$$\text{matrix-valued row}\quad \mathsf{P} = \left({}_{\mu}\mathsf{P}_{i}^{\,j}\right) = {}_{1}\mathsf{P}_{\cdot}^{\,\cdot}\;\Big|\;\dots\;\Big|\;{}_{n}\mathsf{P}_{\cdot}^{\,\cdot} \;=\; \begin{array}{ccc|c|ccc} {}_{1}\mathsf{P}_{1}^{\,1} & \dots & {}_{1}\mathsf{P}_{1}^{\,n} & \dots & {}_{n}\mathsf{P}_{1}^{\,1} & \dots & {}_{n}\mathsf{P}_{1}^{\,n} \\ \hline & \vdots & & \ddots & & \vdots & \\ \hline {}_{1}\mathsf{P}_{n}^{\,1} & \dots & {}_{1}\mathsf{P}_{n}^{\,n} & \dots & {}_{n}\mathsf{P}_{n}^{\,1} & \dots & {}_{n}\mathsf{P}_{n}^{\,n} \end{array}$$

$$\text{global transpose}\quad \mathsf{P}^{\sharp} = \begin{array}{ccc|c|ccc} {}_{1}\mathsf{P}_{1}^{\,1} & \dots & {}_{1}\mathsf{P}_{1}^{\,n} & \dots & {}_{1}\mathsf{P}_{n}^{\,1} & \dots & {}_{1}\mathsf{P}_{n}^{\,n} \\ \hline & \vdots & & \ddots & & \vdots & \\ \hline {}_{n}\mathsf{P}_{1}^{\,1} & \dots & {}_{n}\mathsf{P}_{1}^{\,n} & \dots & {}_{n}\mathsf{P}_{n}^{\,1} & \dots & {}_{n}\mathsf{P}_{n}^{\,n} \end{array}$$

$${}_{\mu}\mathsf{P}_{i}^{T\,j} = {}_{i}\mathsf{P}_{\mu}^{\,j}$$

$$\text{local transpose}\quad \mathsf{P}^{t} = {}_{1}\mathsf{P}_{\cdot}^{t\,\cdot}\;\Big|\;\dots\;\Big|\;{}_{n}\mathsf{P}_{\cdot}^{t\,\cdot}$$

$${}_{\mu}\mathsf{P}_{i}^{t\,j} = {}_{\mu}\mathsf{P}_{j}^{\,i}$$

$$\text{metric}\quad g_{\cdot\cdot} = g = \begin{matrix} g_{11} & \dots & g_{1n} \\ \vdots & \ddots & \vdots \\ g_{n1} & \dots & g_{nn} \end{matrix}$$

$$g_i{}^j = g_{1j}$$

$$\text{dual metric}\quad g^{\cdot\cdot} = g^{-} = \begin{matrix} g^{11} & \dots & g^{1n} \\ \vdots & \ddots & \vdots \\ g^{n1} & \dots & g^{nn} \end{matrix}$$

$$g_i{}^j = g^{ij}$$

$$\text{Christoffel}\quad {}_{\cdot}\Gamma_{\cdot}{}^{\cdot} = {}_1\Gamma_{\cdot}{}^{\cdot} \,\Big|\, \dots \,\Big|\, {}_n\Gamma_{\cdot}{}^{\cdot} = \begin{array}{ccc|c|ccc} {}_1\Gamma_1{}^1 & \dots & {}_1\Gamma_1{}^n & \dots & {}_n\Gamma_1{}^1 & \dots & {}_n\Gamma_1{}^n \\ \hline & \vdots & & \ddots & & \vdots & \\ \hline {}_1\Gamma_n{}^1 & \dots & {}_1\Gamma_n{}^n & \dots & {}_n\Gamma_n{}^1 & \dots & {}_n\Gamma_n{}^n \end{array}$$

$$e_{\cdot} = \begin{bmatrix} {}_1e \\ \vdots \\ {}_ne \end{bmatrix} \text{ column}$$

$$e^{\sharp} = {}_1e \,\Big|\, \dots \,\Big|\, {}_ne \quad \text{row}$$

$$e^{\sharp} \boxtimes g = {}_1e\,g \,\Big|\, \dots \,\Big|\, {}_ne\,g = \begin{array}{ccc|c|ccc} {}_1e\,g_{11} & \dots & {}_1e\,g_{1n} & \dots & {}_ne\,g_{11} & \dots & {}_ne\,g_{1n} \\ \hline & \vdots & & \ddots & & \vdots & \\ \hline {}_1e\,g_{n1} & \dots & {}_1e\,g_{nn} & \dots & {}_ne\,g_{n1} & \dots & {}_ne\,g_{nn} \end{array} \text{ matrix-valued row}$$

$$_{\mu}e^{\sharp} \boxtimes g_i{}^j = {}_{\mu}e\,g_{ij} = {}_{\mu}e\,g_i{}^j$$

$$\text{global transpose}\quad e^{\sharp} \boxtimes^{T} g = \begin{array}{ccc|c|ccc} {}_1e\,g_{11} & \dots & {}_1e\,g_{1n} & \dots & {}_1e\,g_{n1} & \dots & {}_1e\,g_{nn} \\ \hline & \vdots & & \ddots & & \vdots & \\ \hline {}_ne\,g_{11} & \dots & {}_ne\,g_{1n} & \dots & {}_ne\,g_{n1} & \dots & {}_ne\,g_{nn} \end{array}$$

$$_{\mu}e^{\sharp} \boxtimes^{T} g_i{}^j = {}_ie^{\sharp} \boxtimes g_{\mu}{}^j = {}_ie\,g_{\mu j} = {}_ie\,g_{\mu}{}^j$$

$$\text{local transpose}\quad e^{\sharp} \boxtimes^{Tt} g = \begin{array}{ccc|c|ccc} {}_1e\,g_{11} & \dots & {}_ne\,g_{11} & \dots & {}_1e\,g_{n1} & \dots & {}_ne\,g_{n1} \\ \hline & \vdots & & \ddots & & \vdots & \\ \hline {}_1e\,g_{1n} & \dots & {}_ne\,g_{1n} & \dots & {}_1e\,g_{nn} & \dots & {}_ne\,g_{nn} \end{array}$$

$$ {}_{\mu}\overset{\sharp}{\underset{-}{\iota}}\overset{Tt}{\bowtie}_{i}\,\eta^{j} = {}_{\mu}\overset{\sharp}{\underset{-}{\iota}}\overset{T}{\bowtie}_{j}\,\eta^{i} = {}_{j}\overset{\sharp}{\underset{-}{\iota}}\bowtie_{\mu}\,\eta^{i} = {}_{j}\underset{-}{\iota}\,\eta_{\mu i} = {}_{j}\underset{-}{\iota}\,\eta_{\mu}^{\ i} $$

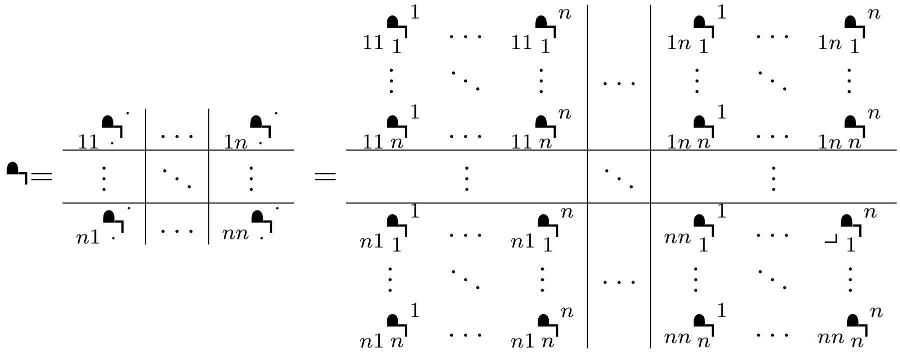
$$ 0/ \qquad 2\,\bar{\eta}\eta = \underbrace{\overset{\sharp}{\underset{-}{\iota}}\overset{T}{\bowtie}\,\eta + \overset{\sharp}{\underset{-}{\iota}}\overset{Tt}{\bowtie}\,\eta - \overset{\sharp}{\underset{-}{\iota}}\bowtie\,\eta}\,\overset{|}{\eta} $$

$$ {}_{\mu}\mathrm{RHS}_{i}^{k} = {}_{\mu}\underbrace{\overset{\sharp}{\underset{-}{\iota}}\overset{T}{\bowtie}\,\eta + \overset{\sharp}{\underset{-}{\iota}}\overset{Tt}{\bowtie}\,\eta - \overset{\sharp}{\underset{-}{\iota}}\bowtie\,\eta}_{i}{}^{j}\,\overset{|}{\eta}_{j}^{\ k} = \left({}_{\mu}\overset{\sharp}{\underset{-}{\iota}}\overset{T}{\bowtie}_{i}\,\eta^{j} + {}_{\mu}\overset{\sharp}{\underset{-}{\iota}}\overset{Tt}{\bowtie}_{i}\,\eta^{j} - {}_{\mu}\overset{\sharp}{\underset{-}{\iota}}\bowtie_{i}\,\eta^{j} \right) \overset{|}{\eta}_{j}^{\ k} $$

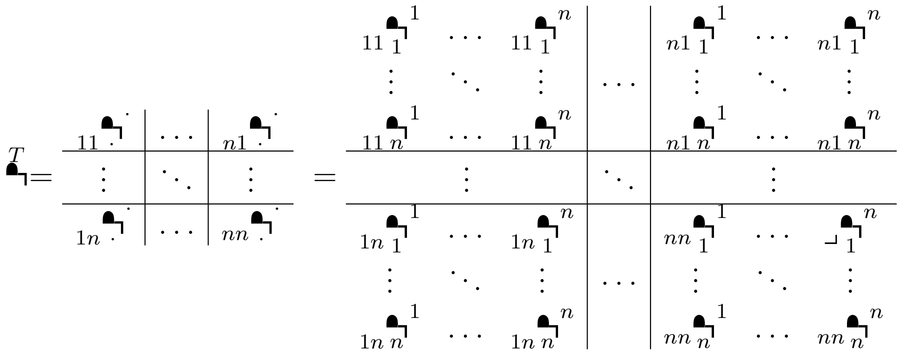
$$ = \left({}_{j}\underset{-}{\iota}\,\eta_{\mu}^{\ i} + {}_{i}\underset{-}{\iota}\,\eta_{\mu}^{\ j} - {}_{\mu}\underset{-}{\iota}\,\eta_{i}^{\ j} \right) \overset{|}{\eta}_{j}^{\ k} = \left({}_{j}\underset{-}{\iota}\,\eta_{\mu i} + {}_{i}\underset{-}{\iota}\,\eta_{\mu j} - {}_{\mu}\underset{-}{\iota}\,\eta_{ij} \right) \eta^{jk} = {}_{\mu}\bar{\eta}_{i}\,\eta^{k} $$

$$ 0/ \qquad 2\,\underbrace{\bar{\eta}\;\eta}\;\eta = \overset{\sharp}{\underset{-}{\iota}}\overset{T}{\bowtie}\,\eta + \overset{\sharp}{\underset{-}{\iota}}\overset{Tt}{\bowtie}\,\eta - \overset{\sharp}{\underset{-}{\iota}}\bowtie\,\eta $$

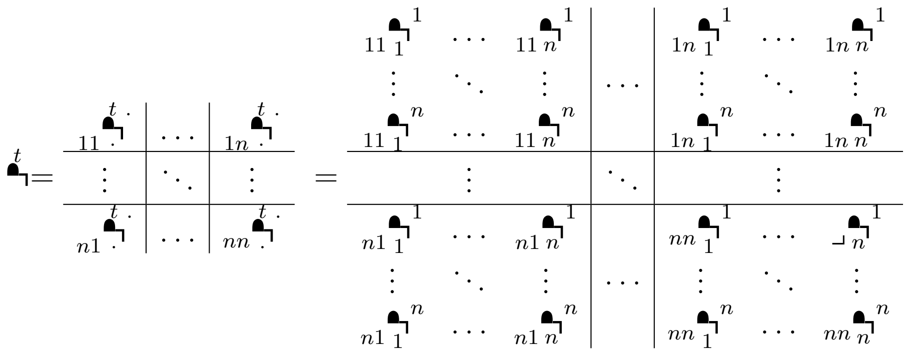
matrix-valued matrix



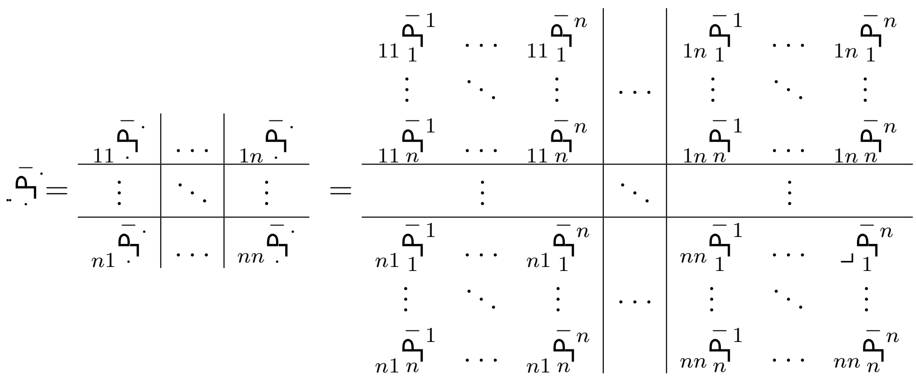
global transpose



local transpose



differential



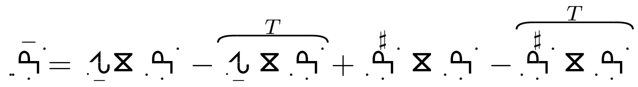
$$\partial\boxtimes\daleth = \begin{bmatrix} {}_{1-}\wr \\ \vdots \\ {}_{n-}\wr \end{bmatrix} \left[\, {}_1\daleth^{\cdot} \;\middle|\; \cdots \;\middle|\; {}_n\daleth^{\cdot} \,\right] = \begin{array}{c|c|c} {}_{1-}\wr_1\daleth^{\cdot} & \cdots & {}_{1-}\wr_n\daleth^{\cdot} \\ \hline \vdots & \ddots & \vdots \\ \hline {}_{n-}\wr_1\daleth^{\cdot} & \cdots & {}_{n-}\wr_n\daleth^{\cdot} \end{array} = \begin{array}{ccc|c|ccc} {}_{1-}\wr_1\daleth_1^1 & \cdots & {}_{1-}\wr_1\daleth_1^n & & {}_{1-}\wr_n\daleth_1^1 & \cdots & {}_{1-}\wr_n\daleth_1^n \\ \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots \\ {}_{1-}\wr_1\daleth_n^1 & \cdots & {}_{1-}\wr_1\daleth_n^n & & {}_{1-}\wr_n\daleth_n^1 & \cdots & {}_{1-}\wr_n\daleth_n^n \\ \hline & \vdots & & \ddots & & \vdots & \\ \hline {}_{n-}\wr_1\daleth_1^1 & \cdots & {}_{n-}\wr_1\daleth_1^n & & {}_{n-}\wr_n\daleth_1^1 & \cdots & {}_{n-}\wr_n\daleth_1^n \\ \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots \\ {}_{n-}\wr_1\daleth_n^1 & \cdots & {}_{n-}\wr_1\daleth_n^n & & {}_{n-}\wr_n\daleth_n^1 & \cdots & {}_{n-}\wr_n\daleth_n^n \end{array} \text{ matrix-valu}$$

$$\text{global transpose } \overbrace{\partial\boxtimes\daleth}^{T} = \begin{array}{ccc|c|ccc} {}_{1-}\wr_1\daleth_1^1 & \cdots & {}_{1-}\wr_1\daleth_1^n & & {}_{n-}\wr_1\daleth_1^1 & \cdots & {}_{n-}\wr_1\daleth_1^n \\ \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots \\ {}_{1-}\wr_1\daleth_n^1 & \cdots & {}_{1-}\wr_1\daleth_n^n & & {}_{n-}\wr_1\daleth_n^1 & \cdots & {}_{n-}\wr_1\daleth_n^n \\ \hline & \vdots & & \ddots & & \vdots & \\ \hline {}_{1-}\wr_n\daleth_1^1 & \cdots & {}_{1-}\wr_n\daleth_1^n & & {}_{n-}\wr_n\daleth_1^1 & \cdots & {}_{n-}\wr_n\daleth_1^n \\ \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots \\ {}_{1-}\wr_n\daleth_n^1 & \cdots & {}_{1-}\wr_n\daleth_n^n & & {}_{n-}\wr_n\daleth_n^1 & \cdots & {}_{n-}\wr_n\daleth_n^n \end{array}$$

$$\daleth^{\sharp} = \begin{bmatrix} {}_1\daleth^{\cdot} \\ \vdots \\ {}_n\daleth^{\cdot} \end{bmatrix} \text{ matrix-valued column}$$

$$\daleth^{\sharp}\boxtimes\daleth = \begin{bmatrix} {}_1\daleth^{\cdot} \\ \vdots \\ {}_n\daleth^{\cdot} \end{bmatrix} \left[\, {}_1\daleth^{\cdot} \;\middle|\; \cdots \;\middle|\; {}_n\daleth^{\cdot} \,\right] = \begin{array}{c|c|c} {}_1\daleth^{\cdot}\,{}_1\daleth^{\cdot} & \cdots & {}_1\daleth^{\cdot}\,{}_n\daleth^{\cdot} \\ \hline \vdots & \ddots & \vdots \\ \hline {}_n\daleth^{\cdot}\,{}_1\daleth^{\cdot} & \cdots & {}_n\daleth^{\cdot}\,{}_n\daleth^{\cdot} \end{array} = \begin{array}{ccc|c|ccc} {}_1\daleth_1^i\,{}_1\daleth_i^1 & \cdots & {}_1\daleth_1^i\,{}_1\daleth_i^n & & {}_1\daleth_1^i\,{}_n\daleth_i^1 & \cdots & {}_1\daleth_1^i\,{}_n\daleth_i^n \\ \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots \\ {}_1\daleth_n^i\,{}_1\daleth_i^1 & \cdots & {}_1\daleth_n^i\,{}_1\daleth_i^n & & {}_1\daleth_n^i\,{}_n\daleth_i^1 & \cdots & {}_1\daleth_n^i\,{}_n\daleth_i^n \\ \hline & \vdots & & \ddots & & \vdots & \\ \hline {}_n\daleth_1^i\,{}_1\daleth_i^1 & \cdots & {}_n\daleth_1^i\,{}_1\daleth_i^n & & {}_n\daleth_1^i\,{}_n\daleth_i^1 & \cdots & {}_n\daleth_1^i\,{}_n\daleth_i^n \\ \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots \\ {}_n\daleth_n^i\,{}_1\daleth_i^1 & \cdots & {}_n\daleth_n^i\,{}_1\daleth_i^n & & {}_n\daleth_n^i\,{}_n\daleth_i^1 & \cdots & {}_n\daleth_n^i\,{}_n\daleth_i^n \end{array} \text{ matri}$$

$$\text{global transpose } \overbrace{\daleth^{\sharp}\boxtimes\daleth}^{T} = \begin{array}{ccc|c|ccc} {}_1\daleth_1^i\,{}_1\daleth_i^1 & \cdots & {}_1\daleth_1^i\,{}_1\daleth_i^n & & {}_n\daleth_1^i\,{}_1\daleth_i^1 & \cdots & {}_n\daleth_1^i\,{}_1\daleth_i^n \\ \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots \\ {}_1\daleth_n^i\,{}_1\daleth_i^1 & \cdots & {}_1\daleth_n^i\,{}_1\daleth_i^n & & {}_n\daleth_n^i\,{}_1\daleth_i^1 & \cdots & {}_n\daleth_n^i\,{}_1\daleth_i^n \\ \hline & \vdots & & \ddots & & \vdots & \\ \hline {}_1\daleth_1^i\,{}_n\daleth_i^1 & \cdots & {}_1\daleth_1^i\,{}_n\daleth_i^n & & {}_n\daleth_1^i\,{}_n\daleth_i^1 & \cdots & {}_n\daleth_1^i\,{}_n\daleth_i^n \\ \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots \\ {}_1\daleth_n^i\,{}_n\daleth_i^1 & \cdots & {}_1\daleth_n^i\,{}_n\daleth_i^n & & {}_n\daleth_n^i\,{}_n\daleth_i^1 & \cdots & {}_n\daleth_n^i\,{}_n\daleth_i^n \end{array} \text{ matrix-valued matrix}$$

1/ 

2/