Besov-type function spaces on patchwise smooth manifolds and regularity analysis of operator equations in these scales

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Adaptive methods for operator equations

Operator equations...

▸ ... arise in models for various problems of modern sciences, e.g., in the context of image analysis, signal processing, acoustics, etc.

▸ ... often cannot be solved analytically.
Adaptive methods for operator equations

Operator equations...
  ▶ ... arise in models for various problems of modern sciences, e.g., in the context of image analysis, signal processing, acoustics, etc.
  ▶ ... often cannot be solved analytically.

Adaptive algorithms ...
  ▶ ... return numerical approximations to the (unknown) exact solution.
  ▶ ... use updating strategies, where additional degrees of freedom are only spent in regions where the current approximation is still “far away” from the exact solution.
  ▶ ... promise higher rates of convergence (compared to conventional, uniform approx. schemes).
  ▶ ... are hard to implement and analyze.
Adaptive wavelet Galerkin schemes …

- … use wavelets to discretize the operator equation under consideration because of their attractive analytical properties.
- … employ compression strategies to sparsify the resulting (infinite) linear system.
- … finally solve a series of finite-dimensional linear systems selected by adaptive refinement and coarsening routines (based on local estimates of the residuum).
- … return an $n$-term approximation using at most $O(n)$ arithmetic operations.

Question

Can we (theoretically) justify this complicated procedure, i.e., does adaptivity really pay off?
Outline

A motivating example
  Boundary integral equations: double layer

Wavelet bases on patchwise smooth manifolds
  Patchwise smooth manifolds
  Wavelets

New Besov-type function spaces
  Classical Besov spaces
  Definition
  Properties
  (In)dependence of the basis

Regularity analysis in Besov-type spaces
  Weighted Sobolev spaces
  A Non-standard embedding
  Besov regularity for the double layer equation

Final remarks and references
A motivating example

Boundary integral equations: double layer
Indirect methods for PDE’s in Ω (or Ω^c) like

\[ \Delta U = 0, \]

\[ U = g \quad \text{on} \quad \Gamma := \partial \Omega, \]

naturally lead to boundary integral equations such as

\[ S_{DL}(v) := \left( \frac{1}{2} \text{Id} - K \right)(v) = g \quad \text{on} \quad \Gamma. \quad (1) \]

Therein \( K \) denotes the harmonic double layer

\[ v \mapsto K(v) := \frac{1}{4\pi} \int_{\partial \Omega} v(x) \frac{\partial}{\partial \eta(x)} \frac{1}{|x - \cdot |^2} \, dS(x). \]
Indirect methods for PDE’s in $\Omega$ (or $\Omega^c$) like

$$\Delta U = 0, \quad U = g \quad \text{on} \quad \Gamma := \partial \Omega,$$

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$$v \mapsto K(v) := \frac{1}{4\pi} \int_{\partial \Omega} v(x) \frac{\partial}{\partial \eta(x)} \left| x - \cdot \right|_2 \, dS(x).$$

Advantages of such reformulations:
- bounded domain of definition $\Gamma$
- reduced dimensionality
We like to solve the *double layer eq.* (1) numerically using (adaptive) wavelet Galerkin boundary element methods.

**Question**

*Does adaptivity pay off for such boundary integral equations (or even more general operator equations on manifolds)?*
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Question

\textit{Does adaptivity pay off for such boundary integral equations (or even more general operator equations on manifolds)?}

A rule of thumb for (elliptic) PDE’s:

\begin{itemize}
  \item On smooth domains there is no need for adaptivity
  \item On (general) Lipschitz domains adaptive schemes outperform uniform schemes
\end{itemize}

because in the second case the \textit{Besov smoothness} of the solution is significantly higher than its \textit{Sobolev regularity}.
Additional challenges related to operator equations on (non-smooth) manifolds:

1. Construction of suitable wavelet bases
2. Definition and analysis of higher-order smoothness spaces
Additional challenges related to operator equations on (non-smooth) manifolds:

1.) Construction of suitable wavelet bases
2.) Definition and analysis of higher-order smoothness spaces

Possible solution:

1.) Restriction to (practically most relevant) patchwise smooth manifolds and lifting of bases from some reference domain (e.g., the unit cube) to the patches
2.) Construction of function spaces based on the decay properties of wavelet coefficients
Wavelet bases on patchwise smooth manifolds
Patchwise smooth manifolds

Here: Lipschitz surfaces $\Gamma = \partial \Omega$ which are boundaries of bounded, simply connected, closed polyhedra $\Omega \subset \mathbb{R}^3$ with finitely-many flat, quadrilateral sides and straight edges

- patchwise decomposition:
  $$\Gamma = \bigcup_{i=1}^{I} \overline{\Gamma_i}, \quad \overline{\Gamma_i} = \kappa_i([0, 1]^2)$$

- local description as boundary $\partial C_n$ of tangent cones $C_n$ subordinate to vertices $\nu_n$ of $\Omega$

(This approach naturally extends to higher dimensions $d$)
Wavelet bases on patchwise smooth manifolds

Using the parametric liftings $\kappa_i : [0, 1]^d \to \Gamma_i$ we define an (equivalent) inner product on $L_2(\Gamma)$ by

$$\langle u, v \rangle := \sum_{i=1}^{l} \langle u \circ \kappa_i, v \circ \kappa_i \rangle_{L_2([0,1]^d)}.$$ 

Then there exist several constructions of $\langle \cdot, \cdot \rangle$-biorthogonal wavelet Riesz bases $\Psi = (\Psi^\Gamma, \tilde{\Psi}^\Gamma)$ which characterize $L_2(\Gamma)$. That is, $\forall u \in L_2(\Gamma)$:

$$u = \sum_{j=0}^{\infty} \sum_{\xi \in \nabla_j^\psi} \langle u, \tilde{\psi}_j^\Gamma, \xi \rangle \psi_j^\Gamma, \xi$$

with

$$\| u \|_{L_2(\Gamma)} \sim \left[ \sum_{j=0}^{\infty} \sum_{\xi \in \nabla_j^\psi} \left| \langle u, \tilde{\psi}_j^\Gamma, \xi \rangle \right|^2 \right]^{1/2}.$$
Here the wavelets $\psi_{j,\xi}^\Gamma$ and $\widetilde{\psi}_{j,\xi}^\Gamma$ are indexed by
- $j$ ... level of resolution,
- $\xi$ ... point from a \textit{multiscale grid} $\nabla^\psi = (\nabla_j^\psi)_{j \in \mathbb{N}_0}$ for $\Gamma$.

They are constructed from wavelets on the unit cube which result from tensor products of (boundary adapted) univariate wavelets based on $D$th-order B-splines.
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They are constructed from wavelets on the unit cube which result from tensor products of (boundary adapted) univariate wavelets based on $D$th-order B-splines.

In addition, they satisfy a couple of attractive properties, e.g.

- normalization and local support
- interior vanishing moments of order $\tilde{D} \in \mathbb{N}$:

$$\langle \mathcal{P}, \tilde{\psi}^\Gamma_{j,\xi} \circ \kappa_i \rangle_{L_2([0,1]^d)} = 0 \quad \forall \mathcal{P} \in \Pi_{\tilde{D}}([0,1]^d)$$

and all $(j, \xi)$ such that $\text{supp} \tilde{\psi}^\Gamma_{j,\xi} \subset \Gamma_i$ for some $i$

- $H^s(\Gamma)$-norm equivalences for $-1/2 < s < \min\{3/2, s_{\Gamma}\}$:

$$\| u \|_{H^s(\Gamma)} \sim \left[ \sum_{j=0}^{\infty} 2^j s^2 \sum_{\xi \in \nabla_j^\psi} \left| \langle u, \tilde{\psi}^\Gamma_{j,\xi} \rangle \right|^2 \right]^{1/2}$$
Examples of such bases are given by

- Composite wavelet bases $\Psi = \Psi_{DS}$ introduced by Dahmen and Schneider (designed for general operator equations),
- Modified composite wavelets $\Psi = \Psi_{HS}$ established by Harbrecht and Stevenson (first choice in the boundary element method for integral equations)
- Bases $\Psi = \Psi_{CTU}$ due to Canuto, Tabacco, and Urban (primarily used in the wavelet element method)
New Besov-type function spaces
Classical Besov spaces

Besov spaces $B^\alpha_q(L_p(\mathbb{R}^d))$ . . .

- . . . essentially generalize Sobolev (Hilbert) spaces $H^s$.
- . . . depend on (at least) 3 parameters: $\alpha$, $p$, $q$.
- . . . are *defined* in various ways (e.g. using harmonic analysis, moduli of smoothness, interpolation, . . .).
- . . . are *characterized* by decay properties of expansion coefficients w.r.t. various building blocks (atoms, quarks, *wavelets*, . . .).
Classical Besov spaces

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Corresponding spaces can also be defined for domains and (nonsmooth) manifolds (as trace spaces or via pullbacks),

**BUT** currently there seems to exist no approach, suitable for numerical applications, to define higher-order Besov smoothness for functions on *patchwise smooth* manifolds!
New Besov-type spaces on $\Gamma$

**Definition (Dahlke, W. 2013 / W. 2014)**

- A tuple of real parameters $(\alpha, p, q)$ is called **admissible** if $0 < p < \infty$ and
  - $\alpha > d \cdot \max\{0, 1/p - 1/2\}$ and $0 < q \leq \infty$, or
  - $\alpha = d \cdot \max\{0, 1/p - 1/2\}$ and $0 < q \leq 2$.
- By $B_{\psi}^\alpha(\mathcal{L}_p(\Gamma))$ we denote the set of all $u \in L_2(\Gamma)$ s.t.

\[
\left\| u \right\|_{B_{\psi}^\alpha(\mathcal{L}_p(\Gamma))} := \left\| \left( \left\langle u, \tilde{\psi}_{j,\xi}^\Gamma \right\rangle \right)_{(j,\xi)} \right\|_{b_{p,q}^\alpha(\nabla \psi)} := \left[ \sum_{j=0}^{\infty} 2^{j(\alpha+d\left[\frac{1}{p}-\frac{1}{2}\right])} q \left( \sum_{\xi \in \nabla_j} \left\| \left\langle u, \tilde{\psi}_{j,\xi}^\Gamma \right\rangle \right\|_p \right)^{q/p} \right]^{1/q}
\]

(with the usual modifications for $q = \infty$) is finite.
Hence,

- *per definition* the function $u$ belongs to the Besov-type space $B_{\Psi,q}^\alpha(L_p(\Gamma)) \iff$ its sequence of expansion coefficients $\left(\langle u, \tilde{\psi}_{j,\xi}^\Gamma \rangle\right)_{(j,\xi) \in \nabla \Psi}$ w.r.t. the wavelet basis $\Psi = (\psi^\Gamma, \tilde{\psi}^\Gamma)$ exhibits a certain rate of decay, i.e., belongs to the sequence space

\[
b_{p,q}^\alpha(\nabla \Psi) := \left\{ a = (a_{(j,\xi)})_{(j,\xi)} \mid \| a \|_{b_{p,q}^\alpha(\nabla \Psi)} < \infty \right\}.
\]

- properties of the scale $B_{\Psi,q}^\alpha(L_p(\Gamma))$ can be derived from corresponding results for $b_{p,q}^\alpha(\nabla \Psi)$. 


Hence,

- per definition the function $u$ belongs to the Besov-type space $B^\alpha_{\Psi, q}(L_p(\Gamma)) \iff$ its sequence of expansion coefficients $(\langle u, \tilde{\psi}^\Gamma_{j, \xi} \rangle)_{(j, \xi) \in \nabla \Psi}$ w.r.t. the wavelet basis $\Psi = (\psi^\Gamma, \tilde{\psi}^\Gamma)$ exhibits a certain rate of decay, i.e., belongs to the sequence space

$$b^\alpha_{p, q}(\nabla \Psi) := \left\{ a = (a(j, \xi))_{(j, \xi)} \mid \| a \|_{b^\alpha_{p, q}(\nabla \Psi)} < \infty \right\}.$$

- properties of the scale $B^\alpha_{\Psi, q}(L_p(\Gamma))$ can be derived from corresponding results for $b^\alpha_{p, q}(\nabla \Psi)$.

Note that (in principle) our approach is applicable for every set $\Gamma$ which allows the construction of a wavelet basis for $L_2(\Gamma)$. This covers (un-)bounded domains in $\mathbb{R}^d$, as well as (non-)smooth manifolds with or without a boundary!
Simple properties of the new scale $B_{\Psi,q}^\alpha(L_p(\Gamma))$

- always quasi-Banach spaces
  (Banach $\iff$ $\min\{p, q\} \geq 1$ / Hilbert $\iff$ $p = q = 2$)

- simplified (quasi-)norms for so-called **adaptivity scale**
  $p = q = \tau := (\alpha_\tau / d + 1/2)^{-1}$, $\alpha_\tau \geq 0$

\[
\| u \|_{B_{\Psi,\tau}^{\alpha_\tau}(L_\tau(\Gamma))} = \left[ \sum_{j=0}^{\infty} \sum_{\xi \in \nabla_j \Psi} \left| \langle u, \tilde{\psi}_{j,\xi}^\Gamma \rangle \right|^\tau \right]^{1/\tau}
\]

and for Hilbert scale $p = q = 2$

\[
\| u \|_{B_{\Psi,2}^\alpha(L_2(\Gamma))} = \left[ \sum_{j=0}^{\infty} 2^j \alpha^2 \sum_{\xi \in \nabla_j \Psi} \left| \langle u, \tilde{\psi}_{j,\xi}^\Gamma \rangle \right|^2 \right]^{1/2}
\]

$\Rightarrow H^s(\Gamma) = B_{\Psi,2}^s(L_2(\Gamma))$ (equivalent norms) for all

$0 \leq s < \min\{3/2, s_\Gamma\}$, e.g. $L_2(\Gamma) = B_{\Psi,2}^0(L_2(\Gamma))$
\[ \frac{1}{\tau} = \frac{\alpha_\tau}{d} + \frac{1}{2}, \quad \alpha_\tau \geq 0 \]

\[ L_2(\Gamma) = B_{\psi,2}^0(L_2(\Gamma)) \quad \text{and} \quad H^s(\Gamma) = B_{\psi,2}^s(L_2(\Gamma)) \]
More advanced properties: Interpolation

Proposition (Dahlke, W. 2013 / W. 2014)

Let \((\alpha_0, p_0, q_0)\) and \((\alpha_1, p_1, q_1)\) denote admissible parameter tuples. For \(0 < \Theta < 1\) we set 
\[ s_\Theta := (1 - \Theta) \alpha_0 + \Theta \alpha_1, \]
\[ \frac{1}{p_\Theta} := \frac{1 - \Theta}{p_0} + \frac{\Theta}{p_1}, \]
and 
\[ \frac{1}{q_\Theta} := \frac{1 - \Theta}{q_0} + \frac{\Theta}{q_1}. \]

- If \(\alpha_0 \neq \alpha_1\) and \(p = p_0 = p_1\), then for all \(0 < q \leq \infty\) and every \(0 < \Theta < 1\) we have

\[ \left( B_{\Psi, q_0}^{\alpha_0} (L^p(\Gamma)), B_{\Psi, q_1}^{\alpha_1} (L^p(\Gamma)) \right)_{\Theta, q} = B_{\Psi, q}^{s_\Theta} (L^p(\Gamma)). \]

- If \(\min\{q_0, q_1\} < \infty\), then for all \(0 < \Theta < 1\) it holds

\[ \left[ B_{\Psi, q_0}^{\alpha_0} (L^{p_0}(\Gamma)), B_{\Psi, q_1}^{\alpha_1} (L^{p_1}(\Gamma)) \right]_{\Theta} = B_{\Psi, q_\Theta}^{s_\Theta} (L^{p_\Theta}(\Gamma)). \]
Besov-type function spaces and regularity analysis

M. Weimar

Outline

Motivation
Double layer equation

Wavelets on manifolds
Manifolds
Wavelets

New Besov-type function spaces
Classical Besov spaces
Definition
Properties
(In)dependence of the basis
Besov regularity
Weighted Sobolev spaces
Non-standard embedding
Regularity: double layer

Final remarks and references

\[
\left[ B_{\psi,q_0}(L_{p_0}(\Gamma)), B_{\psi,q_1}(L_{p_1}(\Gamma)) \right]_{\Theta} = B_{\psi,q_\Theta}(L_{p_\Theta}(\Gamma)), \quad \Theta \in (0, 1)
\]
More advanced properties II: Standard embeddings (on bounded sets $\Gamma$)

Proposition (Dahlke, W. 2013 / W. 2014)

Let $(\alpha + \gamma, p_0, q_0)$ and $(\alpha, p_1, q_1)$ denote admissible parameter tuples. Then we have the continuous embedding

$$B_{\psi, q_0}^{\alpha + \gamma}(L_{p_0}(\Gamma)) \hookrightarrow B_{\psi, q_1}^\alpha(L_{p_1}(\Gamma))$$

if and only if one of the following conditions applies:

- $\gamma > d \cdot \max\{0, \frac{1}{p_0} - \frac{1}{p_1}\}$,
- $\gamma = d \cdot \max\{0, \frac{1}{p_0} - \frac{1}{p_1}\}$ and $q_0 \leq q_1$.

Note that $\gamma < 0$ is not possible!
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More advanced properties III:
Best $n$-term approximation (on bounded sets $\Gamma$)

For $F \hookrightarrow G$ and a dictionary $\mathcal{D} = \{\varphi_1, \varphi_2, \ldots\} \subset G$ consider

$$\sigma_n(F; \mathcal{D}, G) := \sup_{\|f\|_F \leq 1} \inf_{i_1, \ldots, i_n \in \mathbb{N}} \left\| f - \sum_{m=1}^{n} c_m \varphi_{i_m} \right\|_G,$$  

$n \in \mathbb{N}_0$.
More advanced properties III: 
Best \( n \)-term approximation (on bounded sets \( \Gamma \))

For \( F \leftrightarrow G \) and a dictionary \( \mathcal{D} = \{ \varphi_1, \varphi_2, \ldots \} \subset G \) consider

\[
\sigma_n(F; \mathcal{D}, G) := \sup_{\|f|F\| \leq 1, i_1, \ldots, i_n \in \mathbb{N}} \inf_{c_1, \ldots, c_n \in \mathbb{C}} \left\| f - \sum_{m=1}^{n} c_m \varphi_{i_m} \right\|_G, \quad n \in \mathbb{N}_0
\]

Proposition (Dahlke, W. 2013 / W. 2014)

Let \((\alpha + \gamma, p_0, q_0)\) and \((\alpha, p_1, q_1)\) be admissible parameters and set \( \sigma_n := \sigma_n\left( B_{\Psi, q_0}^{\alpha+\gamma}(L^{p_0}(\Gamma)); \Psi^\Gamma, B_{\Psi, q_1}^{\alpha}(L^{p_1}(\Gamma)) \right) \). Then

- \( \gamma > d \cdot \max\left\{ 0, \frac{1}{p_0} - \frac{1}{p_1} \right\} \) implies
  \[
  \sigma_n \sim n^{-\gamma/d},
  \]

- \( \gamma = d \cdot \max\left\{ 0, \frac{1}{p_0} - \frac{1}{p_1} \right\} \) and \( q_0 \leq q_1 \) implies
  \[
  \sigma_n \sim n^{-\min\{\gamma/d, 1/q_0 - 1/q_1\}}.
  \]
approximation rate = \frac{\text{smoothness difference}}{\text{dimension}} = \frac{\gamma}{d}
(In)dependence on the basis

A (purely theoretical) drawback:

- Our Besov-type spaces $B_{\psi,q}^{\alpha}(L_p(\Gamma))$ formally depend on the wavelet basis $\Psi = (\psi^\Gamma, \tilde{\psi}^\Gamma)$ used in their definition.
Almost diagonal matrices between seq. spaces

**Definition (W. 2014)**

A matrix $\mathcal{M} = \{m_{(j,\xi),(k,\eta)}\}_{(j,\xi) \in \nabla^1, (k,\eta) \in \nabla^0}$ is called *almost diagonal* between $b_{p,q}^{\alpha_0}(\nabla^0)$ and $b_{p,q}^{\alpha_1}(\nabla^1)$ if

$$\exists \varepsilon > 0 \text{ such that } \sup_{(j,\xi) \in \nabla^1, (k,\eta) \in \nabla^0} \frac{|m_{(j,\xi),(k,\eta)}|}{\omega(j,\xi),(k,\eta)(\varepsilon)} < \infty,$$

where

$$\omega(j,\xi),(k,\eta)(\varepsilon) := 2^{k\alpha_0 - j\alpha_1} \cdot \frac{\min\{2^{-(j-k)(d/2+\varepsilon)}, 2^{(j-k)(d/2+\varepsilon+\sigma_p)}\}}{[1 + \min\{2^k, 2^j\} \text{dist}(\xi,\eta)]^{d+\varepsilon+\sigma_p}}.$$

In this case we write $\mathcal{M} \in \text{ad}\left(b_{p,q}^{\alpha_0}(\nabla^0), b_{p,q}^{\alpha_1}(\nabla^1)\right)$.

(cf. Frazier and Jawerth, Cohen et al., ... )
Change of basis

Theorem (W. 2014)

Every $\mathcal{M} \in \text{ad}\left(b_{p,q}^{\alpha_0}(\nabla^0), b_{p,q}^{\alpha_1}(\nabla^1)\right)$ induces a bounded linear operator $M: b_{p,q}^{\alpha_0}(\nabla^0) \rightarrow b_{p,q}^{\alpha_1}(\nabla^1)$.

This can be used to prove change of basis embeddings which imply that “similar” wavelet bases generate the same spaces:
Change of basis

**Theorem (W. 2014)**

Every $M \in \text{ad}(b_{p,q}^{\alpha_0}(\nabla^0), b_{p,q}^{\alpha_1}(\nabla^1))$ induces a bounded linear operator $M : b_{p,q}^{\alpha_0}(\nabla^0) \rightarrow b_{p,q}^{\alpha_1}(\nabla^1)$.

This can be used to prove *change of basis embeddings* which imply that “similar” wavelet bases generate the same spaces:

**Theorem (W. 2014)**

Assume $\Gamma$ to be patchwise smooth and let the construction parameters of the bases $\Psi, \Phi \in \{\Psi_{DS}, \Psi_{HS}, \Psi_{CTU}\}$ satisfy

$$\min\{D^\Psi, \tilde{D}^\Psi, \gamma^\Psi, \tilde{\gamma}^\Psi, D^\Phi, \tilde{D}^\Phi, \gamma^\Phi, \tilde{\gamma}^\Phi\} > d/2.$$ 

Then, for all $0 \leq \alpha < \min\{D^\Psi, D^\Phi, \gamma^\Psi, \gamma^\Phi\}$, it holds

$$B_{\Psi,q}^\alpha(L_p(\Gamma)) = B_{\Phi,q}^\alpha(L_p(\Gamma)) \text{ (equiv. quasi-norms)}.$$
Regularity analysis of operator equations in Besov-type spaces:

The double layer equation

\[ S_{DL}(v) = \left( \frac{1}{2} \text{Id} - K \right)(v) = g \quad \text{on} \quad \Gamma = \partial \Omega \]
Weighted Sobolev spaces on $\Gamma = \partial \Omega$, $\Omega \subset \mathbb{R}^3$

For $k \in \mathbb{N}$ and $0 \leq \varrho \leq k$ define

$$X^k_{\varrho}(\partial \Omega) := \frac{C^\infty_{\text{patchwise}}(\partial \Omega)}{|| \cdot ||} \cdot || X^k(\partial \Omega)||$$

with norm

$$\| u \| X^k_{\varrho}(\partial \Omega) := \sum_{n=1}^{N} \| \varphi_n u \| X^k_{\varrho}(\partial C_n)$$

such that

$$\| \delta_n^{k-\varrho} \nabla^k f \|_2 \lesssim \| f \| X^k_{\varrho}(\partial C_n) \|.$$

Therein

- $(\varphi_n)_{n=1}^{N} \ldots$ special resolution of unity on $\partial \Omega$ subordinate to vertices $\nu_n$
- $\delta_n \ldots$ distance to interfaces
- $\nabla^k \ldots$ vector of $k$th-order derivatives

(cf. Elschner, Maz’ya et al., Babuska, Kondratiev, . . . )
A non-standard embedding

**Theorem (Dahlke, W. 2013)**

*Under certain conditions there exists $s^* \in (s, 2s)$ such that*

$$B_{\psi, p}^s(L_p(\partial \Omega)) \cap X_q^k(\partial \Omega) \hookrightarrow B_{\psi, \tau}^{\alpha \tau}(L_\tau(\partial \Omega)) \quad \forall \alpha \tau \in [0, s^*].$$
Besov regularity for the double layer equation

Theorem (Dahlke, W. 2013)

Let \( s \in (0, 1), k \in \mathbb{N}, \) and \( \varrho \in (0, \min\{\varrho_0, k\}) \) for some \( \varrho_0 = \varrho_0(\partial\Omega) \in (1, 3/2). \) Moreover, let \( \alpha, \tau \) and \( \tau \) be given s.t.

\[
\frac{1}{\tau} = \frac{\alpha}{2} + \frac{1}{2} \quad \text{and} \quad 0 \leq \alpha < 2 \cdot \min\{\varrho, k - \varrho, s\}
\]

and assume that \( \tilde{\Psi}_{\partial\Omega} \) has \( \tilde{D} \geq k \) (int.) vanishing moments.

Then for every RHS \( g \in H^s(\partial\Omega) \cap X^k_{\varrho}(\partial\Omega) \)

the double layer eq. has a unique solution \( u \in B_{\Psi,\tau}^{\alpha,\tau}(L_\tau(\partial\Omega)) \).

Furthermore, if \( s' \in [0, s) \), then

\[
\sigma_n(u; H^{s'}(\partial\Omega)) \lesssim n^{-r} \quad \forall r < \left[1 - \frac{s'}{s}\right] \min\{\varrho, k - \varrho, s\}.
\]
Assume for simplicity that \( \min\{\varrho, k - \varrho\} \geq s \). Then we can take \( \alpha_\tau = 2s - \delta \) and \( r = [s - s'] - \delta \) (for all \( \delta > 0 \) small).

**Conclusion**

*Best n-term approximation rate* \( r \) (benchmark for adaptive schemes) *is twice as large* as the rate for uniform approx.!
Final remarks and references
Final remarks

In this talk, we . . .

- introduced new Besov-type spaces $B_{\Psi,q}^{\alpha}(L_p(\Gamma))$ on patchwise smooth manifolds $\Gamma$.
- recovered many typical properties (standard embed., interpolation, and best $n$-term approximation results).
- clarified the dependence on the chosen basis $\Psi = (\Psi_{\Gamma}, \tilde{\Psi}_{\Gamma})$.
- derived a regularity/approximation assertion for some boundary integral equation (double layer eq.) which is of fundamental relevance in practice.
Final remarks

In this talk, we ...  
▶ ...introduced new Besov-type spaces $B^{\alpha}_{\Psi,q}(L_p(\Gamma))$ on patchwise smooth manifolds $\Gamma$.
▶ ...recovered many typical properties (standard embed., interpolation, and best $n$-term approximation results).
▶ ...clarified the dependence on the chosen basis $\Psi = (\Psi^\Gamma, \tilde{\Psi}^\Gamma)$.
▶ ...derived a regularity/approximation assertion for some boundary integral equation (double layer eq.) which is of fundamental relevance in practice.

Work in progress:
▶ other important equations (e.g., single layer)  (✓)
▶ application to real-life problems (Helmholtz eq.)
References


...


Besov-type function spaces and regularity analysis
M. Weimar

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Thank you!