Regularity of boundary integral equations in Besov-type spaces based on wavelet expansions

Markus Weimar
Philipps-University Marburg

based on joint work with Stephan Dahlke (Marburg)

A5@FoCM 2014 conference
Montevideo, December 2014

Research supported by Deutsche Forschungsgemeinschaft DFG (DA 360/19-1)
Outline

Motivation
  Boundary integral equations: double layer

Manifolds, wavelets, and Sobolev spaces
  Patchwise smooth manifolds
  Wavelets
  (Weighted) Sobolev spaces

New Besov-type spaces
  Definition
  Properties
  A non-standard embedding

Besov regularity for the double layer equation

Final remarks and references
Indirect methods for PDE’s in $\Omega$ (or $\Omega^c$) like

$$\Delta U = 0,$$

$$U = g \quad \text{on} \quad \Gamma := \partial \Omega,$$

naturally lead to boundary integral equations such as

$$S_{DL}(v) := \left( \frac{1}{2} \text{Id} - K \right)(v) = g \quad \text{on} \quad \Gamma. \quad (1)$$

Therein $K$ denotes the harmonic double layer

$$v \mapsto K(v) := \frac{1}{4\pi} \int_{\partial \Omega} v(x) \frac{\partial}{\partial \eta(x)} \frac{1}{|x - \cdot|_2} \, dS(x).$$
**Indirect methods** for PDE’s in $\Omega$ (or $\Omega^c$) like

$$\Delta U = 0,$$

$$U = g \quad \text{on} \quad \Gamma := \partial \Omega,$$

naturally lead to *boundary integral equations* such as

$$S_{DL}(v) := \left( \frac{1}{2} \text{Id} - K \right)(v) = g \quad \text{on} \quad \Gamma. \quad (1)$$

Therein $K$ denotes the *harmonic double layer*

$$v \mapsto K(v) := \frac{1}{4\pi} \int_{\partial \Omega} v(x) \frac{\partial}{\partial \eta(x)} \frac{1}{|x - \cdot|_2} \, dS(x).$$

Advantages of such reformulations:

- bounded domain of definition $\Gamma$
- reduced dimensionality
We like to solve the *double layer eq.* (1) numerically using (adaptive) wavelet Galerkin boundary element methods.

**Question**

*Does adaptivity pay off for boundary integral equations (or even more general operator equations on manifolds)?*
We like to solve the *double layer eq.* (1) numerically using (adaptive) wavelet Galerkin boundary element methods.

**Question**

*Does adaptivity pay off for boundary integral equations (or even more general operator equations on manifolds)?*

Rule of thumb for (elliptic) PDE's:

- On smooth domains there is no need for adaptivity
- On (general) Lipschitz domains adaptive schemes outperform uniform schemes

because in the second case the *Besov smoothness* of the solution is significantly higher than its *Sobolev regularity.*
Additional challenges related to operator equations on (non-smooth) manifolds:

1.) Construction of suitable wavelet bases
2.) Definition and analysis of higher-order smoothness spaces
Additional challenges related to operator equations on (non-smooth) manifolds:

1.) Construction of suitable wavelet bases
2.) Definition and analysis of higher-order smoothness spaces

Possible solution:

1.) Restriction to (practically most relevant) patchwise smooth manifolds and lifting of bases from some reference domain (e.g., the unit cube) to the patches
2.) Construction of function spaces based on the decay properties of wavelet coefficients
Patchwise smooth manifolds

Here: Lipschitz surfaces $\Gamma = \partial \Omega$ which are boundaries of bounded, simply connected, closed polyhedra $\Omega \subset \mathbb{R}^3$ with finitely-many flat, quadrilateral sides and straight edges.

- patchwise decomposition:
  $$\Gamma = \bigcup_{i=1}^{I} \Gamma_i, \quad \Gamma_i = \kappa_i([0,1]^2)$$

- local description as boundary $\partial C_n$ of tangent cones $C_n$ subordinate to vertices $\nu_n$ of $\Omega$

(This approach naturally extends to higher dimensions $d \in \mathbb{N}$)
Wavelet bases on patchwise smooth manifolds

Using the parametric *lifting* \( \kappa_i : [0, 1]^d \to \Gamma_i \) we define an (equivalent) inner product on \( L_2(\Gamma) \) by

\[
\langle u, v \rangle := \sum_{i=1}^{l} \langle u \circ \kappa_i, v \circ \kappa_i \rangle_{L_2([0,1]^d)}.
\]

Then there exist several constructions of \( \langle \cdot, \cdot \rangle \)-biorthogonal wavelet Riesz bases \( \Psi = (\Psi_\Gamma, \tilde{\Psi}_\Gamma) \) which characterize \( L_2(\Gamma) \). That is,

\[
u = \sum_{j=0}^\infty \sum_{\xi \in \nabla_j^{\Psi}} \langle u, \tilde{\psi}_{j,\xi} \rangle \psi_{j,\xi} \quad \forall u \in L_2(\Gamma)
\]

with

\[
\| u \|_{L_2(\Gamma)} \sim \left[ \sum_{j=0}^\infty \sum_{\xi \in \nabla_j^{\Psi}} \left| \langle u, \tilde{\psi}_{j,\xi} \rangle \right|^2 \right]^{1/2}.
\]
Here the wavelets $\psi_{j,\xi}^\Gamma$ and $\tilde{\psi}_{j,\xi}^\Gamma$ are indexed by

- $j$ ... level of resolution,
- $\xi$ ... point from a multiscale grid $\nabla^\psi = (\nabla_j^\psi)_{j \in \mathbb{N}_0}$ for $\Gamma$.

In addition, they satisfy a couple of attractive properties, e.g.

- normalization and local support
- interior vanishing moments of order $\tilde{D} \in \mathbb{N}$:

$$\langle \mathcal{P}, \tilde{\psi}_{j,\xi}^\Gamma \circ \kappa_i \rangle_{L_2([0,1]^d)} = 0 \quad \forall \mathcal{P} \in \Pi_{\tilde{D}}([0,1]^d)$$

and all $(j, \xi)$ such that $\text{supp} \tilde{\psi}_{j,\xi}^\Gamma \subset \Gamma_i$ for some $i$

- $H^s(\Gamma)$-norm equivalences for $-1/2 < s < \min\{3/2, s_\Gamma\}$:

$$\|u|H^s(\Gamma)\| \sim \left[ \sum_{j=0}^{\infty} 2^j s^2 \sum_{\xi \in \nabla_j^\psi} \left| \langle u, \tilde{\psi}_{j,\xi}^\Gamma \rangle \right|^2 \right]^{1/2}$$

Typical example: composite wavelet basis by Dahmen/Schneider (see also Harbrecht/Stevenson, Canuto/Tabacco/Urban)
Weighted Sobolev spaces on $\Gamma = \partial \Omega$, $\Omega \subset \mathbb{R}^3$

For $k \in \mathbb{N}$ and $0 \leq \varrho \leq k$ define

$$X^k_{\varrho}(\Gamma) := \frac{\mathcal{C}_{\text{patchwise}}(\Gamma)}{\| \cdot \|_{X^k_{\varrho}(\Gamma)}}$$

with norm $\| u \|_{X^k_{\varrho}(\Gamma)} := \sum_{n=1}^{N} \| \varphi_n u \|_{X^k_{\varrho}(\partial C_n)}$ such that

$$\| \delta^{k-\varrho}_n | \nabla^k f \|_{L^2(\Gamma)} \lesssim \| f \|_{X^k_{\varrho}(\partial C_n)}.$$

Therein

- $(\varphi_n)_{n=1}^{N}$ ... special resolution of unity on $\Gamma = \partial \Omega$
  subordinate to vertices $\nu_n$ of $\Omega$

- $\delta_n$ ... distance to interfaces

- $\nabla^k$ ... vector of $k$th order derivatives

(cf. Elschner, Maz’ya et al., Babuska, Kondratiev, . . . )
Classical Besov spaces

Besov spaces $B^\alpha_q(L^p(\mathbb{R}^d))$ . . .

- . . . essentially generalize Sobolev (Hilbert) spaces $H^s$.
- . . . depend on (at least) 3 parameters: $\alpha$, $p$, $q$.
- . . . are defined in various ways (e.g. using harmonic analysis, moduli of smoothness, interpolation, etc.).
- . . . are characterized by decay properties of expansion coefficients w.r.t. various building blocks such as atoms, quarks, or wavelets.
Classical Besov spaces

Besov spaces $B^\alpha_q(L_p(\mathbb{R}^d))$ . . .

- . . . essentially generalize Sobolev (Hilbert) spaces $H^s$.
- . . . depend on (at least) 3 parameters: $\alpha$, $p$, $q$.
- . . . are defined in various ways (e.g. using harmonic analysis, moduli of smoothness, interpolation, etc.).
- . . . are characterized by decay properties of expansion coefficients w.r.t. various building blocks such as atoms, quarks, or wavelets.

Corresponding spaces can also be defined for domains and (non-smooth) manifolds (as trace spaces or via pullbacks),

**BUT** currently there seems to exist no approach, suitable for numerical applications, to define higher-order (Besov) smoothness for functions on *patchwise smooth* manifolds!
New Besov-type spaces on $\Gamma$

**Definition (Dahlke, W. 2013 / W. 2014)**

- A tuple of real parameters $(\alpha, p, q)$ is called *admissible* if $0 < p < \infty$ and
  - $\alpha > d \cdot \max\{0, 1/p - 1/2\}$ and $0 < q \leq \infty$, or
  - $\alpha = d \cdot \max\{0, 1/p - 1/2\}$ and $0 < q \leq 2$.

- By $B_{\Psi,q}^\alpha(L_p(\Gamma))$ we denote the set of all $u \in L_2(\Gamma)$ s.t.

\[
\left\| u \right\|_{B_{\Psi,q}^\alpha(L_p(\Gamma))} := \left\| \left( \langle u, \tilde{\psi}_{j,\xi}^\Gamma \rangle \right)_{(j,\xi)} \right\|_{b_{p,q}^\alpha(\nabla \Psi)} := \left[ \sum_{j=0}^{\infty} 2^j \left( \alpha + d \left[ \frac{1}{2} - \frac{1}{p} \right] \right) q \left( \sum_{\xi \in \nabla_j} \left| \langle u, \tilde{\psi}_{j,\xi}^\Gamma \rangle \right|^p \right) \right]^{q/p} \left( \sum_{\xi \in \nabla_j} \left| \langle u, \tilde{\psi}_{j,\xi}^\Gamma \rangle \right|^p \right)^{1/q} \]

(with the usual modifications for $q = \infty$) is finite.
Hence,

- **per definition** the function $u$ belongs to the Besov-type space $B^\alpha_{\Psi,q}(L_p(\Gamma)) \iff$ its sequence of expansion coefficients $(\langle u, \tilde{\psi}_{j,\xi}^\Gamma \rangle)_{(j,\xi) \in \nabla \Psi}$ w.r.t. the wavelet basis $\Psi = (\psi^\Gamma, \tilde{\psi}^\Gamma)$ exhibits a certain rate of decay, i.e., belongs to the sequence space

$$b^{\alpha}_{p,q}(\nabla \Psi) := \left\{ a = (a_{(j,\xi)})_{(j,\xi)} \left| \left\| a \right\|_{b^{\alpha}_{p,q}(\nabla \Psi)} < \infty \right. \right\}.$$ 

- properties of the scale $B^\alpha_{\Psi,q}(L_p(\Gamma))$ can be derived from corresponding results for $b^{\alpha}_{p,q}(\nabla \Psi)$.

In principle this approach is applicable for every set $\Gamma$ which allows the construction of a wavelet basis for $L_2(\Gamma)$. This covers (un-)bounded domains in $\mathbb{R}^d$, as well as (non-)smooth manifolds with or without a boundary!
Properties of the new scale $B_{\psi,q}^{\alpha}(L_p(\Gamma))$

- always quasi-Banach spaces
  
  \begin{align*}
  \text{(Banach} \iff \min\{p, q\} \geq 1 \text{ / Hilbert} \iff p = q = 2)\end{align*}

- simplified (quasi-)norms for so-called **adaptivity scale**
  
  \begin{align*}
p = q = \tau := \left(\frac{\alpha_{\tau}}{d} + 1/2\right)^{-1}, \quad \alpha_{\tau} \geq 0\end{align*}

  \begin{align*}
  \left\| u \right\|_{B_{\psi,\tau}^{\alpha}(L_\tau(\Gamma))} & = \left[ \sum_{j=0}^{\infty} \sum_{\xi \in \nabla^j \psi} \left| \left< u, \tilde{\psi}_j^\tau, \xi \right> \right|^\tau \right]^{1/\tau} \\
  \text{and for Hilbert scale } p = q = 2 \end{align*}

  \begin{align*}
  \left\| u \right\|_{B_{\psi,2}^{\alpha}(L_2(\Gamma))} & = \left[ \sum_{j=0}^{\infty} 2^j \alpha^2 \sum_{\xi \in \nabla^j \psi} \left| \left< u, \tilde{\psi}_j^\tau, \xi \right> \right|^2 \right]^{1/2} \\
  \quad \Longrightarrow H^s(\Gamma) & = B_{\psi,2}^{s}(L_2(\Gamma)) \text{ (equivalent norms) for all } 0 \leq s < \min\{3/2, s_\Gamma\}, \text{ e.g. } L_2(\Gamma) = B_{\psi,2}^{0}(L_2(\Gamma))
  \end{align*}
Regularity in Besov-type spaces

M. Weimar

Motivation
Double layer equation

Manifolds, wavelets, and Sobolev spaces

Manifolds
Wavelets
(Weighted) Sobolev spaces

Besov spaces
Definition
Properties
Non-standard embedding

Regularity
Final remarks and references

\[ L^2(\Gamma) = B^0_{\psi,2}(L^2(\Gamma)) \quad \text{and} \quad H^s(\Gamma) = B^s_{\psi,2}(L^2(\Gamma)) \]

▶ typical results on (real/complex) interpolation
typical characterization of standard embeddings

\[ B_{\Psi,q_0}^s(L_{p_0}(\Gamma)) \hookrightarrow B_{\Psi,q_1}^\alpha(L_{p_1}(\Gamma)) \]

(possible only if \( s \geq \alpha \)) and ...

... results on their uniform/best n-term wavelet (tree) approximation (rate: \( r \leq [s - \alpha]/d \))

\[
\text{best } n\text{-term rate } = \frac{\text{smoothness difference}}{\text{dimension}} = \frac{s - \alpha}{d}
\]
A non-standard embedding \((\Gamma = \partial \Omega, \Omega \subset \mathbb{R}^3)\)

**Theorem (Dahlke, W. 2013)**

*Under certain conditions there exists \(s^* \in (s, 2s)\) such that*

\[
B^s_{\Psi, p}(L_p(\Gamma)) \cap X^k_\varrho(\Gamma) \hookrightarrow B^\alpha_{\Psi, \tau}(L_\tau(\Gamma)) \quad \forall \alpha_{\tau} \in [0, s^*).
\]
**Theorem (Dahlke, W. 2013)**

Let \( s \in (0, 1) \), \( k \in \mathbb{N} \), and \( \varrho \in (0, \min\{\varrho_0, k\}) \) for some \( \varrho_0 = \varrho_0(\Gamma) \in (1, 3/2) \). Moreover let \( \alpha_\tau \) and \( \tau \) be given s.t.

\[
\frac{1}{\tau} = \frac{\alpha_\tau}{2} + \frac{1}{2} \quad \text{and} \quad 0 \leq \alpha_\tau < 2 \cdot \min\{\varrho, k - \varrho, s\}
\]

and assume that \( \tilde{\Psi}_\Gamma \) has \( \tilde{D} \geq k \) (int.) vanishing moments. Then for every RHS

\[ g \in H^s(\Gamma) \cap X^k_{\varrho}(\Gamma) \]

the double layer eq. has a unique solution \( u \in B^{\alpha_\tau}_{\Psi, \tau}(L_\tau(\Gamma)) \). Furthermore, if \( s' \in [0, s) \) then

\[
\sigma_n\left(u; H^{s'}(\Gamma)\right) \lesssim n^{-r} \quad \forall 0 \leq r < \left[1 - \frac{s'}{s}\right] \min\{\varrho, k - \varrho, s\}.
\]
Assume for simplicity that \( \min\{\varrho, k - \varrho\} \geq s \). Then we can take \( \alpha_{\tau} = 2s - \delta \) and \( r = [s - s'] - \delta \) (for all \( \delta > 0 \) small).

**Conclusion**

*Best n-term approximation rate \( r \) (benchmark for adaptive schemes) is twice as large as the rate for uniform approx.*
Final remarks

We...

- ...introduced new Besov-type spaces $B^{\alpha}_{\Psi,q}(L_p(\Gamma))$ on patchwise smooth manifolds $\Gamma$.
- ...recovered many typical properties (standard embed., interpolation, and approximation results).
- ...derived a regularity/approximation assertion for some boundary integral equation (double layer eq.) which is of fundamental relevance in practice.

Work in progress:

- dependence on the chosen basis $\Psi = (\Psi_\Gamma, \tilde{\Psi}_\Gamma)$ ✓
- other important equations (e.g. single layer) ✓
- application to real-life problems (Helmholtz eq.)
Regularity in Besov-type spaces
M. Weimar

Motivation
Double layer equation
Manifolds, wavelets, and Sobolev spaces
Manifolds
Wavelets
(Weighted) Sobolev spaces
Besov spaces
Definition
Properties
Non-standard embedding
Regularity
Final remarks and references

References


...


Thank you!