit element e and positive cone Ω, $X \oplus Y$ is a semi-simple real Jordan $*$-algebra with self-adjoint part X and skew-adjoint part Y, and the Peirce decomposition [7] of Z with respect to e has the 1-eigenspace $X \oplus Y$ and the $\frac{1}{2}$-eigenspace V. Now choose a *frame* $e_1, \ldots, e_r$ of minimal idempotents in X satisfying $e_1 + \ldots + e_r = e$ and consider the joint Peirce decomposition [7]

$$X = \sum_{1\le i\le j\le r} X_{ij}, \ \ Y = \sum_{1\le i\le j\le r} Y_{ij}, \ \ V = \sum_{1\le j\le r} V_{0j}.$$

Then for $1 \le j \le r$ and $1 \le i < j \le r$ we have $\ dim\ X_{jj} = 1$ and

$$a := dim\ X_{ij}, \quad b := dim\ V_{0j}, \quad c := dim\ Y_{ij}$$

are independent of i, j and of the frame $e_1, \ldots, e_r$. For the symmetric cones we have $Y = \{0\} = V$. In all other cases (except root system D_2, which is not considered in the sequel) the classification yields $\ dim\ Y_{ij} = a \ \ (1 \le i < j \le r)$. Hence the fine structure of D is completely encoded in the numerical invariants a, b, c. In particular, we have $n_X := dim\ X = r + \frac{a}{2}\, r(r-1)$, $n_Y := dim\ Y = cr + \frac{a}{2}\, r(r-1)$, $n_V := dim\ V = br$. Returning to the bounded realization, the commuting completely integrable holomorphic vector fields

$$M_j := (e_j - \{z\, e_j^*\, z\})\ \frac{\partial}{\partial z} \qquad (1 \leq j \leq r) \tag{3.1}$$

on $D_{\mathbb{C}}$ [12, 7] leave D invariant and can be chosen as a basis of $\mathfrak{a}$. Let $M_1^\#, \ldots, M_r^\# \in \mathfrak{a}^\#$ denote the dual basis satisfying $M_i^\#(M_j) = \delta_{ij}$ for $1 \leq i, j \leq r$. Then, by [7, 15] the positive restricted roots of $\mathfrak{g}$ are the following:

$$M_j^\# - M_i^\#, \quad \text{multiplicity } a \quad (1 \leq i < j \leq r) \tag{3.2}$$

$$M_j^\# + M_i^\#, \quad \text{multiplicity } a \quad (1 \leq i < j \leq r) \tag{3.3}$$

$$2M_j^\#, \quad \text{multiplicity } c \quad (1 \leq j \leq r) \tag{3.4}$$

$$M_j^\#, \quad \text{multiplicity } b \quad (1 \leq j \leq r), \tag{3.5}$$

unless $Z = X$, in which case only (3.2) occurs. This case and also the root type D_2 will be omitted in the sequel. For the half-sum of positive roots ρ we obtain

$$\begin{aligned} 2\rho &= \sum_{\alpha \in \Sigma_+} m_\alpha\, \alpha = \sum_{1 \leq i < j \leq r} a\, \left(M_j^\# - M_i^\# + M_j^\# + M_i^\#\right) + \sum_{1 \leq j \leq r} (c\, 2M_j^\# + b\, M_j^\#) \\ &= \sum_{1 \leq i < j \leq r} 2a\, M_j^\# + \sum_{1 \leq j \leq r} (2c + b) M_j^\# = \sum_{1 \leq j \leq r} 2((j-1)\, a + c + \frac{b}{2}) M_j^\# \end{aligned}$$

and hence

$$\rho = \sum_{1 \leq j \leq r} \left((j-1)\, a + c + \frac{b}{2}\right)\ M_j^\#\ .$$

By [6, Theorem 5.8] there exists a Haar measure dg on G such that

$$\int_G dg\ f(g(0)) = \int_{\mathbb{R}_+^r} dt_1 \cdots dt_r\ f(exp\,(\Sigma_j\, t_j\, M_j)(0)) \prod_{\alpha \in \Sigma_+} \sinh\,(\alpha(\Sigma_j\, t_j\, M_j))^{m_\alpha}$$

holds for K-invariant functions of f on $D = G/K$. Here Σ_+ denotes the set of positive roots and m_α is the multiplicity of $\alpha \in \Sigma_+$. For each tripotent $c = \{c\, c^*\, c\} \in Z$ the vector field

$$M_c := (c - \{z\, c^*\, z\})\ \frac{\partial}{\partial z}$$

satisfies $exp\,(t\, M_c)(0) = \tanh\,(t)\, c$ for all $t \in \mathbb{R}$ [12, 7]. Similarly, we have

$$exp\ \left(\sum_{j=1}^r t_j\, M_j\right)(0) = \sum_{j=1}^r \tanh\,(t_j)\, e_j$$

for the (commuting) vector fields (3.1). Using the coordinates

$$x_j = \tanh^2(t_j) \in [0,1] \tag{3.6}$$

satisfying $\frac{dx_j}{dt_j} = 2x_j^{1/2}(1-x_j)$, the explicit root decomposition (3.2)-(3.5) yields

$$\int_G dg\ f(g(0)) =$$

$$\int_{\mathbb{R}_+^r} \prod_j dt_j\ \sinh(2t_j)^c\ \sinh(t_j)^b \cdot f(\Sigma_j \tanh(t_j)\, e_j) \cdot \prod_{i<j} |\sinh(t_j - t_i)\ \sinh(t_j + t_i)|^a =$$

$$\int_{[0,1]^r} \prod_j \frac{dx_j}{2(1-x_j)x_j^{1/2}} \left(\frac{2x_j^{1/2}}{1-x_j}\right)^c \left(\frac{x_j}{1-x_j}\right)^{b/2} \cdot \prod_{i<j} \left|\frac{x_i - x_j}{(1-x_i)(1-x_j)}\right|^a \cdot f\left(\Sigma_j\, x_j^{1/2}\, e_j\right) =$$

$$2^{r(c-1)} \int_{[0,1]^r} dx_1 \cdots dx_r \cdot \prod_j (1-x_j)^{-1-c-b/2-a(r-1)}\, x_j^{(c-1+b)/2} \cdot \prod_{i<j} |x_i - x_j|^a \cdot f\left(\Sigma_j\, x_j^{1/2}\, e_j\right) =$$

$$2^{n_Y - n_X} \int_{[0,1]^r} f\left(\Sigma_j\, x_j^{1/2}\, e_j\right) \prod_{i<j} |x_i - x_j|^a \cdot \prod_j dx_j\ (1-x_j)^{-(n_X+n_Y+n_V/2)/r}\, x_j^{(n_Y-n_X+n_V)/2r}.$$

Let Ω be the positive cone of the euclidean Jordan algebra X. The Gindikin Γ-function Γ_Ω associated with Ω [5, Chapter VII] has the property [5, p. 123 and p. 104]

$$\Gamma_\Omega\left(\alpha + \frac{n_X}{r}\right) = c_\Omega \int_{\mathbb{R}_+^r} \prod_{i<j} |x_i - x_j|^a \prod_j dx_j\ e^{-x_j}\, x_j^\alpha, \tag{3.7}$$

where c_Ω is a constant depending only on Ω. Similarly for the Beta-integral [5, p. 130 and p. 104] which is symmetric in α and γ:

$$\frac{\Gamma_\Omega(\alpha + \frac{n_X}{r})\ \Gamma_\Omega(\gamma + \frac{n_X}{r})}{\Gamma_\Omega(\alpha + \gamma + \frac{2n_X}{r})} = c_\Omega \int_{[0,1]^r} \prod_{i<j} |x_i - x_j|^a \prod_j dx_j\, x_j^\alpha (1-x_j)^\gamma. \tag{3.8}$$

Lemma 3.1 *For $\xi, \eta \in D \cap X$ we have*

$$K(\xi,\eta)^{-1/2} = \Delta(e - \{\xi\, e^*\, \eta\})^\nu.$$

Proof: In case $D_\mathbb{C}$ is irreducible, we have

$K(\xi,\eta)^{-1/2} = \Delta_\mathbb{C}(\xi,\eta)^{\nu_\mathbb{C}/2} = \Delta_\mathbb{C}(e - \{\xi\, e^*\, \eta\})^{\nu_\mathbb{C}/2},\ \ \Delta(e - \{\xi\, e^*\, \eta\})^{r_\mathbb{C}\,\nu_\mathbb{C}/2r} = \Delta(e - \{\xi\, e^*\, \eta\})^\nu.$

In case $D_\mathbb{C} = D \times \overline{D}$, with D complex hermitian, (1.3) implies

$$K(\xi,\xi,\eta,\eta)^{-1/2} = K(\xi,\eta)^{-1} = \Delta(\xi,\eta)^\nu = \Delta(e - \{\xi\, e^*\, \eta\})^\nu.$$

In both cases, the assertion follows. ∎

Proposition 3.1 *The measure μ_0 normalized by (2.8) is given by*

$$\int_D d\mu_0(\zeta)\, f(\zeta) = c_\Omega\, 2^{n_X - n_Y}\ \frac{\Gamma_\Omega(\nu + \frac{n_X - n_Y}{2r})}{\Gamma_\Omega(\nu - \frac{n_Y + n_V/2}{r})\ \Gamma_\Omega(\frac{n}{2r})}\ \int_G dg\, f(g(0)).$$

Proof: By Lemma 3.1 we have for $\zeta := \Sigma_j \tanh(t_j)\, e_j \in D \cap X$

$$I(\zeta)^{-1} = K(\zeta, \zeta)^{-1/2} = \Delta(e - \{\zeta\, e^*\, \zeta\})^\nu = \prod_j (1 - \tanh^2(t_j))^\nu = \prod_j (1 - x_j)^\nu.$$

Since $n - 2n_Y - n_V = n_X - n_Y$, it follows that

$$\int_G dg\ I(g(0))^{-1} = 2^{n_Y - n_X} \int_{[0,1]^r} \prod_{i<j} |x_i - x_j|^a \cdot$$

$$\prod_j dx_j\ (1 - x_j)^{\nu - (n_X + n_Y + n_V/2)/r}\ x_j^{(n_Y - n_X + n_V)/2r} =$$

$$c_\Omega^{-1}\ 2^{n_Y - n_X}\ \frac{\Gamma_\Omega(\nu - \frac{n_Y + n_V/2}{r})\ \Gamma_\Omega(\frac{n}{2r})}{\Gamma_\Omega(\nu + \frac{n_X - n_Y}{2r})}.$$

∎

The spherical functions of the cone Ω, regarded as a reductive symmetric space, can be expressed in terms of the so-called *Jack polynomials* $J_{\underline{m}}(x_1, \ldots, x_r)$ associated with an integer partition $\underline{m} = (m_1, \ldots, m_r)$. Using the Jack polynomials and the multi-variable *Pochhammer symbol*

$$(\alpha)_{\underline{m}} := \frac{\Gamma_\Omega(\alpha + \underline{m})}{\Gamma_\Omega(\alpha)}$$

one defines the multivariable *hypergeometric series* [5]

$${}_2F_1\begin{pmatrix} \alpha\ \beta \\ \gamma \end{pmatrix} (x_1, \ldots, x_r) = \sum_{\underline{m}} \frac{(\alpha)_{\underline{m}}\, (\beta)_{\underline{m}}}{(\gamma)_{\underline{m}}\, (1)_{\underline{m}}}\ J_{\underline{m}}(x_1, \ldots, x_r).$$

The multivariable hypergeometric function together with the Gindikin Γ-function yields the following Selberg-type integral

$$c_\Omega \cdot \int_{[0,1]^r} \prod_j dz_j\ z_j^\alpha\, (1 - z_j)^\gamma (1 - \frac{z_j}{2})^{-\beta} \prod_{i<j} |z_i - z_j|^a = \tag{3.9}$$

$$\frac{\Gamma_\Omega(\alpha + \frac{n_X}{r})\ \Gamma_\Omega(\gamma + \frac{n_X}{r})}{\Gamma_\Omega(\alpha + \gamma + \frac{2n_X}{r})}\ {}_2F_1\begin{pmatrix} \alpha + \frac{n_X}{r} & \beta \\ \alpha + \gamma + \frac{2n_X}{r} & \end{pmatrix} \left(\frac{1}{2}, \ldots, \frac{1}{2}\right)$$

for (suitably restricted) scalar parameters α, β, γ. Since $K(z, 0) = 1$ in the bounded setting, (2.4) implies

$$(\omega_\nu K_0)(\zeta) = c_\nu^{-1}\, \mathfrak{b}_\nu(\zeta), \tag{3.10}$$

where we define $\mathfrak{b}_\nu(\zeta) := K(s_\zeta(0), \zeta)^{-1/2}$ for all $\zeta \in D$.

Lemma 3.2 *The K-invariant function $\mathfrak{b}_\nu$ on D satisfies*

$$\mathfrak{b}_\nu\left(\sum_{j=1}^r \tanh(t_j)\, e_j\right) = \prod_{j=1}^r \left[\frac{1-\tanh^2(t_j)}{1+\tanh^2(t_j)}\right]^\nu = \prod_{j=1}^r \left(\frac{1-x_j}{1+x_j}\right)^\nu.$$

Proof: By Lemma 3.1 we have $K(\zeta, s_\zeta(0))^{-1/2} = \Delta(e - \{\zeta\, e^*\, s_\zeta(0)\})^\nu$ for $\zeta \in D \cap X$. Applying geodesic reflection, it follows that for $\xi := \sum_{j=1}^r \tanh(t_j)\, e_j \in D \cap X$ we have

$$s_\xi(0) = \sum_{j=1}^r \tanh(2t_j)\, e_j.$$

By orthogonality of $\{e_j\}$, we obtain

$$\mathfrak{b}_\nu(\sum_{j=1}^r \tanh(t_j)\, e_j) = \Delta\left(e - \{(\sum_{j=1}^r \tanh(t_j)\, e_j)\, e^*\, (\sum_{k=1}^r \tanh(2t_k)\, e_k)\}\right)^\nu =$$

$$\prod_{j=1}^r (1 - \tanh(t_j)\ \tanh(2t_j))^\nu = \prod_{j=1}^r \left(1 - 2\frac{\tanh^2(t_j)}{1+\tanh^2(t_j)}\right)^\nu = \prod_{j=1}^r \left(\frac{1-\tanh^2(t_j)}{1+\tanh^2(t_j)}\right)^\nu.$$

∎

Proposition 3.2 *The normalizing constant for the Weyl calculus at parameter ν is*

$$c_\nu = 2^{-\nu}\ {}_2F_1\left(\begin{matrix} \nu - \frac{n_Y + n_V/2}{r} & \nu \\ \nu + \frac{n_X - n_Y}{2r} & \end{matrix}\right)(\frac{1}{2}, \ldots, \frac{1}{2}).$$

Proof: By definition, c_ν is chosen such that $\omega_\nu I = 1$. Since the definition of ω_ν applies directly only to the kernel vectors we use the dual condition $\omega_\nu^* 1 = I$. By (3.10) we have

$$1 = (\omega_\nu^* 1)(0) = \int_D d\mu_0(\zeta)(\omega_\nu K_0)(\zeta) = c_\nu^{-1} \int_D d\mu_0(\zeta)\ \mathfrak{b}_\nu(\zeta)$$

and hence

$$c_\nu = \int_D d\mu_0(\zeta)\ \mathfrak{b}_\nu(\zeta).$$

Applying Proposition 3.1 and Lemma 3.2 yields, putting $z_j = 1 - x_j$,

$$\frac{\Gamma_\Omega\left(\nu - \frac{n_Y + n_V/2}{r}\right)\ \Gamma_\Omega\left(\frac{n}{2r}\right)}{\Gamma_\Omega\left(\nu + \frac{n_X - n_Y}{2r}\right)} \int_D d\mu_0(\zeta)\ \mathfrak{b}_\nu(\zeta) = c_\Omega\ 2^{n_X - n_Y} \int_G dg\ \mathfrak{b}_\nu(g(0)) =$$

$$c_\Omega \int_{[0,1]^r} \prod_{i<j} |z_i - z_j|^a \cdot \prod_j dz_j \cdot z_j^{-(n_X + n_Y + n_V/2)/r} (1 - z_j)^{(n_Y - n_X + n_V)/2r} \left(\frac{z_j}{2 - z_j}\right)^\nu =$$

$$2^{-\nu} c_\Omega \int_{[0,1]^r} \prod_{i<j} |z_i - z_j|^a \cdot \prod_j dz_j\ z_j^{\nu-(n_X + n_Y + n_V/2)/r} (1 - z_j)^{(n_Y - n_X + n_V)/2r} (1 - \frac{z_j}{2})^{-\nu}$$

$$= 2^{-\nu} \frac{\Gamma_\Omega\left(\nu - \frac{n_Y + n_V/2}{r}\right)\ \Gamma_\Omega\left(\frac{n}{2r}\right)}{\Gamma_\Omega\ \left(\nu + \frac{n_X - n_Y}{2r}\right)}\ {}_2F_1\left(\begin{matrix} \nu - \frac{n_Y + n_V/2}{r} & \nu \\ \nu + \frac{n_X - n_Y}{2r} & \end{matrix}\right)(\frac{1}{2}, \ldots, \frac{1}{2}).$$

∎

In the sequel the *Laplace-Beltrami operator* Δ on D (not to be confused with the Jordan determinant) will play a crucial role.

Proposition 3.3 *Expressed in the coordinates*

$$y_j := -\sinh^2(t_j) \qquad (1 \leq j \leq r) \tag{3.11}$$

the Laplace-Beltrami operator Δ *on* D *has the* K*-radial part* $\tilde{\Delta}$ *given by*

$$-\frac{1}{4}\tilde{\Delta} = \sum_{j=1}^r \left\{ y_j (1 - y_j) \frac{\partial}{\partial y_j} + a \sum_{i \neq j} \frac{y_j (1 - y_j)}{y_i - y_j} + \frac{1 + c + b}{2} - (1 + c + \frac{b}{2})\, y_j \right\} \frac{\partial}{\partial y_j}.$$

Proof: For any Riemannian symmetric space G/K, the K-radial part of Δ realized on $A_+ := exp\,(\mathfrak{a}_+)(o)$ has the form [6, Proposition II. 3.9]

$$\tilde{\Delta} = \Delta_A + \sum_{\alpha \in \Sigma_+} m_\alpha \coth(\alpha)\, \alpha_\#$$

where Δ_A is the (euclidean) Laplacian on A and $\alpha_\# \in \mathfrak{a}$ is determined by $\langle \alpha_\# | H \rangle = \alpha(H)$ for all $H \in \mathfrak{a}$. Specializing to the root decomposition (3.2)-(3.5) we obtain

$$\tilde{\Delta} = \sum_j \frac{\partial^2}{\partial t_j^2} + \sum_j \left(c\ \coth(2t_j)\, 2 \frac{\partial}{\partial t_j} + b\ \coth(t_j) \frac{\partial}{\partial t_j} \right) +$$

$$a \sum_{i<j} \coth(t_j - t_i) \left(\frac{\partial}{\partial t_j} - \frac{\partial}{\partial t_i} \right) + \coth(t_j + t_i) \left(\frac{\partial}{\partial t_j} + \frac{\partial}{\partial t_i} \right) =$$

$$\sum_j \left[\frac{\partial}{\partial t_j} + 2c\ \coth(2t_j) + b \coth(t_j) + a \sum_{i \neq j} (\coth(t_j - t_i) + \coth(t_j + t_i) \right] \frac{\partial}{\partial t_j}$$

since $-\coth(t_j - t_i) = \coth(t_i - t_j)$. Since $\frac{dy_j}{dt_j} = -\sinh^2(t_j)$ we have

$$\coth(2t_j)\,\frac{\partial}{\partial t_j} = -\coth(2t_j)\,\sinh(2t_j)\,\frac{\partial}{\partial y_j} = (2y_j - 1)\,\frac{\partial}{\partial y_j},$$

$$\coth(t_j)\,\frac{\partial}{\partial t_j} = -\coth(t_j)\,\sinh(2t_j)\,\frac{\partial}{\partial y_j} = 2(y_j - 1)\,\frac{\partial}{\partial y_j},$$

$$(\coth(t_j - t_i) + \coth(t_j + t_i))\,\frac{\partial}{\partial t_j} = \frac{\sinh(2t_j)\,\sinh(2t_j)}{\sinh(t_j - t_i)\,\sinh(t_j + t_i)}\,\frac{\partial}{\partial y_j},$$

$$\frac{4\cosh^2(t_j)\,\sinh^2(t_j)}{\sinh^2(t_j)\,\cosh^2(t_i) - \cosh^2(t_j)\,\sinh^2(t_i)}\,\frac{\partial}{\partial y_j} = -\frac{4y_j(1 - y_j)}{y_i - y_j}\,\frac{\partial}{\partial y_j}$$

and

$$\left(\frac{\partial}{\partial t_j}\right)^2 = \left(\frac{dy_j}{dt_j}\right)^2 \left(\frac{\partial}{\partial y_j}\right)^2 + \frac{d^2 y_j}{dt_j^2}\,\frac{\partial}{\partial y_j} = -4y_j(1 - y_j)\left(\frac{\partial}{\partial y_j}\right)^2 + (4y_j - 2)\,\frac{\partial}{\partial y_j}.$$

Substituting into the previous expression for $\tilde{\Delta}$ yields the assertion. ∎

4 The Weyl transform for rank 1 domains

The Weyl transform is harder to analyze than the Toeplitz-Berezin transform. Up to now only the simplest case of the unit disk has been treated in detail [10]. In this section we analyze the Weyl transform for an important class of higher-dimensional symmetric domains, namely those of rank 1. This includes the unit ball in $\mathbb{C}^n$, and our main result is new even in this special case.

Let $\mathbb{K}$ denote one of the real division algebras $\mathbb{R}, \mathbb{C}, \mathbb{H}$ or $\mathbb{O}$. Then $Z := \mathbb{K}^m$ becomes a real Jordan triple for $m \geq 1$, with $m = 2$ in case $\mathbb{K} = \mathbb{O}$. The unit balls

$$D := \{(x_1, \ldots, x_m) \in \mathbb{K}^m : \sum_{i=1}^m x_i\, x_i^* < 1\}$$

are precisely the real bounded symmetric domains of rank 1. Here $x \mapsto x^*$ is the natural involution on $\mathbb{K}$. We put

$$a := dim_{\mathbb{R}}\,\mathbb{K}, \quad n := dim_{\mathbb{R}}\,D = am.$$

The Peirce decomposition with respect to the tripotent $e := (1, 0, \ldots, 0) \in \mathbb{K}^m$ has the form

$$\mathbb{K}^m = X \oplus Y \oplus V$$

where

$$X = \mathbb{R} \times \{0\}^{m-1}, \quad Y = \mathbb{R}^{\perp} \times \{0\}^{m-1}, \quad V = \{0\} \times \mathbb{K}^{m-1}$$

and $\mathbb{R}^{\perp} := \{x \in \mathbb{K} : \ x^* = -x\}$. The vector field

$$M := (e - \{z\, e^*\, z\})\, \frac{\partial}{\partial z}$$

generating $\mathfrak{a}$ gives rise to the positive roots

$$2M^{\#}, \quad \text{multiplicity } \ c = a - 1 \tag{4.1}$$

$$M^{\#}, \quad \text{multiplicity } \ b = n - a. \tag{4.2}$$

For the half-sum ρ of positive roots we obtain

$$2\rho = (a-1)\, 2M^{\#} + (n-a)\, M^{\#} = \left(-1 + \frac{n+a}{2}\right) 2M^{\#}.$$

In terms of the coordinate $y := -\sinh^2(t) \in (-\infty, 0]$ the Laplace-Beltrami operator Δ on D has the radial part (Proposition 3.2)

$$-\frac{\tilde{\Delta}}{4} = y\,(1-y) \left(\frac{d}{dy}\right)^2 + \left(\frac{n}{2} - \frac{n+a}{2}\, y\right) \frac{d}{dy} \tag{4.3}$$

when acting on K-invariant functions. Since (4.3) corresponds to the hypergeometric equation it follows that the hypergeometric series

$${}_2F_1\left(\begin{matrix} \alpha & \beta \\ & \gamma \end{matrix}\right)(y) := \sum_{k \geq 0} \frac{(\alpha)_k\ (\beta)_k}{(\gamma)_k}\ \frac{y^k}{k!},$$

yields the eigenfunctions (regular at $y = 0$):

$$-\frac{\tilde{\Delta}}{4}\ {}_2F_1\left(\begin{matrix} \alpha & \beta \\ & n/2 \end{matrix}\right) = \alpha\beta\ {}_2F_1\left(\begin{matrix} \alpha & \beta \\ & n/2 \end{matrix}\right)$$

where

$$\alpha + \beta = \frac{n+a}{2} - 1 = 2\rho. \tag{4.4}$$

Using the coordinate $x := \tanh^2(t)$ we obtain the *spherical function*

$$\phi_\lambda(x^{1/2} e) = \phi_\lambda(\tanh(t)\, e) = {}_2F_1\left(\begin{matrix} \rho+\lambda & \rho-\lambda \\ & n/2 \end{matrix}\right)(y) \tag{4.5}$$

$$= {}_2F_1\left(\begin{matrix} \rho+\lambda & \rho-\lambda \\ & n/2 \end{matrix}\right)\left(\frac{x}{x-1}\right) = (1-x)^{\rho+\lambda}\ {}_2F_1\left(\begin{matrix} \rho+\lambda & \lambda-\rho+\frac{n}{2} \\ & n/2 \end{matrix}\right)(x)$$

with eigenvalue

$$-\frac{\tilde{\Delta}}{4}\,\phi_\lambda = (\rho^2-\lambda^2)\,\phi_\lambda. \tag{4.6}$$

Note that $\rho = 1/2$ for the unit disk. We are interested in the Weyl calculus acting on H_ν, where $D_{\mathbb{C}}$ is the complexification of D. By [7, 12.18] Z has the complexification $Z_{\mathbb{C}}$, of rank $r_{\mathbb{C}}$ and the half sum of positive roots ρ, given by the following table

Z	$Z_{\mathbb{C}}$	$r_{\mathbb{C}}$	ρ
$\mathbb{R}^m$	$\mathbb{C}^m$	1	$\frac{m-1}{4}$
$\mathbb{C}^m$	$\mathbb{C}^m \times \overline{\mathbb{C}}^m$	2	$\frac{m}{2}$
$\mathbb{H}^m$	$\mathbb{C}^{2\times 2m}$	2	$m+\frac{1}{2}$
$\mathbb{O}^2$	$\mathbb{C}^{16}_V$	2	$\frac{11}{2}$.

Here $\mathbb{C}^{16}_V$ denotes the 16-dimensional exceptional Jordan triple not of tube type. For rank 1 domains we have $X = \mathbb{R}$ and $\Omega = (0,\infty)$. Hence Γ_Ω is the usual Γ-function and $c_\Omega = 1$. Since $n_X = 1,\ n_Y = a-1,\ n_V = a(m-1) = n-a,\ 2\rho+1 = (n+a)/2$, Proposition 3.1 shows

$$\int\limits_D d\mu_0(\zeta)\, f(\zeta) = 2^{2-a}\,\frac{\Gamma(\nu+1-\frac{a}{2})}{\Gamma(\nu-2\rho)\ \Gamma(\frac{n}{2})}\int\limits_G dg\ f(g(0))$$

where, for K-invariant functions, we have

$$2^{2-a}\int\limits_G dg\ f(g(0)) = \int\limits_0^1 dx\ f(x^{1/2}e)\cdot(1-x)^{-2\rho-1}\,x^{n/2-1}. \tag{4.7}$$

Specializing Proposition 3.2 yields the normalizing constant

$$c_\nu = 2^{-\nu}\ {}_2F_1\begin{pmatrix} \nu-2\rho & \nu \\ \multicolumn{2}{c}{\nu+1-\frac{a}{2}}\end{pmatrix}(\frac{1}{2}) \tag{4.8}$$

since $n_Y + n_V/2 = a-1+\frac{n-a}{2} = 2\rho$. The main result of this paper, leading to the eigenvalues of the Weyl transform, is the following:

Theorem 4.1 *Let D be a real bounded symmetric domain of rank 1 and dimension n. Then the Weyl calculus ω_ν satisfies*

$$(\omega_\nu^* e_\lambda)(0) = \frac{\Gamma(\nu-\rho+\lambda)\ \Gamma(\nu-\rho-\lambda)}{\Gamma(\nu-2\rho)\ \Gamma(\nu)}\ \frac{{}_2F_1\begin{pmatrix} \nu-\rho+\lambda & \nu-\rho-\lambda \\ \multicolumn{2}{c}{\nu+\frac{n}{2}-2\rho}\end{pmatrix}(\frac{1}{2})}{{}_2F_1\begin{pmatrix} \nu-2\rho & \nu \\ \multicolumn{2}{c}{\nu+\frac{n}{2}-2\rho}\end{pmatrix}(\frac{1}{2})}$$

$$= \frac{\Gamma(\nu-\rho+\lambda)\ \Gamma(\nu-\rho-\lambda)}{\Gamma(\nu-2\rho)\ \Gamma(\nu)} \ \frac{{}_2F_1\left(\begin{array}{cc} \frac{n}{2}-\rho+\lambda & \frac{n}{2}-\rho-\lambda \\ \multicolumn{2}{c}{\nu+\frac{n}{2}-2\rho} \end{array}\right)(\frac{1}{2})}{{}_2F_1\left(\begin{array}{cc} \frac{n}{2}-2\rho & \frac{n}{2} \\ \multicolumn{2}{c}{\nu+\frac{n}{2}-2\rho} \end{array}\right)(\frac{1}{2})}.$$

Proof: Using (4.7), (4.5) and Lemma 3.2, and applying [8, §20.2, p. 399, (6)] to the admissible parameters $\alpha = \rho+\lambda,\ \ \beta = \lambda-\rho+\frac{n}{2},\ \ \gamma = \frac{n}{2},\ \ \rho = \nu+\lambda-\rho,\ \ \sigma = \nu$ and $z = -1$, we obtain

$$2^{2-a}\int_G dg\ \phi_\lambda(g(0))\ \mathfrak{b}_\nu(g(0)) = \int_0^1 dx\, \phi_\lambda(x^{1/2}\, e)\ \mathfrak{b}_\nu(x^{1/2}\, e)(1-x)^{-2\rho-1}x^{n/2-1} =$$

$$\int_0^1 dx\,(1-x)^{\rho+\lambda}\ \ {}_2F_1\left(\begin{array}{cc} \rho+\lambda & \rho-\lambda+\frac{n}{2} \\ \multicolumn{2}{c}{n/2} \end{array}\right)(x)\left(\frac{1-x}{1+x}\right)^{\nu}(1-x)^{-2\rho-1}\ x^{n/2-1} =$$

$$\int_0^1 dx\ \ {}_2F_1\left(\begin{array}{cc} \rho+\lambda & \lambda-\rho+\frac{n}{2} \\ \multicolumn{2}{c}{n/2} \end{array}\right)(x)\,(1-x)^{\nu+\lambda-\rho-1}\ (1+x)^{-\nu}\ x^{n/2-1} =$$

$$\frac{\Gamma(\frac{n}{2})\,\Gamma(\nu-\rho+\lambda)\,\Gamma(\nu-\rho-\lambda)}{\Gamma(\nu+\frac{n}{2}-2\rho)\,\Gamma(\nu)}\ 2^{-\nu}\ \ {}_3F_2\left(\begin{array}{ccc} \nu-\rho+\lambda & \nu-\rho-\lambda & \nu \\ & \nu+\frac{n}{2}-2\rho & \nu \end{array}\right)(\frac{1}{2}),$$

with the ${}_3F_2$-function reducing to ${}_2F_1$. This implies the assertion, since $(\omega_\nu^*\, e_\lambda)(0)$ is a multiple of this integral normalized at $\lambda = \rho$. ∎

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