Integration of permutation-invariant functions

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A completely different approach

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Integration problem for Korobov-type spaces

We study multivariate integration

$$\text{Int}_d(f) := \int_{[0,1]^d} f(x) \, dx, \quad d \in \mathbb{N},$$

for certain subspaces of some RKHS of (1-periodic) functions

$$F_d(r_{\alpha,\beta}) := \left\{ f \in L_2([0,1]^d) \mid \| f \|_d < \infty \right\},$$

where we set

$$\| f \|_d^2 := \sum_{h \in \mathbb{Z}^d} \left| \hat{f}(h) \right|^2 r_{\alpha,\beta}(h)$$

with $\alpha > 1/2$ (smoothness), $\beta > 0$ (weight parameter), and

$$r_{\alpha,\beta}(h) := \prod_{\ell=1}^d \left[ \delta_{0,h_\ell} + (1 - \delta_{0,h_\ell}) \beta^{-1} (2\pi |h_\ell|)^{2\alpha} \right], \quad h \in \mathbb{Z}^d.$$
Permutation-invariant subspaces

For each $d \in \mathbb{N}$ take a subset of coordinates

$$\mathcal{I}_d \subseteq \{1, \ldots, d\}.$$

A function $f \in F_d(r_\alpha, \beta)$ is called $\mathcal{I}_d$-permutation-invariant, if for all $x \in [0, 1]^d$ and every permutation $\sigma \in S_d$ of $\mathcal{I}_d$

$$f(x) = f(\sigma(x)).$$
Symmetrizing the standard Fourier basis

\[ \{ e_h := \exp(2\pi i \cdot h) \mid h \in \mathbb{Z}^d \} \]

leads to an orthonormal basis \( \{ \phi_k \mid k \in \nabla_d \} \),

\[
\phi_k(x) := \sqrt{\frac{r_{\alpha,\beta}^{-1}(k)}{\#S_d \cdot M_d(k)!}} \cdot \sum_{\sigma \in S_d} e_{\sigma(k)}(x),
\]

\[ \nabla_d := \{ k \in \mathbb{Z}^d \mid k_{\ell_1} \leq \ldots \leq k_{\ell_{\#I_d}} \}, \]

of the subspace \( \mathcal{G}_{I_d}(F_d(r_{\alpha,\beta})) \) of \( I_d \)-permutation-invariant functions in \( F_d(r_{\alpha,\beta}) \).

Here \( M_d(k)! \) “counts the repetitions” in \( k \) (under \( S_d \)):

\[
M_d(k)! := \#\{ \sigma \in S_d \mid k = \sigma(k) \}.
\]
Reproducing kernels

We obtain a “permutation-invariant” reproducing kernel

\[ K_{d,I_d}^{\text{perm}}(x, y) = \sum_{k \in \nabla_d} \frac{r_{\alpha, \beta}^{-1}(k)}{M_d(k)!} \sum_{\sigma \in S_d} e_k(\sigma(x) - y) \]

for the subspace out of the classical reproducing kernel

\[ K_d(x, y) = \sum_{h \in \mathbb{Z}^d} r_{\alpha, \beta}^{-1}(h) e_h(x - y) \]

of \( F_d(r_{\alpha, \beta}) = \mathcal{H}(K_d) \).

This new kernel \( K_{d,I_d}^{\text{perm}} \) is

- NOT of product form,
- NOT shift-invariant!
Algorithms and worst case error

Consider a general $n$-point cubature rule with nodes $t^{(j)}$ and weights $w_j$:

$$Q_{n,d}(f) := \frac{1}{n} \sum_{j=0}^{n-1} w_j \cdot f(t^{(j)}) .$$

Its squared worst case error (see e.g. Novak, Woźniakowski) in any RKHS $\mathcal{H}_d = \mathcal{H}(K)$ is given by

$$\text{err}(Q_{n,d}; \mathcal{H}_d)^2 := \sup_{f \in \mathcal{H}_d, \|f\|_{\mathcal{H}_d} \leq 1} | \text{Int}_d(f) - Q_{n,d}(f) |^2 .$$

We are interested in the behavior of

$$e_{\text{wor}}(n, d) := \inf_{Q_{n,d}} \text{err}(Q_{n,d}; \mathcal{H}(K_d^{\text{perm}})),$$

$$n(\varepsilon, d) := \min\{n \in \mathbb{N}_0 | e_{\text{wor}}(n, d) \leq \varepsilon\}, \quad \varepsilon \in (0, 1).$$
Notions of tractability and known results

- **Weak tractability:**
  \[
  \lim_{\varepsilon^{-1} + d \to \infty} \frac{\log n(\varepsilon, d)}{\varepsilon^{-1} + d} = 0
  \]

- **Polynomial tractability:**
  \[
  \exists C, p > 0 \text{ and } q \geq 0 \text{ such that } n(\varepsilon, d) \leq C d^q \varepsilon^{-p} \quad \forall \varepsilon \in (0, 1), d \in \mathbb{N}
  \]

- **Strong polynomial tractability:**
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- **Strong polynomial tractability:** polynomial tract. with \( q = 0 \)

Known results for \( \mathcal{I}_d = \emptyset \) (i.e. no permutation-invariance) and dimension-dependent weights \( \beta = \beta(d) \):

- **WT** \( \iff \lim_{d \to \infty} \beta(d) = 0 \)
- **PT** \( \iff \beta(d) \lesssim \frac{\log(d + 1)}{d} \)
- **SPT** \( \iff \beta(d) \lesssim \frac{1}{d} \)
Random sampling

By averaging $\text{err}(Q_{n,d}; \mathcal{H}(K_{d,I_d}^{\text{perm}}))^2$ over all integration nodes $t^{(j)} \in [0,1]^d$ we conclude:

Theorem (Nuyens, Suryanarayana, W. 2014+)

*For all $n, d \in \mathbb{N}$, $I_d \subseteq \{1, \ldots, d\}$, $\alpha > 1/2$, and $\beta > 0$ there exists an equal weight cubature rule $Q_{n,d}^*$ such that

\[
e_{\text{wor}}(n,d)^2 \leq \text{err}(Q_{n,d}^*, \mathcal{H}(K_{d,I_d}^{\text{perm}}))^2 \leq C_{d,1}(r_{\alpha,\beta}) \cdot n^{-1}.
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Therein the constant

$$C_{d,1}(r_\alpha,\beta) = \sum_{k \in \nabla_d \setminus \{0\}} r_{\alpha,\beta}^{-1}(k) = \sum_{h \in \mathbb{Z}^d \setminus \{0\}} \frac{M_d(h)!}{\#S_d} r_{\alpha,\beta}^{-1}(h)$$

- grows exponentially with $\beta/(2\pi)^{2\alpha} \text{ and in } (d - \#I_d)$,
- is polyn. bounded if $\beta < c_\alpha \text{ and } (d - \#I_d) \in \mathcal{O}(\ln d)$.

$\implies$ (strong) polynomial tractability, if the space allows
(Unshifted) rank-1 lattice rules

Since we deal with Korobov-type spaces, (rank-1) lattice rules, completely determined by their generating vector $\mathbf{z}$ (from $\mathbb{Z}^d_n = \{0, 1, \ldots, n - 1\}^d$), are the method of choice:

$$Q^\text{lat}_{n,d}(f) = \frac{1}{n} \sum_{j=0}^{n-1} f \left( \left\{ \mathbf{z} \frac{j}{n} \right\} \right).$$

Using the “character property” we conclude

$$\text{err}(Q^\text{lat}_{n,d}, \mathcal{H}(K^\text{perm}_{d,I_d}))^2 = \sum_{\mathbf{h} \in \mathcal{L} \setminus \{\mathbf{0}\}} \frac{r_{\alpha,\beta}^{-1}(\mathbf{h})}{\#S_d} \sum_{\sigma \in S_d} 1_{\sigma(\mathbf{h}) \in \mathcal{L}}.$$

where $\mathcal{L}$ denotes the dual lattice induced by $\mathbf{z}$, i.e. the set of all $\mathbf{h} \in \mathbb{Z}^d$ with $\mathbf{h} \cdot \mathbf{z} \equiv 0 \pmod{n}$. 
A negative result

**Theorem (Nuyens, Suryanarayana, W. 2014+)**

Let $\alpha > 1/2$ and $\beta > 0$. Then, independently of the amount of permutation-invariance, the $n$th minimal worst case error among all unshifted lattice rules satisfies

$$e_{\text{wor}}^{\text{lat}}(n, d)^2 \geq \sum_{h \in \mathbb{Z}^d \setminus \{0\}} r_{\alpha, \beta}^{-1}(nh)$$

$$= \left(1 + \frac{2 \cdot \zeta(2\alpha)}{\beta \cdot (2\pi n)^{2\alpha}}\right)^d - 1$$

$$\geq c_{\alpha, \beta} \cdot d \cdot n^{-2\alpha}$$

for all $d \in \mathbb{N}$ and $n \in \mathbb{N}$ prime.

$\implies$ The class of unshifted lattice rules is too small to obtain strong polynomial tractability!
(Randomly) Shifted lattice rules

Let us enlarge the class of algorithms under consideration by adding random shifts $\Delta \in [0, 1)^d$ to given lattice rules $Q_{n,d}^{\text{lat}}$:

$$Q_{n,d}^{\text{sh-lat}}(f) := (Q_{n,d}^{\text{lat}} + \Delta)(f) = \frac{1}{n} \sum_{j=0}^{n-1} f \left( \left\{ \frac{z_j}{n} + \Delta \right\} \right).$$

**Proposition**

For all $Q_{n,d}^{\text{lat}}$ the average over all shifts satisfies

$$E(Q_{n,d}^{\text{lat}})^2 := \int_{[0,1]^d} \text{err}(Q_{n,d}^{\text{lat}} + \Delta; \mathcal{H}(K_{d,I_d}^{\text{perm}}))^2 \, d\Delta$$

$$= \text{err}(Q_{n,d}^{\text{lat}}; \mathcal{H}(K_{d,I_d}^{\text{sh,perm}}))^2 = \sum_{h \in L^d \setminus \{0\}} \frac{M_d(h)!}{\#S_d} r_{\alpha,\beta}(h)$$

and

$$c_{\alpha,\beta} \frac{\max\{d - \#I_d, 1\}}{n^{2\alpha}} \leq E(Q_{n,d}^{\text{lat}})^2 \leq \text{err}(Q_{n,d}^{\text{lat}}; \mathcal{H}(K_{d,I_d}^{\text{perm}}))^2.$$

It grows exponentially with $\beta/(2\pi n)^{2\alpha}$ and in $(d - \#I_d)$.  

References
Theorem (Nuyens, Suryanarayana, W. 2014+)

Let \( d \in \mathbb{N}, \mathcal{I}_d \subseteq \{1, \ldots, d\} \), as well as \( n \in \mathbb{N} \) prime, and set

\[
C_{d,\lambda}(r_{\alpha, \beta}) = \left( \sum_{h \in \mathbb{Z}^d \setminus \{0\}} \frac{M_d(h)!}{\# S_d} r_{\alpha, \beta}^{-1}(h) \right)^{1/\lambda}, \quad \lambda \geq 1.
\]

Then

- there exist \( Q_{n,d}^{\text{lat} \,*} \) such that for all \( 1 \leq \lambda < 2\alpha \)

\[
E(Q_{n,d}^{\text{lat} \,*})^2 \leq 2^\lambda \cdot C_{d,\lambda}(r_{\alpha, \beta}) \cdot n^{-\lambda},
\]

- there exist \( Q_{n,d}^{\text{lat} \,**} \), constructed component-by-component, such that for all \( 1 \leq \lambda < 2\alpha \)

\[
E(Q_{n,d}^{\text{lat} \,**})^2 \leq 2^\lambda \cdot d^{\lambda-1} \cdot C_{d,\lambda}(r_{\alpha, \beta}) \cdot n^{-\lambda}.
\]
Conclusions

Random sampling works $\implies$ **strong polynomial tractability**, if the space allows (e.g. for $I_d = \{1, \ldots, d\}$ and $\beta < c_\alpha$)

Upper bounds for $Q_{n,d}^{\text{lat}}$ with respect to $\mathcal{H}(K_d^{\text{sh,perm}}, I_d)$ imply the existence of good shifts $\Delta^*$ such that the same bounds hold for $Q_{n,d}^{\text{sh-lat}} = Q_{n,d}^{\text{lat}} + \Delta^*$ with respect to $\mathcal{H}(K_d^{\text{perm}}, I_d)$.

- The (shifted) CBC-construction realizes the random sampling error bound up to a factor of 2.
- Unshifted lattice rules cannot achieve such a bound.
Final remarks

For $\mathcal{I}_d = \emptyset$ our bounds match well-known results.

The CBC-algorithm depends on the target dimension, i.e. the obtained lattice rules are not extendable.

For higher-order convergence $n^{-\lambda}$, $\lambda > 1$,

- we lose a factor of $d^{\lambda-1}$ using the CBC-approach, but we (almost) obtain the optimal rate $2\alpha$,
- $C_{d,\lambda}(r_\alpha, \beta)$ grows exponentially for all $\mathcal{I}_d$ and constant $\beta$,
- $C_{d,\lambda}(r_\alpha, \beta)$ is polynomially upper bounded for

$$
(d - \#\mathcal{I}_d) \in O(\ln d) \quad \text{and} \quad \beta = \beta(d) \lesssim \frac{1}{d^{\lambda-1}}.
$$

Finally, everything is semi-constructive.
A completely different approach

Based on results for integration and approximation w.r.t. the average case setting (see Hickernell & Woźniakowski and Wasilkowski) we can show (semi-constructively):

**Theorem (Nuyens, Suryanarayana, W. 2014+)**

For all $\alpha > 1/2$ and $\delta > 0$ there exists $Q_{n,d}^{**}$ such that

\[
\epsilon_{\text{wor}}(n, d)^2 \leq \text{err}(Q_{n,d}^{**}, \mathcal{H}(K_{d}^{\text{perm}}))^2 \leq C_{\alpha,\beta,\delta}(d) \cdot n^{-2\alpha+\delta}.
\]

Therein the constant satisfies

\[
C_{\alpha,\beta,\delta}(d) \leq c_{\alpha,\beta,\delta} \cdot d^q \quad \text{for some} \quad q \geq 0,
\]

provided that $\#I_d$ and $\beta^{-1}$ are large enough.

\[\Rightarrow\quad \text{(strong) polynomial tractability with (almost) opt. rate if the space allows}\]
References


Thank you!