On optimal wavelet approximations in spaces of Besov type

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Approximation theory is...

concerned with

- ... the (approximate) representation of “complicated” objects (e.g., functions) by “simpler” ones,
- ... the (explicit) construction of algorithms based on incomplete information,
- ... qualitative and quantitative error analysis,

and related to

- ... (numerical) analysis,
- ... function spaces,
- ... applications in natural science and technology.
A typical approach:

1.) Fix a “suitable” basis/frame/dictionary,
2.) expand the objects of interest, and
3.) approximate these expansions by finitely many well-chosen terms according to a prescribed strategy (e.g., coefficient thresholding).

Error bounds

- ... are meaningful only for classes/spaces of functions,
- ... involve the number of degrees of freedom (\(\sim\) rate of convergence),
- ... depend on the (quasi-)norms of source and target spaces.
Abstract example

Expansion and truncation:

\[ f = \sum_{\lambda \in \nabla} c_\lambda(f) \cdot \psi_\lambda \]

\[ \downarrow \]

\[ \tilde{f}_N = \sum_{\lambda \in \tilde{\nabla}_N} c_\lambda(f) \cdot \psi_\lambda, \quad N \in \mathbb{N}_0, \]

where

- \( \{\psi_\lambda \mid \lambda \in \nabla\} \) is a wavelet basis for \( \Omega \subset \mathbb{R}^d \) bounded,
- \( \tilde{\nabla}_N \subset \nabla \) refers to \( N \) preferably “large” coefficients
  \( c_\lambda(f) = \langle f, \tilde{\psi}_\lambda \rangle \).

Typical bound:

\[ \sup_{\|f\|_{H^s} \leq 1} \inf \left\| f - \tilde{f}_N \right\|_{L_2} \sim N^{-s/d} \]
We present 3 different strategies for choosing $\widetilde{\nabla}_N$ and derive error bounds for $L_2$ and $H^s$ replaced by more general scales of Besov-type spaces $B^\alpha_q(L_p)$ on bounded domains and (patchwise smooth) manifolds.

Concrete applications cover

- Galerkin approaches to partial differential and boundary integral equations (e.g., classical FEM, adaptive BEM, ...),
- Signal processing (e.g., image compression),
- ...

and our results justify the use of adaptive schemes for these purposes.
Figure: Adaptive wavelet Galerkin solution $\tilde{u}_N$ to Poisson's problem $\Delta u = f$ with homogeneous Dirichlet boundary conditions $u|_{\partial\Omega} = 0$ on the L-shape domain $\Omega \subset \mathbb{R}^2$. 

**Figure:** Original (b/w) Lena and \( \approx 50\% \) compression with discrete cosine transform used by JPEG (left) and wavelet transform used by JPEG 2000 (right).
Outline

Introduction and motivating examples

Wavelet expansions, multiscale and tree structures
  Wavelet expansions
  Multiscale and tree structures

Besov spaces
  Classical Besov spaces on $\mathbb{R}^d$
  Definition of Besov-type spaces on $\Gamma$
  Simple properties

Optimal wavelet approximations
  Quantities of interest
  Main results

Approximation spaces

Summary and references
Wavelet expansions, multiscale and tree structures
Wavelets

Let $\Gamma$ denote a bounded $d$-dimensional domain or manifold.

We assume to be given a fixed biorthogonal wavelet Riesz bases $\Psi = (\psi_\Gamma, \widetilde{\psi}_\Gamma)$ which characterizes $L_2(\Gamma)$. That is, for all $u \in L_2(\Gamma)$,

$$u = \sum_{j=0}^{\infty} \sum_{\xi \in \nabla_j} \langle u, \widetilde{\psi}_{j, \xi} \rangle \psi_{j, \xi} \quad \text{on} \quad \Gamma$$

with

$$\| u \|_{L_2(\Gamma)} \sim \left[ \sum_{j=0}^{\infty} \sum_{\xi \in \nabla_j} | \langle u, \widetilde{\psi}_{j, \xi} \rangle |^2 \right]^{1/2}.$$

Here the wavelets $\psi_{j, \xi}$ and $\widetilde{\psi}_{j, \xi}$ are indexed by

- $j$ ... level of resolution,
- $\xi$ ... point from a multiscale grid $\nabla^\Psi = (\nabla^\Psi_j)_{j \in \mathbb{N}_0}$ for $\Gamma$ which encodes location and type.
Among other useful properties we can assume

- $\# \nabla_j \Psi \sim 2^{jd}$,
- local supports,
- normalization in $L_2$,
- some “vanishing moments”,
- some “smoothness”

such that, e.g., $H^s(\Gamma)$-norm equivalences

$$\| u \|_{H^s(\Gamma)} \sim \left[ \sum_{j=0}^{\infty} 2^{js} 2 \sum_{\xi \in \nabla_j} \left| \langle u, \tilde{\psi}_j^\Gamma, \xi \rangle \right|^2 \right]^{1/2}$$

hold for all $s$ in some range around zero.
Such bases can be constructed if we restrict ourselves to (the practically most relevant cases of)

- bounded, flat Lipschitz domains in $\mathbb{R}^d$, or
- Lipschitz manifolds that can be decomposed in finitely-many smooth $2^d$-sided patches

$$\Gamma = \bigcup_{i=1}^{l} \bar{\Gamma}_i, \quad \bar{\Gamma}_i = \kappa_i([0, 1]^d),$$

e.g., surfaces of bounded, simply connected, closed domains in $\mathbb{R}^3$: 
Bases on manifolds are constructed via some \textit{lifting and gluing procedure} out of wavelets on the unit cube which in turn result from tensor products of (boundary adapted) univariate wavelets based on B-splines.
Bases on manifolds are constructed via some *lifting and gluing procedure* out of wavelets on the unit cube which in turn result from tensor products of (boundary adapted) univariate wavelets based on B-splines.

Explicit basis constructions are given by

- composite wavelet bases $\Psi = \Psi_{DS}$ introduced by Dahmen and Schneider (designed for general operator equations),
- modified composite wavelets $\Psi = \Psi_{HS}$ established by Harbrecht and Stevenson (first choice in the boundary element method for int. eq.)
- bases $\Psi = \Psi_{CTU}$ due to Canuto, Tabacco, and Urban (primarily used in the so-called wavelet element method)

In these constructions biorthogonality is realized w.r.t. an (equivalent) inner product on $L_2(\Gamma)$

$$\langle u, v \rangle := \sum_{i=1}^{l} \langle u \circ \kappa_i, v \circ \kappa_i \rangle_{L_2([0,1]^d)}.$$
Multiscale and tree structures

The set of equivalence classes

\[ \left\{ (j, y) \in \mathbb{N}_0 \times \Gamma \mid y = y_\xi \text{ for some } \xi \in \nabla_j^\Psi \right\} \]

is arranged in an infinite (master) tree:

- finitely many roots corresponding to coarsest level \( j = 0 \)
- every knot at level \( j \) has \( \sim 2^d \) children on next finer level \( j + 1 \) (wavelets of smaller support located nearby).

**Definition**

\( T \subset \nabla^\Psi = \{(j, \xi) \mid j \in \mathbb{N}_0, \xi \in \nabla_j^\Psi\} \) is called tree if for all \( (j, \xi) \in T \) with \( j > 0 \) there exists \( (j - 1, \xi') \in T \) such that \( (j - 1, y_{\xi'}) \) is the parent of \( (j, y_\xi) \) in the above sense.
Infinite master tree for Haar wavelets on $\Gamma = [0, 1]$

0

$\frac{1}{2}$

$\frac{1}{4}$

$\frac{3}{4}$

$\frac{7}{8}$

$\frac{15}{16}$

...
and some tree according to the definition given above:

```
    0
   j = 0
  /\ 1
 1/2  
 /\ 1
/\ 1/2
j = 1
/\ 1
1/2  
/\ 1/2
j = 2
/\ 1
3/4  
/\ 1/2
5/8  
/\ 1/2
3/4  
/\ 1/2
7/8  
/\ 1/2
11/16
/\ 3/4
3/4
```

level
Function spaces of Besov type
Classical Besov spaces

Besov spaces $B_\alpha^q(L_p(\mathbb{R}^d))$ ...

- ... are (quasi-) Banach spaces.
- ... essentially generalize Sobolev (Hilbert) spaces $H^s$.
- ... depend on (at least) 3 parameters: $\alpha$, $p$, $q$.
- ... are defined in various ways (e.g. using harmonic analysis, moduli of smoothness, interpolation, ...).
- ... are characterized by decay properties of expansion coefficients w.r.t. various building blocks (atoms, quarks, wavelets, ...).
Corresponding spaces can be defined for **bounded domains** and non-smooth **manifolds** (as trace spaces or via pullbacks), but then additional difficulties appear.

For numerical applications it is therefore convenient to define higher-order Besov smoothness by the decay of wavelet coefficients as follows:
Definition (Dahlke, W. 2013 / W. 2014)

- A tuple of real parameters $\left(\alpha, p, q\right)$ is called **admissible** if $0 < p < \infty$ and
  - $\alpha > d \cdot \max\{0, 1/p - 1/2\}$ and $0 < q \leq \infty$, or
  - $\alpha = d \cdot \max\{0, 1/p - 1/2\}$ and $0 < q \leq 2$.
- $B_{\Psi}^{\alpha,q}(L_p(\Gamma))$ denotes the set of all $u \in L_2(\Gamma)$ s.t.

\[
\left\| u \right\|_{B_{\Psi}^{\alpha,q}(L_p(\Gamma))} := \left\| \left( \langle u, \tilde{\psi}_{j,\xi}^{\Gamma} \rangle \right)_{(j,\xi)} \right\|_{b_{p,q}(\nabla\Psi)}^{q/p} \quad 1/q
\]

\[
= \left[ \sum_{j=0}^{\infty} 2^j \left( \alpha + d \left[ \frac{1}{p} - \frac{1}{2} \right] \right) q \left( \sum_{\xi \in \nabla\psi} \left| \langle u, \tilde{\psi}_{j,\xi}^{\Gamma} \rangle \right|^p \right)^{p/q} \right]^{1/q}
\]

(with the usual modifications for $q = \infty$) is finite.
Hence,

- **per definition** the function $u$ belongs to the Besov-type space $B_{\psi,q}^\alpha(L_p(\Gamma)) \iff$ its sequence of expansion coefficients $\left(\langle u, \widetilde{\psi}_{j,\xi}^\Gamma \rangle\right)_{(j,\xi)\in \nabla \psi}$ w.r.t. the wavelet basis $\psi = (\psi^\Gamma, \widetilde{\psi}^\Gamma)$ exhibits a certain rate of decay, i.e., belongs to the sequence space

$$b_{p,q}^\alpha(\nabla \psi) := \left\{ a = (a(j,\xi))_{(j,\xi)} \mid \| a \|_{b_{p,q}^\alpha(\nabla \psi)} < \infty \right\}.$$ 

- properties of the scale $B_{\psi,q}^\alpha(L_p(\Gamma))$ can be derived from corresponding results for $b_{p,q}^\alpha(\nabla \psi)$. 
Hence,

- *per definition* the function $u$ belongs to the Besov-type space $B_{\psi, q}^{\alpha}(L_p(\Gamma)) \iff$ its sequence of expansion coefficients $\left( \langle u, \tilde{\psi}_{j, \xi} \rangle \right)_{(j, \xi) \in \nabla \psi}$ w.r.t. the wavelet basis $\psi = (\psi^\Gamma, \tilde{\psi}^\Gamma)$ exhibits a certain rate of decay, i.e., belongs to the sequence space

$$b_{p, q}^\alpha (\nabla \Psi) := \left\{ a = (a(j, \xi))_{(j, \xi)} \left| \left\| a \right\| b_{p, q}^\alpha (\nabla \Psi) < \infty \right. \right\}. $$

- properties of the scale $B_{\psi, q}^{\alpha}(L_p(\Gamma))$ can be derived from corresponding results for $b_{p, q}^\alpha (\nabla \Psi)$.

In principle, our approach is applicable for every set $\Gamma$ which allows the construction of a wavelet basis for $L_2(\Gamma)$. This covers (un-)bounded domains in $\mathbb{R}^d$, as well as (non-)smooth manifolds with or without a boundary!
Simple properties of the new scale $B^{\alpha}_{\psi,q}(L^p(\Gamma))$

- always quasi-Banach spaces
  (Banach $\iff$ min\{p, q\} $\geq$ 1 / Hilbert $\iff$ $p = q = 2$)
- simplified (quasi-)norms for so-called **adaptivity scale**
  $p = q = \tau := (\alpha_\tau/d + 1/2)^{-1}$, $\alpha_\tau \geq 0$

\[
\left\| u \right\|_{B^{\alpha_\tau}_{\psi,\tau}(L^\tau(\Gamma))} = \left[ \sum_{j=0}^{\infty} \sum_{\xi \in \nabla_j^\psi} \left| \langle u, \check{\psi}_j^\Gamma, \xi \rangle \right|^\tau \right]^{1/\tau}
\]

and for Hilbert scale $p = q = 2$

\[
\left\| u \right\|_{B^{\alpha}_{\psi,2}(L^2(\Gamma))} = \left[ \sum_{j=0}^{\infty} 2^j \alpha^2 \sum_{\xi \in \nabla_j^\psi} \left| \langle u, \check{\psi}_j^\Gamma, \xi \rangle \right|^2 \right]^{1/2}
\]

$\rightsquigarrow H^s(\Gamma) = B^s_{\psi,2}(L^2(\Gamma))$ (equivalent norms) whenever $s$ is small enough, e.g., $L^2(\Gamma) = B^0_{\psi,2}(L^2(\Gamma))$. 
Optimal wavelet approximation

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Introduction

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Wavelets and trees
Wavelet expansions
Multiscale and tree structures
Besov spaces
Classical Besov spaces
Besov-type spaces
Simple properties
Wavelet approximations
Quantities of interest
Main results
Approximation spaces
Summary and references

\[ H^s = B_{\psi,2}^s(L_2) \]

\[ \frac{1}{\tau} = \frac{\alpha_\tau}{d} + \frac{1}{2}, \quad \alpha_\tau \geq 0 \]

\[ L_2(\Gamma) = B_{\psi,2}^0(L_2(\Gamma)) \quad \text{and} \quad H^s(\Gamma) = B_{\psi,2}^s(L_2(\Gamma)) \]
Optimal wavelet approximations
Quantities of interest I

**Definition (Unconstrained best \( N \)-term approximation)**

Let \( G \) denote a (quasi-)normed space and let \( \mathcal{B} \subset G \).

- For \( N \in \mathbb{N}_0 \) the infimum

\[
\sigma_{N}^{\text{unc}}(f; \mathcal{B}, G) := \inf \left\| f - \sum_{\lambda \in \Lambda} c_{\lambda} f_{\lambda} \right\|_G
\]

taken over all index sets \( \Lambda \) with \( \#\Lambda \leq N \), all \( f_{\lambda} \in \mathcal{B} \), and \( c_{\lambda} \in \mathbb{C} \) \( (\lambda \in \Lambda) \) defines the error of the (unconstrained) **best \( N \)-term approximation** to \( f \) w.r.t. the **dictionary** \( \mathcal{B} \) in the (quasi-)norm of \( G \).
Definition (Unconstrained best $N$-term approximation)

Let $G$ denote a (quasi-)normed space and let $B \subset G$.

- For $N \in \mathbb{N}_0$ the infimum
  \[
  \sigma_{N}^{\text{unc}} (f; B, G) := \inf \left\| f - \sum_{\lambda \in \Lambda} c_\lambda f_\lambda \right\|_G
  \]
  taken over all index sets $\Lambda$ with $\# \Lambda \leq N$, all $f_\lambda \in B$, and $c_\lambda \in \mathbb{C}$ ($\lambda \in \Lambda$) defines the error of the (unconstrained) best $N$-term approximation to $f$ w.r.t. the dictionary $B$ in the (quasi-)norm of $G$.

- For (quasi-)normed spaces $F \hookrightarrow G$ we set
  \[
  \sigma_{N}^{\text{unc}} (F; B, G) := \sup_{\| f \|_F \leq 1} \sigma_{N}^{\text{unc}} (f; B, G), \quad N \in \mathbb{N}_0.
  \]
Quantities of interest II+III

Definition (tree and uniform approximations)

In addition, let the dictionary $\mathcal{B}$...

- ...carry a master tree structure. Then

$$
\sigma_{N}^{\text{tree}}(f; \mathcal{B}, G) := \inf \left\| f - \sum_{\lambda \in \Lambda} c_{\lambda} f_{\lambda} \right\|_{G}
$$

restricted to trees $\Lambda$ of size as most $N$ refers to the error of the **best $N$-term tree approximation** to $f$. 

Mathematical notation used in the document: $\sigma_{N}^{\text{tree}}(f; \mathcal{B}, G)$ represents the error of the best $N$-term tree approximation, where $\Lambda$ is the set of tree structures, $c_{\lambda}$ are the coefficients, and $G$ is the norm.
Quantities of interest II+III

Definition (tree and uniform approximations)

In addition, let the dictionary $\mathcal{B}$...

- ...carry a master tree structure. Then

$$\sigma^\text{tree}_N(f; \mathcal{B}, G) := \inf \left\| f - \sum_{\lambda \in \Lambda} c_{\lambda} f_{\lambda} \right\|_G$$

restricted to trees $\Lambda$ of size as most $N$ refers to the error of the best $N$-term tree approximation to $f$.

- ...carry a multiscale structure, i.e., $\mathcal{B} = \bigcup_{j=0}^{\infty} \mathcal{B}_j$. Then

$$\sigma^\text{unif}_N(f; \mathcal{B}, G) := \inf \left\| f - \sum_{\lambda \in \Lambda} c_{\lambda} f_{\lambda} \right\|_G$$

restricted to multiscale index sets $\Lambda = \bigcup_{j=0}^{J(N)} \Lambda_j$ with

$$J(N) := \min \left\{ J \in \mathbb{N}_0 \mid \sum_{j=0}^{J} \# \mathcal{B}_j \leq N \right\}$$

refers to the error of the best uniform $N$-term approximation.

($\sigma^\text{tree}_N(F; \mathcal{B}, G)$ and $\sigma^\text{unif}_N(F; \mathcal{B}, G)$ are defined like before)
We stress that

- for best $N$-term (tree) approximation the sets of possible approximants to $f$,

\[ \left\{ \sum_{\lambda \in \Lambda} c_{\lambda} f_\lambda \left| f_\lambda \in \mathcal{B}, c_{\lambda} \in \mathbb{C}, \lambda \in \Lambda \right. \right\} \subset G, \]

form *highly nonlinear manifolds* which might depend on $f$ (via $\Lambda$), whereas

- in best uniform $N$-term approximation we deal with *linear subspaces* of $G$ which do not depend on $f$.

Best $N$-term (tree) approximation should be more powerful than uniform approximation.
We like to study the asymptotic decay of

\[ \sigma_N^{\text{meth}}(F; \mathcal{B}, G), \quad \text{meth} \in \{\text{unc}, \text{tree}, \text{unif}\}, \quad (1) \]

for wavelet approximations in Besov-type spaces, i.e., for

- spaces \( F \) and \( G \) taken from the scale \( B_{\psi, q}^\alpha(L_p(\Gamma)) \),
- the dictionary \( \mathcal{B} = \Psi \Gamma \) consisting of primal wavelets.
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for wavelet approximations in Besov-type spaces, i.e., for

- spaces \( F \) and \( G \) taken from the scale \( B^\alpha_{\psi, q}(L_p(\Gamma)) \),
- the dictionary \( B = \psi^\Gamma \) consisting of primal wavelets.

It suffices to analyze (1) for

- sequence spaces \( F \) and \( G \) in the scale \( b^\alpha_{p, q}(\nabla \psi) \),
- the canonical dictionary \( B_{\text{seq}} = \{e_{j, \xi} \mid (j, \xi) \in \nabla \psi\} \),

since

\[
f = \sum_{j=0}^{\infty} \sum_{\xi \in \nabla^j \psi} \langle f, \tilde{\psi}^\Gamma_{j, \xi} \rangle \psi^\Gamma_{j, \xi} \quad \in B^\alpha_{\psi, q}(L_p(\Gamma))
\]

\[\longleftrightarrow\]

\[
x = \sum_{j=0}^{\infty} \sum_{\xi \in \nabla^j \psi} x_{j, \xi} e_{j, \xi} \quad \in b^\alpha_{p, q}(\nabla \psi).
\]
Standard embeddings (on bounded sets $\Gamma$)

When do we have embeddings between function/sequence spaces of Besov-type?
Standard embeddings (on bounded sets $\Gamma$)

When do we have embeddings between function/sequence spaces of Besov-type?

**Proposition (Dahlke, W. 2013 / W. 2014)**

Let $(\alpha + \gamma, p_0, q_0)$ and $(\alpha, p_1, q_1)$ denote admissible parameter tuples. Then we have the continuous embedding

$$B_{\psi, q_0}^{\alpha + \gamma}(L_{p_0}(\Gamma)) \hookrightarrow B_{\psi, q_1}^{\alpha}(L_{p_1}(\Gamma))$$

if and only if one of the following conditions applies:

(E1) $\gamma > d \cdot \max\left\{0, \frac{1}{p_0} - \frac{1}{p_1}\right\},$

(E2) $\gamma = d \cdot \max\left\{0, \frac{1}{p_0} - \frac{1}{p_1}\right\}$ and $q_0 \leq q_1$.

This embedding is compact if and only if (E1) holds.

Note that $\gamma < 0$ is not possible and that the assertion holds likewise for the associated sequence spaces.
Figure: Standard embeddings for Besov-type spaces

(E2) = non-compact embeddings = “along the arrows”
(E1) = compact embeddings = “everything in between”
Main results

**Theorem**

Let $\alpha, \gamma \in \mathbb{R}, 0 < p_0, p_1 < \infty$, and $0 < q_0, q_1 \leq \infty$. Then

$(E1)$ implies

$$
\sigma_N^{\text{meth}} \sim \begin{cases} 
N^{-\gamma/d} & \text{if } \text{meth} = \text{unc}, \\
N^{-\gamma/d} & \text{if } \text{meth} = \text{tree}, \\
N^{-(\gamma/d - \max\{0, 1/p_0 - 1/p_1\})} & \text{if } \text{meth} = \text{unif},
\end{cases}
$$

$(E2)$ implies

$$
\sigma_N^{\text{meth}} \sim \begin{cases} 
N^{-\min\{\gamma/d, 1/q_0 - 1/q_1\}} & \text{if } \text{meth} = \text{unc}, \\
1 & \text{if } \text{meth} = \text{tree}, \\
1 & \text{if } \text{meth} = \text{unif},
\end{cases}
$$

where we set $\sigma_N^{\text{meth}} = \sigma_N^{\text{meth}}(b_{p_0,q_0}^{\alpha + \gamma}(\nabla \psi); B_{\text{seq}}, b_{p_1,q_1}^\alpha(\nabla \psi))$. 
Conclusions

- As expected we always have
  \[ \sigma_N^{\text{unc}} \lesssim \sigma_N^{\text{tree}} \lesssim \sigma_N^{\text{unif}}. \]

- Uniform and tree approximation fail for non-compact embeddings.

- For compact embeddings tree approximation is as powerful as unconstrained best \( N \)-term approximation.

- Optimally:
  \[ \text{rate of conv.} = \frac{\text{difference in smoothness}}{\text{dimension}} \]
Conclusions

- As expected we always have
  \[ \sigma^\text{unc}_N \lesssim \sigma^\text{tree}_N \lesssim \sigma^\text{unif}_N. \]
- Uniform and tree approximation fail for non-compact embeddings.
- For compact embeddings tree approximation is as powerful as unconstrained best \( N \)-term approximation.
- Optimally:
  rate of conv. = difference in smoothness / dimension

Implications for numerical schemes:

- Nonlinear/adaptive methods pay off whenever \( p_0 < p_1 \).
- Almost no dependence on fine indicee \( q \), especially for uniform methods

\( \rightsquigarrow \) Regularity analysis (e.g., of solutions to PDEs)
  should focus on \( B^s_\infty(L^2) \) rather than on \( H^s = B^s_2(L^2) \)!
- Regularity in the “adaptivity scale” is not sufficient for algorithms based on tree approximation.
Figure: Besov-type sequence spaces $b_{\alpha+\gamma(p)}^\alpha(p)(\nabla \Psi)$ with constant rate of uniform approximation $r_i \equiv r(p) = \gamma(p)/d - \max\{0, 1/p - 1/p_1\}$ in the (quasi-)norm of $b_{p_1,q_1}^{\alpha+\gamma(p)}(\nabla \Psi)$ for $0 < r_0 < r_1 < r_2$ and $p_0 < p_1 < 2 < \tilde{p}_0$.

Note that here the largest space for rate $r_0$ is $b_{p_1,\infty}^{\alpha+dr_0}(\nabla \Psi)$. 

\[ \alpha + dr_2 \]
\[ \alpha + dr_1 \]
\[ \alpha + \gamma(p_0) \]
\[ \alpha + dr_0 \]
\[ b_{p_0,q_0}^{\alpha+\gamma(p_0)} \]
\[ b_{p_0,q_0}^{\alpha+\gamma(p_0)} \]
\[ b_{p_0,q_0}^{\alpha+\gamma(p_0)} \]
\[ b_{p_1,q_0}^{\alpha+\gamma(p_1)} \]
\[ b_{p_1,q_0}^{\alpha+\gamma(p_1)} \]
\[ b_{p_1,q_0}^{\alpha+\gamma(p_1)} \]
\[ b_{p_1,q_1}^{\alpha} \]
\[ \frac{1}{p} \]
\[ \frac{1}{p_0} \]
\[ \frac{1}{2} \]
\[ \frac{1}{p_1} \]
\[ \frac{1}{p_2} \]
Figure: Corresponding spaces for best $N$-term (tree) approximation.

Here the largest spaces are the end points at the right-hand side (which are excluded for tree approximation!)
Some more remarks

- Some of the results are well-known (at least for special cases or related problems), e.g.,
  - Best $N$-term for bounded domains: Hansen/Sickel
  - Tree approximation in $L_p$ on bounded domains: Cohen/Dahmen/Daubechies/DeVore
  - Uniform approximation between Hilbert spaces
Some more remarks

- Some of the results are well-known (at least for special cases or related problems), e.g.,
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  - Uniform approximation between Hilbert spaces

- For large ranges of parameters the choice of the subspaces for uniform approximation is optimal, because the Kolmogorov $N$-widths yield the same rate

\[
d_N := d_N(b_{p_0,q_0}^{\alpha+\gamma}(\nabla\psi); b_{p_1,q_1}^{\alpha}(...))
\]

\[
= \inf_{\text{lin. subspace } L_n \text{ of dim } N} \sup_{\|x\| \leq 1} \inf_{y \in L_n} \|x - y\| b_{p_0,q_0}^{\alpha+\gamma}(\nabla\psi)
\]

\[
\sim N^{-(\gamma/d-\max\{0,1/p_0-1/p_1\})}
\]

if (E1) and either $1/p_0 \geq 1/p_1 \geq 1/2$ or $1/p_0 \leq 1/p_1$. 

If the results are known, their applications and implications may be discussed further.
Approximation spaces
Abstract definition

\textbf{Definition}

Let \( G \) denote a (quasi-)normed space, \( r > 0 \), as well as \( 0 < q \leq \infty \), and fix an approximation method \( \text{meth} \). Then

\[ \mathcal{A}_{q}^{r, \text{meth}}(G) := \{ f \in G \mid \| f \| \mathcal{A}_{q}^{r, \text{meth}}(G) < \infty \} \]

with \( \| f \| \mathcal{A}_{q}^{r, \text{meth}}(G) \) given by

\[ \left( \sum_{N=1}^{\infty} \left[ N^r \cdot \sigma_{N-1}^{\text{meth}}(f; \mathcal{B}, G) \right]^q \frac{1}{N} \right)^{1/q} \]

if \( q < \infty \)

and

\[ \sup_{N \in \mathbb{N}} N^r \cdot \sigma_{N-1}^{\text{meth}}(f; \mathcal{B}, G) \]

if \( q = \infty \)

is called \textit{approximation space} of order \( r \) w.r.t. \( G \) and \( \text{meth} \).

\( \mathcal{A}_{\infty}^{r, \text{meth}}(G) \) is the largest subspace of \( G \) s.t. \( \sigma_{N}^{\text{meth}}(\cdot; \mathcal{B}, G) \) decreases at least with rate \( r \), as \( N \to \infty \).
Results

Theorem (Uniform approximation)

Let $\alpha \in \mathbb{R}$, $0 < p_1 < \infty$, and $0 < q_1 \leq \infty$. Then

$$A_r^{\alpha, \text{unif}}(b_{p_1, q_1}^\alpha(\nabla \psi)) = b_{p_1, q}^{\alpha+dr}(\nabla \psi)$$

for all $r > 0$ and $0 < q \leq \infty$.

- Here all approximation spaces are Besov-type spaces
- As noticed before:

$$A_r^{\alpha, \text{unif}}(H^\alpha(\Gamma)) = B_{\psi, \infty}^{\alpha+dr}(L_2(\Gamma)) \iff H^{\alpha+dr}(\Gamma).$$
Theorem (Unconstrained best $N$-term, part 1)

Let $\alpha \in \mathbb{R}$, $0 < p_1 < \infty$, as well as $0 < q_1 \leq \infty$ and consider

$$A_q := A_q^{r, \text{unc}}(b^{\alpha}_{p_1, q_1} (\nabla \psi))$$

for $r > 0$ and $0 < q \leq \infty$. Then

\begin{align}
\left. b^{\alpha+d(r+\varepsilon)}_{(r+\varepsilon+1/p_1)^{-1},(r+\varepsilon+1/q_1)^{-1}}(\nabla \psi) \right. \\
\left. \rightarrow \ A_\mu \rightarrow b^{\alpha+dr}_{(r+1/p_1)^{-1},(r+1/q_1)^{-1}}(\nabla \psi) \rightarrow A_\infty \right. \\
\left. \rightarrow \ A_q \right. \\
\left. \quad b^{\alpha+d(r-\varepsilon)}_{(r-\varepsilon+1/p_1)^{-1},(r-\varepsilon+1/q_1)^{-1}}(\nabla \psi) \right.
\end{align}

(2) (3)

for all $0 < \varepsilon \leq r$ and some $0 < \mu < \infty$.

In fact, $A_q$ is the real interpolation space of (2) and (3) with the parameters $1/2$ and $q$. 
Theorem (Unconstrained best $N$-term, part 2)

- The above embedding

$$b^\alpha_{(r+1/p_1)^{-1},(r+1/q_1)^{-1}}(\nabla \psi) \hookrightarrow A^{r, \text{unc}}_\infty (b^\alpha_{p_1,q_1}(\nabla \psi)) \quad (4)$$

is optimal.

- If $A^{r, \text{unc}}_q (b^\alpha_{p_1,q_1}(\nabla \psi))$ is a space of Besov type, so it is the one in (4).

- Indeed, if $p_1 = q_1$, then

$$A^{r, \text{unc}}_{(r+1/p_1)^{-1}} (b^\alpha_{p_1,p_1}(\nabla \psi)) = b^\alpha_{(r+1/p_1)^{-1},(r+1/p_1)^{-1}}(\nabla \psi).$$
Theorem (Unconstrained best $N$-term, part 2)

- The above embedding

$$b^{\alpha + dr}_{(r+1/p_1)^{-1},(r+1/q_1)^{-1}}(\nabla \Psi) \hookrightarrow A^{r, \text{unc}}_{\infty}(b^{\alpha}_{p_1,q_1}(\nabla \Psi)) \quad (4)$$

is optimal.

- If $A^{r, \text{unc}}_{q}(b^{\alpha}_{p_1,q_1}(\nabla \Psi))$ is a space of Besov type, so it is the one in (4).

- Indeed, if $p_1 = q_1$, then

$$A^{r, \text{unc}}_{(r+1/p_1)^{-1}}(b^{\alpha}_{p_1,p_1}(\nabla \Psi)) = b^{\alpha + dr}_{(r+1/p_1)^{-1},(r+1/p_1)^{-1}}(\nabla \Psi).$$

Optimality of (4) (almost) justifies regularity investigations w.r.t. the adaptivity scale

$$B^{\alpha_\tau}_{\psi,\tau}(L_\tau(\Gamma)), \quad \frac{1}{\tau} = \frac{\alpha_\tau}{d} + \frac{1}{2}, \quad \alpha_\tau > 0,$$

for problems with errors measured in $L_2(\Gamma) = B^{0}_{\psi,2}(L_2(\Gamma))$. 
Theorem (Best $N$-term tree approximation)

Let $\alpha \in \mathbb{R}$, $0 < p_1 < \infty$, as well as $0 < q_1 \leq \infty$ be given. Moreover, fix $r > 0$ and $0 < q \leq \infty$. Then

1. $A^{r, \text{tree}}_q (b^{\alpha}_{p_1,q_1}(\nabla \psi))$ is not of Besov type,

2. for all $0 < q_0 \leq \infty$ and $-1/p_1 < \lambda < \infty$ we have

$$b^{\alpha+d\lambda}_{(\lambda+1/p_1)^{-1},q_0}(\nabla \psi) \nleftrightarrow A^{r, \text{tree}}_q (b^{\alpha}_{p_1,q_1}(\nabla \psi)),$$

3. it holds

$$b^{\alpha+dr}_{(\tilde{\lambda}+1/p_1)^{-1},\tilde{q}_0}(\nabla \psi) \nleftrightarrow b^{\alpha+dr}_{(\lambda+1/p_1)^{-1},q_0}(\nabla \psi) \nleftrightarrow A^{r, \text{tree}}_q (b^{\alpha}_{p_1,q_1}(\nabla \psi)),$$

whenever $-1/p_1 < \tilde{\lambda} \leq \lambda < r$ and $0 < \tilde{q}_0 \leq q_0 \leq \infty$ and this cannot be improved further.

Adaptivity scale is not suited for regularity analysis, but spaces close by
Summary and references
Summary

In this talk, we...

- introduced new Besov-type spaces $B_{\Psi,q}^{\alpha}(L_p(\Gamma))$ based on wavelet expansions.
- investigated sharp asymptotic rates of approximation widths w.r.t. three different strategies of approximation.
- determined optimal embeddings into corresponding approximation spaces.
Selected references

Our contribution


Our contribution


Thank you!