# LOVÁSZ-SAKS-SCHRIJVER IDEALS AND COORDINATE SECTIONS OF DETERMINANTAL VARIETIES

ALDO CONCA AND VOLKMAR WELKER

Abstract. Motivated by questions in algebra and combinatorics we study two ideals associated to a simple graph $G$:

- the Lovász-Saks-Schrijver ideal defining the $d$-dimensional orthogonal representations of the graph complementary to $G$ and
- the determinantal ideal of the $(d+1)$-minors of a generic symmetric with 0s in positions prescribed by the graph $G$.

In characteristic 0 these two ideals turn out to be closely related and algebraic properties such as being radical, prime or a complete intersection transfer from the Lovász-Saks-Schrijver ideal to the determinantal ideal. For Lovász-Saks-Schrijver ideals we link these properties to combinatorial properties of $G$ and show that they always hold for $d$ large enough. For specific classes of graphs, such a forests, we can give a complete picture and classify the radical, prime and complete intersection Lovász-Saks-Schrijver ideals.

## Introduction

Let $\Bbbk$ be a field, $n \geq 1$ be an integer and set $[n] := \{1, \ldots, n\}$. For a simple graph $G = ([n], E)$ with vertex set $[n]$ and edge set $E$ we study the following two classes of ideals associated to $G$.

- Lovász-Saks-Schrijver ideals:
  For an integer $d \geq 1$ we consider the polynomial ring $\Bbbk[y_{i,\ell} \mid i \in [n], \ell \in [d]]$. For every edge $e = \{i,j\} \in \binom{[n]}{2}$ we set
  $$f_e^{(d)} = \sum_{\ell=1}^{d} y_{i\ell}\, y_{j\ell}.$$
  The ideal
  $$L_G^{\Bbbk}(d) = (\, f_e^{(d)} \mid e \in E\,) \subseteq \Bbbk[y_{i,\ell} \mid i \in [n], \ell \in [d]]$$
  is called the Lovász-Saks-Schrijver ideal, LSS-ideal for short, of $G$ with respect to $\Bbbk$. The ideal $L_G^{\Bbbk}(d)$ defines the variety of orthogonal representations of the graph complementary to $G$ (see [28, 27]).
- Coordinate sections of generic (symmetric) determinantal ideals:
  Consider the polynomial ring $\Bbbk[x_{ij} \mid 1 \leq i \leq j \leq n]$ and let $X$ be the generic $n \times n$ symmetric matrix, that is, the $(i,j)$-th entry of $X$ is $x_{ij}$ if $i \leq j$ and $x_{ji}$ if $i > j$. Let $X_G^{\mathrm{sym}}$ be the matrix obtained from $X$ by replacing the entries in positions $(i,j)$ and $(j,i)$ for $\{i,j\} \in E$ with 0. For an integer $d$ let $I_d^{\Bbbk}(X_G^{\mathrm{sym}}) \subseteq \Bbbk[x_{ij} \mid 1 \leq i \leq j \leq n]$ be the ideal of $d$-minors of $X_G^{\mathrm{sym}}$. The ideal $I_d^{\Bbbk}(X_G^{\mathrm{sym}})$ defines a coordinate hyperplane section of the generic symmetric determinantal variety. Similarly, we consider ideals defining coordinate hyperplane sections of the generic determinantal varieties and the generic skew-symmetric Pfaffian varieties.

We observe in Section 7 that the ideal $I_{d+1}^{\Bbbk}(X_G^{\mathrm{sym}})$ and the ideal $L_G^{\Bbbk}(d)$ are closely related. Indeed, if $\Bbbk$ has characteristic 0 classical results from invariant theory can be employed to show that $I_{d+1}^{\Bbbk}(X_G^{\mathrm{sym}})$ is radical (resp. is prime, resp. has the expected height) provided $L_G^{\Bbbk}(d)$ is

*Date*: January 24, 2018.
2010 *Mathematics Subject Classification.* Primary .
*Key words and phrases.*
The work on this paper was partly supported by a DAAD Vigoni project and by INdAM.

radical (resp. is prime, resp. is a complete intersection). We also exhibit similar relations between variants of $L_G^{\Bbbk}(d)$ and ideals defining coordinate sections of determinantal and Pfaffian ideals.

These facts turn the focus on algebraic properties of the LSS-ideals $L_G^{\Bbbk}(d)$. In particular, we analyze the questions: when is $L_G^{\Bbbk}(d)$ a radical ideal? when is it a complete intersection? when is it a prime ideal? We show that for $d$ large enough all three properties hold for $L_G^{\Bbbk}(d)$. Among others, we are able to give necessary conditions that lead to a full classification of graphs for which $L_G^{\Bbbk}(d)$ is a complete intersection or prime in case of small $d$. In characteristic 0 we deduce sufficient conditions for $I_{d+1}^{\Bbbk}(X_G^{\mathrm{sym}})$ to be radical, prime or of expected height. To our knowledge coordinate sections of determinantal varieties have been systematically studied only in the case of maximal minors, for example the results in [5, 17, 18].

The study of the properties of LSS-ideals has its roots in the work Lovász on orthogonal graph embeddings (see [27] for references, motivation and an overview) and we think it is interesting in its own. An orthogonal embedding of a graph $G$ in $\mathbb{R}^d$ is a map $\phi : V \to \mathbb{R}^d$ where $\phi(i) \perp \phi(j)$ if $\{i, j\}$ is an edge in the edge set $\bar{E}$ of the graph $\bar{G}$ complementary to $G$. Thus by definition the real variety associated to the LSS-ideal $L_{\bar{G}}^{\mathbb{R}}(d)$ of the complementary graph $\bar{G}$ coincides with the set of all orthogonal graph embeddings of $G$ in $\mathbb{R}^d$. Note that the variety includes degenerate embeddings that are not injective or send vertices to the zero vector. Since the geometry of the variety of orthogonal graph embeddings was first studied in [28] the ideals $L_G^{\Bbbk}(d)$ carry the name Lovász-Saks-Schrijver ideals. Indeed, many of our algebraic results are inspired by results from [27, 28] about the geometry of the real variety of (general position) orthogonal embeddings.

For $d = 1, 2$ LSS-ideals are well understood objects. For $d = 1$ the LSS-ideal $L_G^{\Bbbk}(1)$ is called edge ideal of the graph $G$. As a squarefree monomial ideal it is clearly a radical ideal with respect to every field. It is prime only when $E$ is empty and a complete intersection if and only if $G$ is a matching, i.e. any two edges in $E$ have empty intersection. Starting from $d = 2$ the properties of being radical, prime and complete intersection become more subtle. For the results in this case see [22]. For $d > 2$ we know of no general results beyond the ones described in Section 2.

In Section 1.2 we generalize LSS-ideals to hypergraphs. We are able to state a few of the results from Section 2 in this generality. But most questions on hypergraph LSS-ideals remain unanswered. Nevertheless, extending the link between LSS-ideals for graphs and ideals of minors, hypergraph LSS-ideals for uniform hypergraphs can be related to the closure of the space of symmetric tensors of bounded rank with prescribed 0s in the their expansion in the standard basis (see Proposition 8.10).

The paper is organized as follows. In Section 1 we introduce basic concepts and notation from graph theory and Gröbner bases theory. Then in Section 2 we formulate our main results on LSS-ideals and sketch some of the proofs. In Section 3 we provide the proofs for the results showing persistence of the properties complete intersection and primeness. In particular, it follows that for fixed $d$ there are graphs which are minimal obstructions to these properties. In Section 4 we exhibit some of these obstructions and prove their necessity. For small $d$ we give complete characterizations of graphs with prime or complete intersection LSS-ideals. In Section 5 we define a new combinatorial invariant for graphs. We use it to prove that $L_G^{\Bbbk}(d)$ is radical, complete intersection or prime for $d$ large enough. In Section 6 we define our notation for ideals of minors and Pfaffians of generic matrices and recall classical results about their relation to invariant theory. Then in Section 7 we use this connection to invariant theory to prove that if $L_G^{\Bbbk}(d)$ is radical or prime then same property hold for the ideal of $(d+1)$ minors of generic matrices with positions of 0s prescribed by a graph $G$. In addition, we give obstructions on $G$ preventing the corresponding ideals of minors to be prime. Finally, in Section 8 we pose questions and state open problems.

## 1. Notations and generalities

1.1. **Graph and Hypergraph Theory.** In the following we introduce graph theory notation. We mostly follow the conventions from [12]. For us a graph $G = (V, E)$ is a simple graph on a finite vertex set $V$. In particular, $E$ is a subset of the set of 2-element subsets $\binom{V}{2}$ of $V$. In most of the cases we assume that $V = [n] := \{1, \ldots, n\}$. A subgraph of a graph $G = (V, E)$ is a graph $G' = (V, E')$ such that $E' \subseteq E$. More generally, a hypergraph $H = (V, E)$ is a pair consisting of a finite set of vertices $V$ and a set $E$ of subsets of $V$. We are only interested in the situation when the sets in $E$ are inclusionwise incomparable. Such a set of subsets is called a clutter. In particular, if $G = (V, E)$ is a graph then $G$ is a hypergraph and $E$ is a clutter.

For $d, m, n > 0$ we will use the following notations:

- $K_n$ denotes the complete graph on $n$ vertices $([n], \binom{[n]}{2})$,
- $K_{m,n}$ denotes the complete bipartite graph $([m] \cup [\tilde{n}], \{\{i, \tilde{j}\} \ : \ i \in [m], \tilde{j} \in [\tilde{n}] \ \}$ with bipartition $[m]$ and $[\tilde{n}] = \{\tilde{1}, \dots, \tilde{n}\}$.
- $B_n$ denotes the subgraph of $K_{n,n}$ obtained by removing the edges $\{i, \tilde{i}\}$ with $i = 1, \dots, n$.
- For $n > 2$ we denote by $C_n$ the $n$-cycle, i.e. the subgraph of $K_n$ with edges $\{1, 2\}, \{2, 3\}, \dots,$ $\{n-1, n\}, \{n, 1\}$.
- For $n > 1$ we denote by $P_n$ the path of length $n-1$, i.e. the subgraph of $K_n$ with edges $\{1, 2\}, \{2, 3\}, \dots, \{n-1, n\}$.

We denote by $\bar{G} = (V, \bar{E})$ with $\bar{E} = \binom{V}{2} \setminus E$ the graph complementary to $G = (V, E)$. Let $W \subseteq V$. We write $G_W = (W, \{e \in E \ : \ e \subseteq W\})$ for the graph induced by $G$ on vertex set $W$ and $G - W$ for the subgraph induced by $G$ on $V \setminus W$. In case $W = \{v\}$ for some $v \in V$ we simply write $G - v$ for $G - \{v\}$.

A graph $G = (V, E)$ on a vertex set $V$ of size $\geq k + 1$ is called $k$-(vertex)connected if for every $W \subset V$ with $|W| = k - 1$ the graph $G - W$ is connected. By $\deg(v) := \Big| \{e \in E \mid v \in e\} \Big|$ we denote the degree of the vertex $v$ and by $\Delta(G) = \max_{v \in V} \deg(v)$ the maximal degree of a vertex in $G$. Clearly, if $G$ is $k$-connected then every vertex has degree at least $k$ and $\Delta(\bar{G}) \leq n - k - 1$, where $n = |V|$. Finally, we denote by $\omega(G)$ the clique number of $G$, i.e. the largest $a$ such that $G$ contains a copy of a complete subgraph $K_a$.

The following well known fact follows directly from the definitions.

**Lemma 1.1.** *Given a graph $G = ([n], E)$ and an integer $1 \leq d \leq n$ the following conditions are equivalent:*

(1) *$\bar{G}$ is $(n - d)$-connected.*
(2) *$G$ does not contain a subgraph $K_{a,b}$ with $a + b = d + 1$.*

1.2. **Basics on LSS-ideals and their generalization to hypergraphs.** Let $H = (V, E)$ be a hypergraph. We define for $e \in E$

$$f_e^{(d)} = \sum_{\ell=1}^{d} \prod_{i \in e} y_{i\ell}.$$

If $E$ is a clutter we call the ideal

$$L_H^{\Bbbk}(d) = (\ f_e^{(d)} \ \mid \ e \in E\ ) \subseteq \Bbbk[y_{i,\ell} \mid i \in [n], \ell \in [d]]$$

the LSS-ideal of the hypergraph $H$.

It will sometimes be useful to consider $L_H^{\Bbbk}(d)$ as a multigraded ideal. For that we equip $\Bbbk[y_{i,\ell} \mid (i, \ell) \in [n] \times [d]]$ with the multigrading induced by $\deg(y_{i,\ell}) = \mathfrak{e}_i$ for the $i$-th unit vector $\mathfrak{e}_i$ in $\mathbb{Z}^n$ and $(i, \ell) \in V \times [d]$. Clearly, for $e \in E$ the polynomial $f_e^{(d)}$ is multigraded of degree $\sum_{i \in e} \mathfrak{e}_i$. In particular, $L_H^{\Bbbk}(d)$ is $\mathbb{Z}^n$-multigraded. The following remark is an immediate consequence of the fact that if $E$ is a clutter the two polynomials $f_e^{(d)}$ and $f_{e'}^{(d)}$ corresponding to distinct edges $e, e' \in E$ have incomparable multidegrees.

**Remark 1.2.** Let $H = ([n], E)$ be a hypergraph such that $E$ is clutter. The generators $f_e^{(d)}$, $e \in E$, of $L_H^{\Bbbk}(d)$ form a minimal system of generators. In particular, $L_H^{\Bbbk}(d)$ is a complete intersection if and only if the polynomials $f_e^{(d)}$, $e \in E$, form a regular sequence.

The following alternative description of $L_G^{\Bbbk}(d)$ for a graph $G$ turns out to be helpful in some places.

**Remark 1.3.** Let $G = ([n], E)$ be a graph. Consider the $n \times d$ matrix $Y = (y_{i,\ell})$. Then $L_G^{\Bbbk}(d)$ is the ideal generated by the entries of the matrix $YY^T$ in positions $(i, j)$ with $\{i, j\} \in E$. Here $Y^T$ denotes the transpose of $Y$.

Similarly, for a bipartite graph $G$, say a subgraph of $K_{m,n}$, one considers two sets of variables $y_{ij}$ with $(i,j) \in [m] \times [d]$, $y_{\tilde{i}j}$ with $(i,j) \in [\tilde{n}] \times [d]$ and the matrices $Y = (y_{ij})$ and $\widetilde{Y} = (y_{\tilde{i}j})$. Then $L_G^{\Bbbk}(d)$ coincides with the ideal generated by the entries of the product matrix $Y\widetilde{Y}^T$. in positions $(i,j)$ for $\{i,\tilde{j}\} \in E$

1.3. **Gröbner Bases.** We use the following notations and facts from Gröbner bases theory, see for example [4] for proofs and details. Consider the polynomial ring $\Bbbk[y_{ij} \mid (i,j) \in [n] \times [d]]$. For a vector $\mathfrak{w} = (w_{ij})_{(i,j)\in[n]\times[d]} \in \mathbb{R}^{nd}$ and a polynomial

$$f(\underline{y}) = \sum_{\alpha \in \mathbb{N}^{[n]\times[d]}} a_\alpha \underline{y}^\alpha$$

we set $m_{\mathfrak{w}}(f) = \max_{a_\alpha \neq 0}\{\alpha \cdot \mathfrak{w}\}$ and

$$\mathrm{in}_{\mathfrak{w}}(f) = \sum_{\alpha\cdot\mathfrak{w} = m_{\mathfrak{w}}(f)} a_\alpha \cdot \underline{y}^\alpha.$$

The latter is called the initial term of $f$ with respect to $\mathfrak{w}$. For a fixed term order $\prec$ and $\mathfrak{w} \in \mathbb{R}^{[n]\times[d]}$ we set $\underline{y}^\alpha \prec_\omega \underline{y}^\beta$ if and only if either $\alpha \cdot \mathfrak{w} < \beta \cdot \mathfrak{w}$ or $\alpha \cdot \mathfrak{w} = \beta \cdot \mathfrak{w}$ and $\underline{y}^\alpha \prec \underline{y}^\beta$.

The following will allows us to deduce properties of ideals from properties of their initial ideals.

**Proposition 1.4.** *Let $I$ be a homogeneous ideal in the polynomial ring $S$, $\prec$ a term order on $S$ and $\mathfrak{w} \in \mathbb{R}^{nd}$. If $\mathrm{in}_\prec(I)$ or $\mathrm{in}_{\mathfrak{w}}(I)$ is radical (resp. a complete intersection, prime) then so is $I$. Moreover, if for generators $f_1, \ldots, f_r$ of $I$ the initial terms $\mathrm{in}_\prec(f_1), \ldots, \mathrm{in}_\prec(f_r)$ (resp. $\mathrm{in}_{\mathfrak{w}}(f_1), \ldots, \mathrm{in}_{\mathfrak{w}}(f_r)$) form a regular sequence then $f_1, \ldots, f_r$ form a regular sequence and are a Gröbner basis for $I$.*

## 2. Results and counterexamples for Lovász-Saks-Schrijver ideals

For the first part of this section let $G = (V,E)$ be a graph. We start by studying radicality of $L_G^{\Bbbk}(d)$. As mentioned in the Introduction $L_G^{\Bbbk}(1)$ is always radical for trivial reasons. For $d = 2$ the following result from [22] gives a complete answer.

**Theorem 2.1** (Thm. 1.1 [22])**.** *Let $G = ([n], E)$ be a graph. If $\mathrm{char}\, \Bbbk \neq 2$ then the ideal $L_G^{\Bbbk}(2)$ is radical. If $\mathrm{char}\, \Bbbk = 2$ then $L_G^{\Bbbk}(2)$ is radical if and only if $G$ is bipartite.*

The next examples show that $L_G^{\Bbbk}(3)$ need not be radical. In the examples we assume that $\Bbbk$ has characteristic 0 but we consider it very likely that the ideals are not radical over any field.

A quick criterion implying that an ideal $J$ in a polynomial ring $S$ is not radical is to identify an element $g \in S$ such that $J : g \neq J : g^2$. We call such a $g$ a witness (of the fact that $J$ is not radical). Of course the potential witnesses must be sought among the elements that are somehow "closely related" to $J$. Alternatively, one can try to compute the radical of $J$ or even its primary decomposition directly and read off whether $J$ is radical. But these direct computations are extremely time consuming for LSS-ideals and did not terminate on our computers in the examples below. Nevertheless, in all examples we have quickly identified witnesses.

**Example 2.2.** We present three examples of graphs $G$ such that $L_G^{\Bbbk}(3)$ is not radical over any field $\Bbbk$ of characteristic 0. The first example has 6 vertices and 9 edges and it is the smallest example we have found (both in terms of edges and vertices). The second example has 7 vertices and 10 edges and it is a complete intersection. This shows that $L_G^{\Bbbk}(3)$ can be a complete intersection without being radical. The third example is bipartite, a subgraph of $K_{5,4}$ with 12 edges, and is the smallest bipartite example we have found. In all cases, since the LSS-ideal $L_G^{\Bbbk}(3)$ has integral coefficients, we may assume that $\Bbbk = \mathbb{Q}$ and exhibit a witness $g$, i.e. a polynomial $g$ such that $L_G^{\Bbbk}(3) : g \neq L_G^{\Bbbk}(3) : g^2$. The latter inequality can be checked with the help of CoCoA [1] or Macaulay 2 [19].

(1) Let $G$ be the graph with 6 vertices and 9 edges depicted in Figure 1(1), i.e. with edges

$$E = \{\{1,2\},\{1,3\},\{1,4\},\{1,5\},\{2,3\},\{2,4\},\{2,6\},\{3,5\},\{4,6\}\}.$$

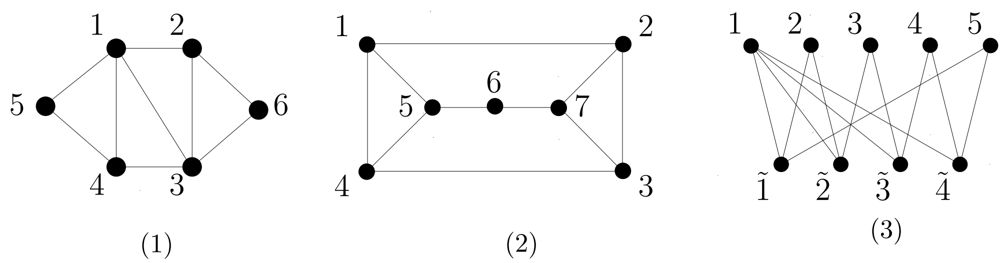


FIGURE 1. Graphs $G$ with non-radical $L_G^{\Bbbk}(3)$

Here the witness can be chosen as follows. Denote by $Y = (y_{ij})$ a generic $6 \times 3$ matrix. As discussed in Remark 1.3 the ideal $L_G^{\mathbb{Q}}(3)$ is generated by the entries of $YY^T$ corresponding to the positions in $E$. Now $g$ can be taken as the 3-minor of $Y$ with row indices $1, 5, 6$.

(2) Let $G$ be the graph with 7 vertices and 10 edges depicted in Figure 1(2), i.e. with edges
$$E = \{\{1,2\},\{1,4\},\{1,5\},\{2,3\},\{2,7\},\{3,4\},\{3,7\},\{4,5\},\{5,6\},\{6,7\}\}.$$
Here the witness can be chosen as follows. Denote by $Y = (y_{ij})$ a generic $7 \times 3$ matrix. Again as discussed in Remark 1.3 the ideal $L_G^{\mathbb{Q}}(3)$ is generated by the entries of $YY^T$ corresponding to the positions in $E$. Now $g$ can be taken as the 3-minor of $Y$ with row indices $1, 2, 4$. The fact that $L_G^{\mathbb{Q}}(3)$ is a complete intersection can be checked quickly with CoCoA [1] or Macaulay 2 [19].

(3) Let $G$ be the subgraph of the complete bipartite graph $K_{5,4}$ depicted in Figure 1(3), i.e. with edges
$$E = \{\{1,\tilde{1}\},\{1,\tilde{2}\},\{1,\tilde{3}\},\{1,\tilde{4}\},\{2,\tilde{1}\},\{2,\tilde{2}\},\{3,\tilde{2}\},\{3,\tilde{3}\},\{4,\tilde{3}\},\{4,\tilde{4}\},\{5,\tilde{1}\},\{5,\tilde{4}\}\}.$$
Denote by $X = (x_{ij})$ a generic $5 \times 3$ matrix and by $Y = (y_{ij})$ a generic $3 \times 4$ matrix. As explained in Remark 1.3 the ideal $L_G^{\mathbb{Q}}(3)$ is generated by the entries of $XY$ corresponding to the positions in $E$. Now the witness $g$ can be taken to be the 3-minor of $X$ corresponding to the column indices $1, 2, 4$.

The following result shows that the properties complete intersection and prime for $L_G^{\Bbbk}(d)$ are closely linked and persist once they occur.

**Theorem 2.3.** *Let $G = ([n], E)$ be a graph. Then:*

(1) *If $L_G^{\Bbbk}(d)$ is prime then $L_G^{\Bbbk}(d)$ is a complete intersection.*
(2) *If $L_G^{\Bbbk}(d)$ is a complete intersection then $L_G^{\Bbbk}(d+1)$ is prime.*
(3) *If $L_G^{\Bbbk}(d)$ is prime then $L_G^{\Bbbk}(d+1)$ is prime.*
(4) *If $L_G^{\Bbbk}(d)$ is a complete intersection then $L_G^{\Bbbk}(d+1)$ is a complete intersection.*
(5) *If $L_G^{\Bbbk}(d)$ is a complete intersection then $L_{G'}^{\Bbbk}(d)$ is a complete intersection for every subgraph $G'$ of $G$.*
(6) *If $L_G^{\Bbbk}(d)$ is a prime then $L_{G'}^{\Bbbk}(d)$ is prime for every subgraph $G'$ of $G$.*

The proof of the theorem consists of several steps that we first briefly sketch and then present in full detail in Section 3.

*Sketch of the proof.* To prove (1) and (2) we interpret $L_G^{\Bbbk}(d)$ as the defining ideal of the symmetric algebra of a module over the quotient of the polynomial ring by $L_{G-n}^{\Bbbk}(d)$. Then we show that the statement follows by induction on $n$ employing a result of Avramov [2, Prop. 3] characterizing complete intersection symmetric algebras and a result of Huneke [24, Thm 1.1] characterizing symmetric algebras that are domains.

To prove (3) and (4) we consider the vector $\mathfrak{v} = (\mathfrak{v}_{ij})_{(i,j)\in[n]\times[d+1]} \in \mathbb{R}^{n(d+1)}$ with entries $\mathfrak{v}_{ij} = 1$ if $(i,j) \in [n]\times[d]$ and $\mathfrak{v}_{i,d+1} = 0$ for every $i \in [n]$. Observe that $\mathrm{in}_{\mathfrak{v}}(f_e^{(d+1)}) = f_e^{(d)}$ for all $e \in E$. Therefore, $\mathrm{in}_{\mathfrak{v}}(L_G^{\Bbbk}(d+1)) \subseteq L_G^{\Bbbk}(d)$. Either $L_G^{\Bbbk}(d)$ is a complete intersection by assumption or by (1) in case the assumption is that $L_G^{\Bbbk}(d)$ is prime. This implies that $\mathrm{in}_{\mathfrak{v}}(L_G^{\Bbbk}(d+1)) = L_G^{\Bbbk}(d)$. Then the assertions follow by the transfer of properties from $\mathrm{in}_{\mathfrak{v}}(J)$ to $J$ as recalled in Proposition 1.4.

Assertion (5) is obvious. For (6) one observes that by (1) $L_G^{\Bbbk}(d)$ is also a complete intersection. It is a general fact that if a regular sequence of homogeneous polynomials generates a prime ideal then so does every subsequence. $\square$

**Remark 2.4.** There is no persistence result for the property of being radical. Indeed, we already have seen that $L_G^{\Bbbk}(1)$ is always radical and $L_G^{\Bbbk}(2)$ is always radical in case $\mathrm{char}\,\Bbbk \neq 2$. On the other hand Example 2.2 gives examples of a non-radical $L_G^{\Bbbk}(3)$. Simple examples also show that radicality is not inherited by subgraphs.

On the other hand radicality is inherited by induced subgraphs. This follows from the fact that for every subset $W \subseteq V$ one has

$$L_{G_W}^{\Bbbk}(d) = L_G^{\Bbbk}(d) \cap \Bbbk[y_{ij} : i \in W, \quad j \in [d]].$$

as can be checked using the multigraded structure.

We will now see that $L_G^{\Bbbk}(d)$ is a complete intersection and prime (and hence radical) for $d$ large enough.

We prove this fact in Section 5. Indeed, more generally we show that for $d \gg 1$ and a hypergraph $H = (V,E)$, where $E$ is a clutter, the hypergraph LSS-ideals $L_H^{\Bbbk}(d)$ is radical and a complete intersection.

As a vehicle we define a purely (hyper)graph theoretic invariant $\mathrm{pmd}(H) \in \mathbb{N}$ called the positive matching decomposition of $H$ and show the following.

**Theorem 2.5.** *Let $H = (V,E)$ be a hypergraph for a clutter $E$. Then:*

(1) *For all $d \geq \mathrm{pmd}(H)$ the ideal $L_H^{\Bbbk}(d)$ has a radical complete intersection initial ideal.*
(2) *For all $d \geq \mathrm{pmd}(H)$ the ideal $L_H^{\Bbbk}(d)$ is a radical complete intersection.*
(3) *If $G = (V,E)$ is graph then for all $d \geq \mathrm{pmd}(G) + 1$ the ideal $L_G^{\Bbbk}(d)$ is prime.*
(4) *If $G = (V,E)$ is a graph then $\mathrm{pmd}(G) \leq \min\{2|V|-3, |E|\}$ Furthermore, if $G$ is a bipartite graph then $\mathrm{pmd}(G) \leq \min\{|V|-1, |E|\}$.*

The proof of the theorem consists of several steps that we first briefly sketch and then present in full details in Section 5.

*Sketch of the proof.* We show that for every $d \geq \mathrm{pmd}(G)$ there exists a vector $\mathfrak{v} = (\mathfrak{v}_{ij})_{(i,j)\in[n]\times[d]} \in \mathbb{R}^{nd}$ such that the set $\{\mathrm{in}_{\mathfrak{v}}(f_e^{(d)}) : e \in E\}$ consists of pairwise coprime monomials. It follows that $\mathrm{in}_{\mathfrak{v}}(L_G^{\Bbbk}(d)) = (\mathrm{in}_{\mathfrak{v}}(f_e^{(d)}) : e \in E)$ and hence $\mathrm{in}_{\mathfrak{v}}(L_G^{\Bbbk}(d))$ is radical and a complete intersection. This complete the proof of (1). By Proposition 1.4 then (2) is implied. Therefore, in the graph case Theorem 2.3(2) implies (3). The claim (4) is derived by simple estimates using the combinatorial structure of the graph. $\square$

For complete graphs $G = K_n$ and $\mathrm{char}\,\Bbbk = 0$ we provide asymptotic (in terms of $n$) results on when $L_G^{K_n}(d)$ is radical, complete intersection or prime in Proposition 7.6 and Corollary 7.7 using the transfer of properties from $L_G^{K_n}(d)$ to $I_{d+1}^{\Bbbk}(X_G^{\mathrm{sym}})$ and bounds derived using Gröbner basis arguments in Corollary 5.5.

For the case of complete bipartite graphs $K_{m,n}$ results by De Concini and Strickland [13] or Musili and Seshadri [31] imply (1)-(3) of the following theorem.

**Theorem 2.6.** *Let $G = K_{m,n}$. Then:*

(1) *$L_G^{\Bbbk}(d)$ is radical for every $d$.*
(2) *$L_G^{\Bbbk}(d)$ is a complete intersection if and only if $d \geq m + n - 1$.*
(3) *$L_G^{\Bbbk}(d)$ is prime if and only if $d \geq m + n$.*
(4) $\mathrm{pmd}(G) = m + n - 1$.

*Proof.* Taking into account Remark 1.3, the assertions (1), (2), and (3) follow form general results on the variety of complexes proved from [13] and, with different techniques, from [31]. It has been observed by Tchernev [33] that the assertions in [13] that refer to the so-called Hodge algebra structure of the variety of complexes in [13] are not correct. However, those assertions can be replaced with statements concerning Gröbner bases as it is done, for example, in a similar case in [33]. Hence, (1),(2) and (3) can still be deduced from the arguments in [13]. Alternative proofs of (2) and (3) are given also in Section 4. Assertion (4) is proved in Section 5. □

Seeing Theorem 2.6(2) and (3) one may wonder if the assertion (2) in Theorem 2.3 can be reversed. The next example shows that in general this is not the case:

**Example 2.7.** Let $G = C_n$ be the $n$-cycle. If $n$ is even and $n \neq 4$ then the ideal $L_G^{\Bbbk}(d)$ is prime if and only if it is a complete intersection if and only if $d \geq 3$. These assertions are special cases of the subsequent Theorem 2.9.

In view of Theorem 2.3 for fixed $d$ and $\Bbbk$ the graphs $G$ for which $L_G^{\Bbbk}(d)$ is a complete intersection or prime define (downward) monotone graph properties. Thus by persistence there are graphs $G$ and numbers $d$ such that $L_G^{\Bbbk}(d)$ is not prime, $L_{G'}^{\Bbbk}(d)$ is prime for each proper subgraph $G'$ of $G$ and $L_G^{\Bbbk}(d+1)$ is prime. Such a pair $G$ and $d$ can be considered as a "minimal obstruction" to primeness. Similarly, we have minimal obstructions to being a complete intersection. The next results are first steps towards a classification of minimal obstructions. The results are partly inspired by theorems from Lovász's book [27, Ch. 9.4].

**Proposition 2.8.** *Let $G = ([n], E)$. Then we have:*

(1) *If $L_G^{\Bbbk}(d)$ is prime then $G$ does not contain a subgraph isomorphic to $K_{a,b}$ with $a+b = d+1$, i.e. $\bar{G}$ is $(n-d)$-connected. Furthermore, if $d > 3$ and* $\operatorname{char} \Bbbk = 0$ *then $G$ does not contain a subgraph isomorphic to $B_d$.*
(2) *If $L_G^{\Bbbk}(d)$ is a complete intersection then $G$ does not contain a subgraph isomorphic to $K_{a,b}$ with $a+b = d+2$, i.e. $\bar{G}$ is $(n-d+1)$-connected. Furthermore, if $d > 2$ and* $\operatorname{char} \Bbbk = 0$ *then $G$ does not contain a subgraph isomorphic to $B_{d+1}$.*

Further obstructions are derived from Proposition 7.6 in Corollary 7.8. For example, in characteristic 0, if $L_G^{\Bbbk}(6)$ is prime then $G$ cannot contain $K_6$. But for small values of $d$ the implications of Proposition 2.8 are actually equivalences.

**Theorem 2.9.** *Let $G$ be a graph.*

(1) *For $d \leq 3$ then the following are equivalent:*
   (a) *$L_G^{\Bbbk}(d)$ is prime.*
   (b) *$G$ does not contain a subgraph isomorphic to $K_{a,b}$ with $a+b = d+1$.*
   (c) *$\bar{G}$ is $(n-d)$-connected.*
(2) *For $d \leq 2$ the following conditions are equivalent:*
   (a) *$L_G^{\Bbbk}(d)$ is a complete intersection*
   (b) *$G$ does not contain a subgraph $K_{a,b}$ with $a+b = d+2$ and when $d = 2$ the graph $G$ does not contain an even cycle.*
   (c) *$\bar{G}$ is $(n-d+1)$-connected and when $d = 2$ the graph $G$ does not contain an even cycle.*

For forests (i.e. graphs without cycles) we can give a quite complete picture.

**Theorem 2.10.** *Let $G$ be a forest and $\Bbbk$ any field. Then:*

(1) *$L_G^{\Bbbk}(d)$ is radical for all $d$.*
(2) *$L_G^{\Bbbk}(d)$ is a complete intersection if and only if $d \geq \Delta(G)$.*
(3) *$L_G^{\Bbbk}(d)$ is prime if and only if $d \geq \Delta(G)+1$.*

The main tool in the proof of Theorem 2.10 is the notion of Cartwright-Sturmfels ideals developed in [9, 10, 11] and inspired by [7, 8]. Indeed it turns out that for a forest $G$ the ideal $L_G^{\Bbbk}(d)$ is a Cartwright-Sturmfels ideal.

## 3. Stabilization of algebraic properties of $L_G^{\Bbbk}(d)$

In this section we prove Theorem 2.3 and state some of its consequences. Before embarking in the proofs we need to recall important results on properties of the symmetric algebra of a module. We will state the results in the way that suit our needs best.

Recall that, given a ring $R$ and an $R$-module $M$ presented as the cokernel of an $R$-linear map

$$f : R^m \to R^n$$

the symmetric algebra $\mathrm{Sym}_R(M)$ of $M$ is (isomorphic to) the quotient of $\mathrm{Sym}_R(R^n) = R[x_1, \ldots, x_n]$ by the ideal $J$ generated by the entries of $A\,(x_1, \ldots, x_n)^T$ where $A$ is the $m \times n$ matrix representing $f$. Vice versa every quotient of $R[x_1, \ldots, x_n]$ by an ideal $J$ generated by homogeneous elements of degree 1 in the $x_i$'s is the symmetric algebra of an $R$-module.

Part (1) of the following is a special case of [2, Prop. 3] and part (2) a special case of [24, Thm 1.1]. Here and in the rest of the paper we denote for a matrix $A$ with entries in a ring $R$ and a number $t$ by $I_t(A)$ the ideal generated by the $t$-minors of $A$ in $R$.

**Theorem 3.1.** *Let $R$ be a complete intersection. Then*

(1) $\mathrm{Sym}_R(M)$ *is a complete intersection if and only if* height $I_t(A) \geq m - t + 1$ *for all* $t = 1, \ldots, m$.
(2) $\mathrm{Sym}_R(M)$ *is a domain and* $I_m(A) \neq 0$ *if and only if $R$ is a domain, and* height $I_t(A) \geq m - t + 2$ *for all* $t = 1, \ldots, m$.

*The equivalent conditions of (2) imply those of (1).*

**Remark 3.2.** Let $G = ([n], E)$ be a graph. The (multigraded) ideal $L_G^{\Bbbk}(d) \subseteq S = \Bbbk[y_{i,j} : i \in [n],\ \ j \in [d]]$ is generated by elements that have degree at most one in each block of variables. Hence $L_G^{\Bbbk}(d)$ can be seen as an ideal defining a symmetric algebra in various ways.

For example, set $G_1 = G - n$, $U = \{i \in [n-1] | \{i, n\} \in E\}$, $u = |U|$, $S' = \Bbbk[y_{i,j} : i \in [n-1],\ \ j \in [d]]$ and $R = S'/L_{G_1}^{\Bbbk}(d)$. Then $S/L_G^{\Bbbk}(d)$ is the symmetric algebra of the cokernel of the $R$-linear map

$$R^u \to R^d$$

associated to the $u \times d$ matrix $A = (y_{ij})$ with $i \in U$ and $j = 1, \ldots, d$.

**Remark 3.3.** In order to apply Theorem 3.1 to the case described in Remark 3.2 it is important to observe that for every $G$ no minors of the matrix $(y_{ij})_{(i,j) \in [n] \times [d]}$ vanish modulo $L_G^{\Bbbk}(d)$. This is because $L_G^{\Bbbk}(d)$ is contained in the ideal $J$ generated by the monomials $y_{ik}y_{jk}$ and the terms in the minors of $(y_{ij})$ do not belong to $J$ for obvious reasons.

We will formulate our next results in terms of the following algebraic invariants. Given an algebraic property $\mathcal{P}$ of ideals and a graph $G$ we set

$$\mathrm{asym}_{\Bbbk}(\mathcal{P}, G) = \inf\{d\ :\ L_G^{\Bbbk}(d') \text{ has property } \mathcal{P} \text{ for all } d' \geq d\}.$$

Here we are of course interested in the properties $\mathcal{P} \in \{\text{radical}, \text{c.i.}, \text{prime}\}$. Other properties of ideals such as defining a normal ring or a UFD are interesting as well but will not be treated here.

Before we use this new notation we provide the proof of Theorem 2.3(4) and (5).

*Proof of Theorem 2.3(4) and (5).* Assume $L_G^{\Bbbk}(d)$ is a complete intersection. Then each minimal generating set is a regular sequence. By Remark 1.2 the $f_e^{(d)}$ for $e \in E$ form a minimal generating set and hence a regular sequence. In particular, each subset is a regular sequence as well. From this (5) follows.

For (4) we consider the vector $\mathfrak{w}$ assigning weight 1 to all variables $y_{i\ell}$ for $(i, \ell) \in V \times [d]$ and weight 0 to all other variables. Then $\mathrm{in}_{\mathfrak{w}}(f_e^{(d')}) = f_e^{(d)}$ for every $d' \geq d$ and $e \in E$. Hence the initial forms of the generators of $L_G(d')$ form a regular sequence. It follows that the $f_e^{(d')}$, $e \in E$, form a regular sequence and hence $L_G^{\Bbbk}(d')$ is a complete intersection. $\square$

In terms of $\mathrm{asym}_{\Bbbk}(\text{c.i.}, G)$ Theorem 2.3(4) yields the following corollary.

**Corollary 3.4.** *Let $G = (V, E)$ be a graph. Then*

$$\mathrm{asym}_{\Bbbk}(\text{c.i.}, G) = \inf\{d \ : \ L_G^{\Bbbk}(d) \text{ is c.i.}\}.$$

Now we are in position to prove Theorem 2.3 (1) and (6).

*Proof of Theorem 2.3 (1) and (6).* First we show that (1) implies (6). From Remark 1.2 we know that the $f_e^{(d)}$, $e \in E$, form a minimal system of generators. Thus if $L_G^{\Bbbk}(d)$ is a complete intersection then these generators form a regular sequence. If a regular sequence generates a prime ideal in a standard graded algebra or in a local ring then so does every subset of the sequence. Now (6) follows.

To prove (1) we argue by induction on the number $n$ of vertices. As usual we assume $V = [n]$. The case $n \leq 2$ is trivial. We use the notation from Remark 3.2. Note, that $S'/L_{G_1}^{\Bbbk}(d)$ is an algebra retract of $S/L_G^{\Bbbk}(d)$. Therefore $L_{G_1}^{\Bbbk}(d) = L_G^{\Bbbk}(d) \cap S'$ and so $L_{G_1}^{\Bbbk}(d)$ is prime. By induction it follows that $L_{G_1}^{\Bbbk}(d)$ is a complete intersection. Since $u$ is the degree of the vertex $n$ in $G$ we have that $K_{1,u} \subset G$ and hence, by Proposition 2.8, $1 + u < d + 1$, i.e. $u < d$. By virtue of Remark 3.3 we have that the minors of the matrix $A$ are non-zero in $S'/L_{G_1}^{\Bbbk}(d)$. In particular, $I_u(A) \neq 0$ in $S'/L_{G_1}^{\Bbbk}(d)$ and hence (2) in Theorem 3.1 holds. Then (1) in Theorem 3.1 holds as well, i.e. $L_G^{\Bbbk}(d)$ is a complete intersection. □

As an immediate corollary of Theorem 2.3(1) we obtain.

**Corollary 3.5.** *Let $G = (V, E)$ be a graph. Then* $\mathrm{asym}_{\Bbbk}(\text{c.i.}, G) \leq \mathrm{asym}_{\Bbbk}(\text{prime}, G)$.

Before we can proceed to the proof of Theorem 2.3(2), we need another technical lemma.

**Lemma 3.6.** *Let $A$ be an $m \times n$ matrix with entries in a Noetherian ring $R$. Assume $m \leq n$. Let $S = R[x] = R[x_1, \dots, x_m]$ be a polynomial ring over $R$ and let $B$ be the $m \times (n + 1)$ matrix with entries in $S$ obtained by adding the column $(x_1, \dots, x_m)^T$ to $A$. Then we have* $\mathrm{height}\, I_1(B) = \mathrm{height}\, I_1(A) + m$ *and*

$$\mathrm{height}\, I_t(B) \geq \min\{\mathrm{height}\, I_{t-1}(A), \mathrm{height}\, I_t(A) + m - t + 1\}$$

*for all $1 < t \leq m$.*

*Proof.* Set $u = \min\{\mathrm{height}\, I_{t-1}(A), \mathrm{height}\, I_t(A)+m-t+1\}$. Let $P$ be a prime ideal of $S$ containing $I_t(B)$. We have to prove that $\mathrm{height}\, P \geq u$. If $P \supseteq I_{t-1}(A)$ then $\mathrm{height}\, P \geq \mathrm{height}\, I_{t-1}(A) \geq u$. If $P \not\supseteq I_{t-1}(A)$ then we may assume that the $(t-1)$-minor $F$ corresponding to the first $(t-1)$ rows and column of $A$ is not in $P$. Hence, $\mathrm{height}\, P = \mathrm{height}\, PR_F[x]$ and $PR_F[x]$ contains $I_t(A)R_F[x]$ and $(x_j - F^{-1}G_j : j = t, \dots, m)$ with $G_j \in R[x_1, \dots, x_{t-1}]$. Since the elements $x_j - F^{-1}G_j$ are algebraically independent over $R_F$ we have

$$\mathrm{height}\, PR_F[x] \geq \mathrm{height}\, I_t(A)R_F + (m - t + 1) \geq \mathrm{height}\, I_t(A) + (m - t + 1).$$

□

Now we turn to the proof of Theorem 2.3(2) and (3).

*Proof or Theorem 2.3 (2) and (3).* First, we show that (2) implies (3). If $L_G^{\Bbbk}(d)$ is prime then by (1) $L_G^{\Bbbk}(d)$ is a complete intersection. Now by (2) it follows that $L_G^{\Bbbk}(d+1)$ is prime. This completes the proof of (3).

For the proof of (2) we argue by induction on the number $n$ of vertices.

If $n \leq 2$ the assertion is obvious. Assume $n > 2$. Set $G_1 = G - n$, $U = \{i \in [n] | \{i, n\} \in E\}$, $u = |U|$, $Y = (y_{ij})_{(i,j) \in U \times [d+1]}$, $S = \Bbbk[y_{ij} : i \in [n],\ j \in [d + 1]]$, $S' = \Bbbk[y_{ij} : i \in [n - 1],\ j \in [d + 1]]$ and $R = S'/L_{G_1}^{\Bbbk}(d + 1)$. By construction, $S/L_G^{\Bbbk}(d + 1)$ is the symmetric algebra of the $R$-module presented as the cokernel of the map $R^u \to R^{d+1}$ associated to $Y$.

By assumption $L_G^{\Bbbk}(d)$ is a complete intersection and hence $L_{G_1}^{\Bbbk}(d)$ is a complete intersection as well. It then follows by induction that $L_{G_1}^{\Bbbk}(d + 1)$ is prime and hence $R$ is a domain. By Proposition 2.8 we have $u \leq d$ and by Remark 3.3 $I_u(Y) \neq 0$ in $R$. Therefore, by Theorem 3.1(2) we have

$$L_G^{\Bbbk}(d + 1) \text{ is prime } \Leftrightarrow \mathrm{height} I_t(Y) \geq u - t + 2 \text{ in } R \text{ for every } t = 1, \dots, u.$$

Equivalently, we have to prove that

$$\text{height}\Big(I_t(Y) + L^{\Bbbk}_{G_1}(d+1)\Big) \geq u - t + 2 + g \text{ in } S' \text{ for every } t = 1, \ldots, u$$

where $g = \text{height}\, L^{\Bbbk}_{G_1}(d+1) = |E| - u$.

Consider the weight vector $\mathfrak{w} \in \mathbb{R}^{n \times d+1}$ defined by $\mathfrak{w}_{ij} = 1$ and $\mathfrak{w}_{i\,d+1} = 0$ for all $j \in [d]$ and $i \in [n]$. By construction the initial terms of the standard generators of $\text{in}_{\mathfrak{w}}(L^{\Bbbk}_{G_1}(d+1))$ are the standard generators of $L^{\Bbbk}_{G_1}(d)$.

Since the standard generators of $I_t(Y)$ coincide with their initial terms with respect to $\text{in}_{\mathfrak{w}}$ it follows that $\text{in}_{\mathfrak{w}}(I_t(Y)) \supseteq I_t(Y)$ (indeed equality holds but we do not need this fact).

Therefore, $\text{in}_{\mathfrak{w}}(I_t(Y) + L^{\Bbbk}_{G_1}(d+1)) \supseteq I_t(Y) + L^{\Bbbk}_{G_1}(d)$ and it is enough to prove that

$$\text{height}\Big(I_t(Y) + L^{\Bbbk}_{G_1}(d)\Big) \geq u - t + 2 + g \text{ in } S' \text{ for every } t = 1, \ldots, u$$

or, equivalently,

$$\text{height} I_t(Y) \geq u - t + 2 \text{ in } R' \text{ for every } t = 1, \ldots, u$$

where $R' = S'/L^{\Bbbk}_{G_1}(d)$.

The variables $y_{1\,d+1}, \ldots, y_{n-1\,d+1}$ do not appear in the generators of $L^{\Bbbk}_{G_1}(d)$. Hence $R' = R''[y_{1\,d+1)}, \ldots, y_{n-1\,d+1}]$ with $R'' = \Bbbk[y_{ij} : (i,j) \in [n-1] \times [d]]/L^{\Bbbk}_{G-n}(d)$. Let $Y'$ be the matrix $Y$ with the $d+1$-st column removed. Then $S/L^{\Bbbk}_{G}(d)$ can be regarded as the symmetric algebra of the $R''$-module presented as the cokernel of

$$0 \to (R'')^u \quad \xrightarrow{Y'} \quad (R'')^d. \tag{1}$$

By assumption $S/L^{\Bbbk}_{G}(d)$ is a complete intersection. Hence by Theorem 3.1(1) we know

$$\text{height}\, I_t(Y') \geq u - t + 1 \text{ in } R'' \text{ for every } t = 1, \ldots, u$$

Since $Y$ is obtained from $Y'$ by adding a column of variables over $R''$ by Lemma 3.6 we have:

$$\text{height}, I_t(Y) \geq \min\{\text{height}, I_{t-1}(Y'), \text{height}, I_t(Y') + u - t + 1\} \geq u - t + 2$$

in $R'$ and for all $t = 1, \ldots, m$. □

Theorem 2.3(2) and (3) together with Corollary 3.5 directly imply the following.

**Corollary 3.7.** *Let $G = (V, E)$ be a graph.*
*Then*

$$\text{asym}_{\Bbbk}(\text{prime}, G) = \inf\{d \;:\; L^{\Bbbk}_{G}(d) \textit{ is prime}\}$$

*and*

$$\text{asym}_{\Bbbk}(\text{c.i.}, G) \leq \text{asym}(\text{prime}, G) \leq \text{asym}_{\Bbbk}(\text{c.i.}, G) + 1.$$

The following proposition is an immediate consequence of Theorem 2.3 and Theorem 2.5.

**Proposition 3.8.** *Let $G = (V, E)$ be a graph. Then*

$$\text{asym}_{\Bbbk}(\text{radical}, G) \leq 2\,|V| - 3, \quad \text{asym}_{\Bbbk}(\text{c.i.}, G) \leq 2\,|V| - 3 \textit{ and } \text{asym}_{\Bbbk}(\text{prime}, G) \leq 2\,|V| - 2.$$

*If $G$ is bipartite then*

$$\text{asym}_{\Bbbk}(\text{radical}, G) \leq |V| - 1, \quad \text{asym}_{\Bbbk}(\text{c.i.}, G) \leq |V| - 1 \textit{ and } \text{asym}_{\Bbbk}(\text{prime}, G) \leq |V|.$$

These bounds are not tight in general as the following example shows.

**Example 3.9.** Using CoCoA [1] or Macaulay 2 [19] one can check for fields $\Bbbk$ of characteristic 0 that $L^{\Bbbk}_{K_4}(2)$ is not a complete intersection while $L^{\Bbbk}_{K_4}(3)$ is. Hence by Theorem 2.3 (4) we have $\text{asym}_{\Bbbk}(\text{c.i.}, K_4) = 3$. Similarly, one can check that $L^{\Bbbk}_{K_4}(d)$ is not prime for $d = 3$ and hence Theorem 2.3 implies that $\text{asym}_{\Bbbk}(\text{prime}, K_4) = 4$. Finally, one checks that $L^{\Bbbk}_{K_4}(d)$ is radical for all $1 \leq d \leq 3$ and hence $\text{asym}_{\Bbbk}(\text{radical}, K_4) = 1$.

In Corollary 7.7 we will be able to analyze the asymptotic behavior of $\text{asym}_{\Bbbk}(\text{prime}, K_n)$ for $n \to \infty$.

## 4. Obstructions to algebraic properties

In this section we prove Theorem 2.9 and study necessary and sufficient conditions for $L_G^{\Bbbk}(d)$ to be radical, complete intersections or prime. First, we turn to necessary conditions which yield lower bounds on $\mathrm{asym}_{\Bbbk}(\mathrm{radical}, G)$, $\mathrm{asym}_{\Bbbk}(\mathrm{prime}, G)$ and $\mathrm{asym}_{\Bbbk}(\mathrm{c.i.}, G)$.

We start with the proof of Proposition 2.8.

*Proof of Proposition 2.8.* First, we show that (1) implies (2). If $L_G^{\Bbbk}(d)$ is complete intersection then by Theorem 2.3 (2) it follows that $L_G^{\Bbbk}(d+1)$ is prime. Hence if $G$ contains $K_{a,b}$ with $a+b > d+1$ or $B_{d+1}$ (in case $\mathrm{char}\,\Bbbk = 0$ and $d > 2$) then $G$ violates the conditions from (1) for primeness of $L_G^{\Bbbk}(d+1)$. This implies (2).

For (1) we first prove that if $L_G^{\Bbbk}(d)$ is prime then $G$ does not contain $K_{a,b}$ for $a+b > d$. Suppose for contradiction that $L_G^{\Bbbk}(d)$ is prime and $G$ contains $K_{a,b}$ for some $a+b > d$. We may decrease either $a$ or $b$ or both and assume right away that $a+b = d+1$, $a, b \geq 1$. In particular $a, b \leq d$.

We assume that $V = [n]$ and that $K_{a,b}$ is a subgraph of $G$, where after renaming vertices here $K_{a,b}$ is the complete bipartite graph with edges $\{i, a+j\}$ for $i \in [a]$ and $j \in [b]$. Set $R = S/L_G^{\Bbbk}(d)$ and $Y = (y_{i\,\ell}) \in R^{a \times d}$ and $Z = (z_{\ell,i}) \in R^{d \times b}$ with $z_{\ell,i} = y_{i+a,\ell}$. Since $K_{a,b}$ is a subgraph of $G$ we have $YZ = 0$ in $R$

By assumption $R$ is a domain and $YZ = 0$ can be seen as a matrix identity over the field of fractions of $R$.

Hence

$$\mathrm{rank}(Y) + \mathrm{rank}(Z) \leq d.$$

From $a+b = d+1$ it follows that $\mathrm{rank}(Y) < a$ or $\mathrm{rank}(Z) < b$. This implies that any $a$-minor of $Y$ or any $b$-minor of $Z$ is zero, i.e., $I_a(Y) = 0$ or $I_b(Z) = 0$ as ideals of $R$.

But by Remark 3.3 none of the minors of $Y$ and $Z$ are in $L_G^{\Bbbk}(d)$. This is a contradiction and hence $L_G^{\Bbbk}(d)$ is not prime.

It remains to be shown that if $\mathrm{char}\,\Bbbk = 0$ and $d > 3$ then $G$ does not contain a copy of $B_d$.

Here, we unfortunately have to resort to Proposition 7.4 and Lemma 7.13(iii) from Section 7. But it is easily seen that its derivation is independent of results from preceding sections.

By Theorem 2.3(6) we know that if $L_G^{\Bbbk}(d)$ is prime then so is $L_{B_d}^{\Bbbk}(d)$. Then Proposition 7.4 implies that $I_{d+1}(X_G^{\mathrm{gen}})$ is prime for a generic matrix $X$ of arbitrary size and this contradicts Lemma 7.13(iii). □

Next we provide the proof of Theorem 2.9.

*Proof of Theorem 2.9 part (1).* By Lemma 1.1 conditions (b) and (c) in Theorem 2.9(1) are equivalent. Hence it suffices to prove the equivalence of (a) and (b).

For $d = 1$ the statements are obvious: $L_G^{\Bbbk}(1)$ is prime if and only if $G$ has no edges set which is equivalent to not containing $K_{11}$.

For $d = 2$ we know by Proposition 2.8 that (a) implies (b). When (b) holds then the edges of $G$ are pairwise disjoint. It follows that the monomial ideal $L_G^{\Bbbk}(1)$ is a complete intersection. Then by Theorem 2.3 (2) assertion (a) follows.

For $d = 3$ again by Proposition 2.8 condition (a) implies (b). To prove that (b) implies (a) we may assume that $\Bbbk$ is algebraically closed. Then, since the tensor product over $\Bbbk$ of $\Bbbk$-algebras that are domains is a domain (see Corollary to Proposition 1 in Bourbaki's Algebra [6, Chapter v, 17] or [30, Prop. 5.17]) we may assume that the graph is connected. A connected graph satisfying (b) is either an isolated vertex or a path $P_n$ with $n \geq 2$ vertices or a cycle $C_n$ of length $n \neq 4$. Hence we have to prove that $L_G^{\Bbbk}(3)$ is prime when $G = P_n$ with $n \geq 2$ or $G = C_n$ with $n \geq 3$ and $n \neq 4$. If $G = P_n$ then $\mathrm{pmd}(P_n) \geq 2$ (indeed $\mathrm{pmd}(P_n) \geq 2$ for $n > 2$). This can be seen easily form the definition or by using Lemma 5.2(iv) to check that a maximal matching and its complement form a positive matching decomposition. Hence by Theorem 2.5(3) it follows that $L_{P_n}^{\Bbbk}(3)$ is prime.

Now let $G = C_n$ be the $n$-cycle for $n \geq 3$ and $n \neq 4$ and set $m = n-1$. To prove that $L_{C_n}^{\Bbbk}(3)$ is prime we use the symmetric algebra perspective. Observe that $C_n - n$ is $P_m$ for $m = n-1$. Set $J = L_{P_m}^{\Bbbk}(3)$, $S = \Bbbk[y_{ij} : i \in [m] \ \ j \in [3]]$ and $R = S/J$. We have already proved that $J$ is a prime complete intersection of height $m-1$. We have to prove that the symmetric algebra of the cokernel of the $R$-linear map:

$$R^2 \xrightarrow{Y} R^3 \text{ with } Y = \begin{pmatrix} y_{11} & y_{12} & y_{13} \\ y_{m1} & y_{m2} & y_{m3} \end{pmatrix}$$

is a domain. Since by Remark 3.3 $I_2(Y) \neq 0$ in $R$, taking into consideration Remark 3.2 we may apply Theorem 3.1. Therefore, it is enough to prove that

$$\operatorname{height} I_1(Y) \geq 3 \text{ and } \operatorname{height} I_2(Y) \geq 2 \text{ in } R.$$

Equivalently, it is enough to prove that

$$\begin{aligned} \operatorname{height} I_1(Y) + J &\geq m+2 \text{ and} \qquad (2)\\ \operatorname{height} I_2(Y) + J &\geq m+1 \text{ in } S. \qquad (3) \end{aligned}$$

We prove first (2). Since $\operatorname{height} I_1(Y) = 6$ in $S$ then (2) is obvious for $m \leq 4$. For $m > 4$ observe that $I_1(Y) + J$ can be written as $I_1(Y) + H$ where $H$ is the LSS-ideal of the path of length $m-2$ on vertices $2, 3, \ldots, m-1$. Because $I_1(Y)$ and $H$ use disjoint set of variables, we have

$$\operatorname{height} I_1(Y) + H = 6 + m - 3 = m + 3$$

and this proves (2). Now we note, en passant, that the condition $\operatorname{height} I_2(Y) \geq 1$ holds in $R$ because $R$ is a domain and $I_2(Y) \neq 0$. Hence we deduce from Theorem 3.1(1) that $L^{\Bbbk}_{C_n}(3)$ is a complete intersection for all $n \geq 3$.

It remains to prove (3). Since $\operatorname{height} I_2(Y) = 3$ in $S$ the assertion in obvious for $m = 2$, i.e. $n = 3$. Hence we may assume $m \geq 4$ (here we use $n \neq 4$). Then let $P$ be a prime ideal of $S$ containing $I_2(Y) + J$. We have to prove that $\operatorname{height} P \geq m+1$. If $P$ contains $I_1(Y)$ then $\operatorname{height} P \geq m+2$ by (2). So we may assume that $P$ does not contain $I_1(Y)$, say $y_{11} \notin P$, and prove that $\operatorname{height} PS_x \geq m+1$ where $x = y_{11}$. Since $I_2(Y)S_x = (y_{m2} - x^{-1}y_{m1}y_{12}, y_{m3} - x^{-1}y_{m1}y_{13})$ we have

$$\begin{aligned} f^{(3)}_{m-1,m} &= y_{m-1,1}y_{m1} + y_{m-1,2}y_{m2} + y_{m-1,3}y_{m3} \\ &= y_{m-1,1}y_{m1} + y_{m-1,2}x^{-1}y_{m1}y_{12} + y_{m-1,3}x^{-1}y_{m1}y_{13} \\ &= x^{-1}y_{m1}f^{(3)}_{1,m-1} \mod I_2(Y)S_x \end{aligned}$$

Since $f^{(3)}_{m-1,m} \in J$, we have $y_{m1}f^{(3)}_{1,m-1} \in PS_x$. Then we have that either $y_{m1} \in PS_x$ or $f^{(3)}_{1,m-1} \in PS_x$. In the first case $PS_x$ contains $y_{m1}, y_{m2}, y_{m3}$ and the LSS-ideal associated to the path of length $m-2$ with vertices $1, \ldots, m-1$. Hence $\operatorname{height} PS_x \geq 3 + m - 2 = m + 1$ as desired. Finally, if $f^{(3)}_{1,m-1} \in PS_x$ we have that $PS_x$ contains the ideal $L^{\Bbbk}_{C_{m-1}}(3)$ associated to the $(m-1)$-cycle with vertices $1, \ldots, m-1$ and we have already observed that this ideal is a complete intersection. Since $y_{m2} - x^{-1}y_{m1}y_{12}, y_{m3} - x^{-1}y_{m1}y_{13}$ are in $PS_x$ as well it follows that $\operatorname{height} PS_x \geq 2 + m - 1 = m + 1$. □

*Proof of Theorem 2.9 part (2).* By Lemma 1.1 conditions (b) and (c) in Theorem 2.9 (2) are equivalent. Hence it suffices to prove the equivalence of (a) and (b). We prove first that (a) implies (b). By Proposition 2.8 that (a) implies that $G$ does not contain $K_{a,b}$ with $a+b = d+1$. Suppose then by contradiction that $G$ does contain $C_{2m}$. Hence $L^{\Bbbk}_{C_{2m}}(2)$ is a complete intersection of height $2m$. But the generators of $L^{\Bbbk}_{C_{2m}}(2)$ are (up to sign) among the 2-minors of the matrix:

$$\begin{pmatrix} y_{11} & -y_{22} & y_{31} & \cdots & y_{2m-1,1} & -y_{2m,2} \\ y_{12} & y_{21} & y_{32} & \cdots & y_{2m-1,2} & y_{2m,1} \end{pmatrix}$$

and the ideal of 2-minors of such a matrix has height $2m-1$, a contradiction.

Now we prove that (b) implies (a). We may assume that $\Bbbk$ is algebraically closed. Since the tensor product over a perfect field $\Bbbk$ of reduced $\Bbbk$-algebras is reduced [6, Thm 3, Chapter V, 15], we may assume that $G$ is connected. A connected graph satisfying (b) is either an isolated vertex, or a path or an odd cycle. We have already observed that $\operatorname{pmd}(P_n) \leq 2$. By Theorem 2.5 it follows that $L^{\Bbbk}_{P_n}(2)$ is a complete intersection. It remains to prove that $L^{\Bbbk}_{C_{2m+1}}(2)$ is a complete intersection (of height $2m+1$). Note that $L^{\Bbbk}_{P_{2m+1}}(2) \subset L^{\Bbbk}_{C_{2m+1}}(2)$ and we know already that $L^{\Bbbk}_{P_{2m+1}}(2)$ is a complete intersection of height $2m$. Hence it remains to prove that $f^{(2)}_{1,2m+1}$ does not belong to any minimal prime of $L^{\Bbbk}_{P_{2n+1}}(2)$. The generators of $L^{\Bbbk}_{P_{2n+1}}(2)$ are (up to sign) the adjacent 2-minors of the matrix:

$$Y = \begin{pmatrix} y_{11} & -y_{22} & y_{31} & \cdots & y_{2m-1,1} & -y_{2m,2} & y_{2m+1,1} \\ y_{12} & y_{21} & y_{32} & \cdots & y_{2m-1,2} & y_{2m,1} & y_{2m+1,2} \end{pmatrix}$$

The minimal primes of $L^{\Bbbk}_{P_{2n+1}}(2)$ are described in the proof of [15, Thm.4.3], see also [23] and [21]. By the description given in [15] it is easy to see that all minimal primes of $L^{\Bbbk}_{P_{2n+1}}(2)$ with the exception of $I_2(Y)$ are contained in the ideal $Q = (y_{ij} : 2 < i < 2m+1 \ \ 1 \leq j \leq 2)$. Clearly $f^{(2)}_{1,2m+1} \notin Q$. Finally one has $f^{(2)}_{1,2m+1} \notin I_2(Y)$ since the monomial $y_{11}y_{2m+1,1}$ is divisible by no monomials in the support of the generators of $I_2(Y)$. □

We proceed with the proof of Theorem 2.10. For the proof we first formulate a result that is a special case of a more general statement. For this we need to introduce the concept of Sturmfels-Cartwright ideals. This concept was coined in [9] inspired by earlier work in [8] and [7]. It was further developed and applied to various classes of ideals in [10] and [11].

Consider for $d_1, \dots, d_n \geq 1$ the polynomial ring $S = \Bbbk[y_{ij} : i \in [n], j \in [d_i]]$ with multigrading $\deg y_{ij} = \mathfrak{e}_i \in \mathbb{Z}^n$. The group $G = \mathrm{GL}_{d_1}(\Bbbk) \times \cdots \times \mathrm{GL}_{d_n}(\Bbbk)$ acts naturally on $S$ as the group of $\mathbb{Z}^n$-graded $K$-algebra automorphism. The Borel subgroup of $G$ is $B = U_{d_1}(\Bbbk) \times \cdots \times U_{d_n}(\Bbbk)$ where $U_d(\Bbbk)$ denotes the subgroup of $\mathrm{GL}_d(\Bbbk)$ of the upper triangular matrices. A $\mathbb{Z}^n$-graded ideal $J$ is Borel fixed if $g(J) = J$ for every $g \in B$. A $\mathbb{Z}^n$-graded ideal $I$ of $S$ is called a Cartwright-Sturmfels ideal if there exists a radical Borel fixed ideal $J$ with the same multigraded Hilbert-series.

**Theorem 4.1.** *For $d_1, \dots, d_n \geq 1$ let $S = \Bbbk[y_{ij} : i \in [n], j \in [d_i]]$ be the polynomial ring with $\mathbb{Z}^n$ multigrading induced by $\deg y_{ij} = \mathfrak{e}_i \in \mathbb{Z}^n$. and $G = (V, E)$ be a forest (i.e. a graph without cycles). For each $e = \{i, j\} \in E$ let $f_e \in S$ be a $\mathbb{Z}^n$-graded polynomial of degree $\mathfrak{e}_i + \mathfrak{e}_j$. Then $I = (f_e \ : \ e \in E)$ is a Cartwright-Sturmfels ideal. In particular, $I$ and all its initial ideals are radical.*

*Proof.* First, we observe that we may assume that the generators $f_e$ of $I$ form a regular sequence. To this end we introduce new variables and add to each $f_e$ a monomial $m_e$ in the new variables of degree $e$ so that $m_e$ and $m_{e'}$ are coprime if $e \neq e'$. The new polynomials $f_e + m_e$ with $e \in E$ form a regular sequence by Proposition 1.4 since their initial terms with respect to an appropriate term order are the pairwise coprime monomials $m_e$. The ideal $I$ arises as a multigraded linear section of the ideal $(f_e + m_e : e \in E)$ by setting all new variables to 0. By [8, Thm. 1.16(5)] the family of Cartwright-Sturmfels ideals is closed under any multigraded linear section. Hence it is enough to prove the statement for the ideal $(f_e + m_e : e \in E)$. Equivalently we may assume right away that the generators $f_e$ of $I$ form a regular sequences.

The multigraded Hilbert series of a multigraded $S$-module $M$ can by written as

$$\frac{K_M(z_1, \dots, z_n)}{\prod_{i=1}^n (1 - z_i)^{d_i}}.$$

The numerator $K_M(z_1, \dots, z_n)$ is a Laurent polynomial polynomial with integral coefficients called the $K$-polynomial of $M$. Since the $f_e$'s form a regular sequence the $K$-polynomial of $S/I$ is the polynomial:

$$F(z) = F(z_1, \dots, z_n) = \prod_{\{i,j\} \in E} (1 - z_i z_j) \in \mathbb{Q}[z_1, \dots, z_n].$$

To prove that $I$ is Cartwright-Sturmfels we have to prove that there is a Borel-fixed radical ideal $J$ such that the $K$-polynomial of $S/J$ is $F(z)$. Taking into consideration the duality between Cartwright-Sturmfels ideals and Cartwright-Sturmfels* ideals discussed in [9], it is enough to exhibit a monomial ideal $J$ whose generators are in the polynomial ring $S' = \Bbbk[y_1, y_2, \dots, y_n]$

equipped with the (fine) $\mathbb{Z}^n$-grading $\deg y_i = \mathfrak{e}_i \in \mathbb{Z}^n$ such that the $K$-polynomial of $J$ regarded as an $S'$-module is $F(1-z_1,\dots,1-z_n)$, that is,

$$\prod_{\{i,j\}\in E}(z_i+z_j-z_iz_j).$$

We claim that, under the assumption that $([n],E)$ is a forest, the ideal

$$J=\prod_{\{i,j\}\in E}(y_i,y_j)$$

has the desired property. In other words, we have to prove that the tensor product

$$T_E=\bigotimes_{\{i,j\}\in E}T_{\{i,j\}}$$

of the truncated Koszul complexes:

$$T_{\{i,j\}}:0\to S'(-\mathfrak{e}_i-\mathfrak{e}_j)\to S'(-\mathfrak{e}_i)\oplus S'(-\mathfrak{e}_j)\to 0$$

associated to $y_i, y_j$ resolves the ideal $J$. Consider a leaf $\{a,b\}$ of $E$. Set $E'=E\setminus\{\{a,b\}\}$,

$$J'=\prod_{\{i,j\}\in E'}(y_i,y_j)$$

and $J''=(y_a,y_b)$. Then by induction on the number of edges we have that $T_{E'}$ resolves the ideal $J'$. Then the homology of $T_E$ is $\mathrm{Tor}^{S'}_{*}(J',J'')$. Since since $\{a,b\}$ is a leaf one of the two variables $y_a,y_b$ does not appear at all in the generators of $J'$. Hence $y_a,y_b$ forms a regular $J'$-sequence. Then $\mathrm{Tor}^{S'}_{\geq 1}(J',J'')=0$ and hence $T_E$ resolves $J'\otimes J''$. Finally, $J'\otimes J''=J'J''$ since $\mathrm{Tor}^{S'}_{1}(J',S/J'')=0$. This concludes the proof that the ideal $I$ is a Cartwright-Sturmfels ideal. Every initial ideal of a Cartwright-Sturmfels ideal is a Cartwright-Sturmfels ideal as well because the property just depends on the Hilbert series. In particular, every initial ideal of a Cartwright-Sturmfels ideal is radical. □

Now we are ready to prove Theorem 2.10:

*Proof of Theorem 2.10.* (1) Setting $d_1=\cdots=d_n=d$ and $f_e=f_e^{(d)}$ in Theorem 4.1 we have that $L^{\Bbbk}_G(d)$ is a Cartwright-Sturmfels ideal and hence radical.

(2) If $L^{\Bbbk}_G(d)$ is a complete intersection then, by Proposition 2.8, $G$ does not contain a copy of $K_{1,d+1}$ as a subgraph, that is, $\Delta(G)\leq d$. Vice versa if $d\geq\Delta(G)$ one proves that $L^{\Bbbk}_G(d)$ is a complete intersection by using induction on the number of vertices of $G$ and the symmetric algebra point of view. As $G$ is a forest we may assume that $\{n-1,n\}$ is a leaf. Then $G_1=G-n$ is a forest with $\Delta(G_1)\leq\Delta(G)\leq d$. Hence by induction $L^{\Bbbk}_{G_1}(d)$ is a complete intersection. Set

$$R=\Bbbk[y_{i,j}:i\in[n-1],j\in[d]]/L^{\Bbbk}_{G_1}(d).$$

We may interpret $L^{\Bbbk}_G(d)$ as the ideal defining the symmetric algebra of the $R$-module defined as the cokernel of the $R$-linear map $R\to R^d$ associated to the matrix $A=(y_{n-1,1},\dots,y_{n-1,d})$. Hence, by virtue of Theorem 3.1(1), it is enough to prove that

$$\text{height}\,(y_{n-1,1},\dots,y_{n-1,d})\geq 1 \text{ in } R$$

equivalently,

$$\text{height}\,(y_{n-1,1},\dots,y_{n-1,d})+L^{\Bbbk}_{G_1}(d)>\text{height}\,L^{\Bbbk}_{G_1}(d)$$

which is true because at most $\Delta(G)-1$ of the generators of $L^{\Bbbk}_{G_1}(d)$ are contained in $(y_{n-1,1},\dots,y_{n-1,d})$.

(3) If $L^{\Bbbk}_G(d)$ is prime then, by Proposition 2.8, $G$ does not contain a copy of $K_{1,d}$ as a subgraph, that is, $\Delta(G)\leq d-1$. Vice versa, we know by (2) that $L^{\Bbbk}_G(d)$ is a complete intersection for $d\geq\Delta(G)$. Hence by Theorem 2.3 $L^{\Bbbk}_G(d)$ is prime for $d\geq\Delta(G)+1$.

□

Hence for a forest $G$ we have a complete picture of asymptotic behaviour:

**Corollary 4.2.** *Let $G$ be a forest and $\Bbbk$ any field. Then:* $\operatorname{asym}_{\Bbbk}(\text{radical}, G) = 1$, $\operatorname{asym}_{\Bbbk}(\text{c.i.}, G) = \Delta(G)$ *and* $\operatorname{asym}_{\Bbbk}(\text{prime}, G) = \Delta(G) + 1$.

## 5. Positive matching decompositions

In this section we introduce the concepts of positive matchings and positive matching decomposition and prove Theorem 2.5.

**Definition 5.1.** Given a hypergraph $H = (V, E)$ a positive matching of $G$ is a subset $M \subset E$ of pairwise disjoint sets (i.e., a matching) such that there exists a weight function $w : V \to \mathbb{R}$ satisfying:

$$\begin{aligned} \sum_{i \in A} w(i) > 0 \text{ if } \quad & A \in M \\ \sum_{i \in A} w(i) < 0 \text{ if } \quad & A \in E \setminus M. \end{aligned} \tag{4}$$

We illustrate the definition for subgraphs of $K_4$. The edge set $M = \{\{1, 2\}, \{3, 4\}\}$ is a matching in $K_4$ but it is not positive. Nevertheless, the matching $M$ is positive in the graph with edge set $\{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{3, 4\}\}$ with respect to the weight function $w(1) = 0, w(2) = 1, w(3) = -2$ and $w(4) = 3$.

The next lemma summarizes some elementary properties of positive matchings.

**Lemma 5.2.** *Let $H = (V, E)$ be a hypergraph such that $E$ is a clutter, $M \subseteq E$ and $V_M = \bigcup_{A \in M} A$.*

(i) *$M$ is a positive matching for $H$ if and only if $M$ is a positive matching for the induced hypergraph $H_{V_M} = (V_M, \{A \in E \mid A \subseteq V_M\})$.*
(ii) *If $M$ is a positive matching on $H$ and $A \in E$ is such that $M \cup \{A\}$ is a matching and there is no $B \in M$ with $B \cap V_M \neq \emptyset \neq B \cap A$ then $M \cup \{A\}$ is a positive matching on $H$.*
(iii) *If $H$ is a bipartite graph with bipartition $V = V_1 \cup V_2$ then $M$ is a positive matching if and only if $M$ is a matching and directing the edges $e \in E$ from $V_1$ to $V_2$ if $e \in M$ and from $V_2$ to $V_1$ if $e \in E \setminus M$ yields an acyclic orientation.*

*Proof.* (i) Clearly a weight function on $V$ for which $M$ is a positive matching restricts to $V_M$ making $M$ a positive matching on $H_{V_M}$. Conversely, setting $w(i) = -\max\{|A| : A \in M\} \cdot \max\{w(j) : j \in V_M\}$ for $i \in V \setminus V_M$ extends a weight function on $V_M$ to $V$ and shows that $M$ is a positive matching on $H$.

(ii) Let $w$ we a weight function for which $M$ is positive. Set $T = \max\{1, w(\ell) : \ell \in V_M\} > 0$. We define $w' : V \to \mathbb{R}$ as follow. If $\ell \in V_M$ we set $w'(\ell) = w(\ell)$. If $\ell \in A$ we set $w'(\ell) = T$ and for $\ell \in V \setminus (V_M \cap A)$ we set $w'(\ell) = -|\max\{|A|, |B| : B \in M\} \cdot T - 1$. One easily checks that $M \cup \{A\}$ is a positive matching with respect to $w'$.

(iii) We change the coordinates $w(i)$ to $-w(i)$ for $i \in V_2$ in the inequalities defining a positive matchings. As a simple reformulation of (4) we get that in these coordinates a matching $M$ is positive if and only if there is a weight function such that for $\{i, j\} \in E$, $i \in V_1$, $j \in V_2$ we have

$$\begin{aligned} w(i) > w(j) \text{ if } \quad & \{i, j\} \in M, \\ w(i) < w(j) \text{ if } \quad & \{i.j\} \in E \setminus M. \end{aligned} \tag{5}$$

This is equivalent to the existence of a region in the arrangement of hyperplanes $w(i) = w(j)$ for $\{i, j\} \in E$ in $\mathbb{R}^V$ satisfying (5). But it is well known that the regions in this arrangement are in one to one correspondence with the acyclic orientations of $G$ (see [20, Lemma 7.1]).

$\square$

Now we are in position to introduce the key concept of this section.

**Definition 5.3.** Let $H = (V, E)$ be a hypergraph for which $E$ is a clutter. A positive matching decomposition (or pm-decomposition) of $G$ is a partition $E = \bigcup_{i=1}^{p} E_i$ into pairwise disjoint subsets such that $E_i$ is a positive matching on $(V, E \setminus \cup_{j=1}^{i-1} E_j)$ for $i = 1, \ldots, p$. The $E_i$ are called the parts of the pm-decomposition. The smallest $p$ for which $G$ admits a pm-decomposition with $p$ parts will be denoted by $\mathrm{pmd}(\mathcal{H})$.

Note that by definition one has $\mathrm{pmd}(H) \leq |E|$ because of the obvious pm-decomposition $\bigcup_{A \in E}\{A\}$. On the other hand $\mathrm{pmd}(G)$ is smaller than $|E|$ for most clutters. Next we prove the bound from Theorem 2.5(4).

*Proof of Theorem 2.5(4).* First, consider the case when $G = (V, E)$ is an arbitrary graph. Set $n = |V|$. We may assume that $G$ is the complete graph $K_n$ because any pm-decomposition of $K_n$ induces a pm-decomposition on its subgraphs. For $\ell = 1, \ldots, 2n-3$ we set $E_\ell = \{\{i, j\} : i + j = \ell + 2\}$. For example, for $n = 7$ we have

$$\begin{array}{llll} E_1 = \{12\}, & E_2 = \{13\}, & E_3 = \{14, 23\}, & E_4 = \{15, 24\}, \\ E_5 = \{16, 25, 34\}, & E_6 = \{17, 26, 35\}, & E_7 = \{27, 36, 45\}, & E_8 = \{37, 46\}, \\ E_9 = \{47, 56\}, & E_{10} = \{57\}, & E_{11} = \{67\} & \end{array}$$

where for simplicity we have written $ij$ for the edge $\{i, j\}$.

Clearly one has $E = \cup_{\ell=1}^{2n-3} E_\ell$ and this is a pm-decomposition of $K_n$ since when a new edge is inserted the smallest index involved in that edge satisfies the condition of Lemma 5.2 (ii) with respect to the current data. For example for $n = 7$ when we insert the edge 27 in $E_7$ the vertex 2 satisfies the condition of Lemma 5.2 (ii) with respect to the matching $\{36, 45\}$, i.e. the edges $23, 24, 25, 26$ are already used in an earlier step of the construction.

Now consider the case when $G = (V, E)$ is a bipartite graph. Let $V = V_1 \cup V_2$ be a the bipartition for $G$. We may assume $V_1 = [m]$ and $V_2 = [\tilde{n}] = \{\tilde{1}, \ldots, \tilde{n}\}$ for numbers $m, n \geq 1$ and we may assume that $G = K_{m,n}$. We show that $E = \bigcup_{\ell=1}^{p} E_\ell$ with $p = m + n - 1$ and $E_\ell = \{\{i, \tilde{j}\} : \ i + j = \ell + 1\}$ is positive matching decomposition of $K_{m,n}$. That is, we show that $E_\ell$ is a positive matching on $E \setminus \bigcup_{k=1}^{\ell-1} E_k$ for $\ell = 1, \ldots, m + n - 1$.

For $\ell = 1$ the assertion is obvious since $E_1$ contains a single edge. Now assume $\ell \geq 2$. By Lemma 5.2(iii) it suffices to show that directing the edges in $E_\ell$ from $[m]$ to $[\tilde{n}]$ and the edges in $E \setminus \bigcup_{k=1}^{\ell} E_k$ in the other direction yields an acyclic orientation. Assume the resulting directed graph has a cycle. As a cycle in a bipartite graph is of even length this cycle must contain at least two edges of type $i \to \tilde{j}$ for $\{i, \tilde{j}\} \in E_\ell$ or equivalently $i + j = \ell + 1$. We assume $\{i, \tilde{j}\} \in E_\ell$ is chosen with this property so that $j$ is maximal. The next edge in the directed cycle is an edge $\tilde{j} \to i'$. We must have $i' < i$ or $i' > i$. If $i' < i$ then the following edge in the cycle $i' \to \tilde{j}'$ must again satisfy $i' + j' = \ell + 1$. But by $i' < i$ and $i + j = \ell + 1$ it follows that $j' > j$ a contradiction. Analogously, consider the edge $\tilde{j}'' \to i$ preceding $i \to \tilde{j}$. Again by construction $j'' + i > \ell + 1$. But then $j'' > j$ again a contradiction. Hence there is no cycle and $E_\ell$ is a positive matching on $E \setminus \bigcup_{k=1}^{\ell-1} E_k$. □

In the bipartite case, Theorem 2.6(4) shows that $\mathrm{pmd}(K_{m,n}) = m + n - 1$ and computer experiments show that $\mathrm{pmd}(K_n) = 2n - 3$ holds for small value of $n$.

Next we connect positive matching decompositions to algebraic properties of LSS-ideals.

**Lemma 5.4.** *Let $H = (V, E)$ be a hypergraph such that $E$ is a clutter, $d \geq p := \mathrm{pmd}(\mathcal{H})$ and $E = \bigcup_{\ell=1}^{p} E_\ell$ a positive matching decomposition. Then there exists a term order $<$ on $S$ such that for every $\ell$ and every $A \in E_\ell$ we have*

$$\mathrm{in}_<(f_A^{(d)}) \quad = \quad \prod_{i \in A} y_{i\ell}. \tag{6}$$

*Proof.* To define $<$ we first define weight vectors $\mathfrak{w}_1, \ldots, \mathfrak{w}_p \in \mathbb{R}^{V \times [d]}$. For that purpose we use the weight functions $w_\ell : V \to \mathbb{R}$, associated to each matching $E_\ell$, $\ell = 1, \ldots, p$. The weight vector $\mathfrak{w}_\ell$ is defined as follows:

- $\mathfrak{w}_\ell(y_{ik}) = 0$ if $k \neq \ell$ and

- $\mathfrak{w}_\ell(y_{i\ell}) = w_\ell(i)$.

By definition the weight of the monomial $\prod_{i\in A} y_{ik}$ with respect to $\mathfrak{w}_\ell$ is 0 if $k \neq \ell$ and $\sum_{i\in A} w_\ell(i)$ if $k=\ell$. Hence, by definition, the weight of $\prod_{i\in A} y_{i\ell}$ is positive if $A \in E_\ell$ and negative if $A \in E \setminus \cup_{v=1}^{\ell} E_v$.

It follows that:

$$\mathrm{in}_{\mathfrak{w}_1}(f_A^{(d)}) = \begin{cases} \prod_{i\in A} y_{i1} & \text{if } A \in E_1 \\ \displaystyle\sum_{k=2}^{d}\prod_{i\in A} y_{ik} & \text{if } A \in E\setminus\{E_1\}. \end{cases} \tag{7}$$

We define the term order $<$ as follows: $y^\alpha < y^\beta$ if

(1) $|\alpha| < |\beta|$ or
(2) $|\alpha| = |\beta|$ and $\mathfrak{w}_\ell(y^\alpha) < \mathfrak{w}_\ell(y^\beta)$ for the smallest $\ell$ such that $\mathfrak{w}_\ell(y^\alpha) \neq \mathfrak{w}_\ell(y^\beta)$ or
(3) $|\alpha| = |\beta|$ and $\mathfrak{w}_\ell(y^\alpha) = \mathfrak{w}_\ell(y^\beta)$ for all $\ell$ and $y^\alpha <_0 y^\beta$ for an arbitrary but fixed term order $<_0$.

A simple induction using (7) now shows that for all $\ell$ and for all $A \in E$ we have $\mathrm{in}_<(f_A^{(d)}) = \prod_{i\in A} y_{i\ell}$. □

We conclude this section with the proofs of Theorem 2.5(1)-(3) and Theorem 2.6(4) and a simple corollary.

*Proof of Theorem 2.5.* Let $p = \mathrm{pmd}(H)$ and $E = \bigcup_{\ell=1}^{p} E_\ell$ a pm-decomposition of $H$.

Hence by Lemma 5.4 there is a term order $<$ satisfying (6). Since each $E_\ell$, $\ell = 1,\ldots,p$, is a matching (6) implies that the initial monomials of the generators $f_A^{(d)}$, $A \in E$, of $L_H(d)$ are pairwise coprime. Hence $\mathrm{in}_<(L_H^{\Bbbk}(d)) = (\mathrm{in}_<(f_e^{(d)} \ : \ e \in E)$ is a radical complete intersection. This prove (1). Then (2) follows from (1) and Proposition 1.4 and (3) follows from Theorem 2.3. □

Now we can also complete the proof of Theorem 2.6(4).

*Proof of Theorem 2.6(4).* From Theorem 2.5(4) we know that $\mathrm{pmd}(K_{m,n}) \leq m+n-1$. From Theorem 2.5(3) we know that $L^{\Bbbk}_{K_{m,n}}(d)$ is prime for $d \geq \mathrm{pmd}(K_{m,n})+1$. From Theorem 2.9(3) we know that if $L^{\Bbbk}_{K_{m,n}}(d)$ is prime then $K_{m,n}$ does not contain a subgraph $K_{a,b}$ with $a+b = d+1$. The latter then implies $d \geq n+m$. Hence $\mathrm{pmd}(K_{m,n}) \geq m+n-1$ and therefore $\mathrm{pmd}(K_{m,n}) = m+n-1$. □

Using the fact Theorem 2.3(6) that primeness is inherited by subgraphs the following is an immediate consequence of Theorem 2.5.

**Corollary 5.5.** (i) *Let $G$ be a subgraph of $K_n$ then $L_G^K(d)$ is a radical complete intersection for $d \geq \min\{2n-3, |E|\}$ and prime for $d \geq \min\{2n-3,|E|\}+1$.*
(ii) *Let $G$ be a subgraph of $K_{m,n}$ then $L_G^K(d)$ is a radical complete intersection for $d \geq \min\{m+n-1, |E|\}$ and prime for $d \geq \min\{m+n-1,|E|\}+1$.*

## 6. Determinantal rings from the point of view of invariant theory

The goal of this section is to recall some classical results in invariant theory (see for example [34]) that have been treated in modern terms by De Concini and Procesi in [14]. In particular, we recall how determinantal/Pfaffian rings arise as invariant ring of group actions. We assume throughout that the base field $\Bbbk$ is of characteristic 0.

6.1. **Generic determinantal rings as rings of invariants (gen).** We take an $n \times m$ matrix of variables $X_{m,n}^{\mathrm{gen}} = (x_{ij})_{(i,j)\in[m]\times[n]}$ and consider the ideal $I_{d+1}^{\Bbbk}(X_{m,n}^{\mathrm{gen}})$ of $S^{\mathrm{gen}} = \Bbbk[x_{ij} \ : \ (i,j) \in [m]\times[n]]$ generated by the $(d+1)$-minors of $X_{m,n}^{\mathrm{gen}}$. Consider two matrices of variables $Y$ and $Z$ of size $m \times d$ and $d \times n$ and the following action of $\mathfrak{G} = \mathrm{GL}_d(\Bbbk)$ on the polynomial ring $\Bbbk[Y,Z]$: The matrix $A \in \mathfrak{G}$ acts by the $\Bbbk$-algebra automorphism of $\Bbbk[Y,Z]$ that sends $Y \to YA$ and $Z \to A^{-1}Z$. The entries of the product matrix $YZ$ are clearly invariant under this action. Hence the ring of invariants $\Bbbk[Y,Z]^{\mathfrak{G}}$ contains the subalgebra $\Bbbk[YZ]$ generated by the entries of the product $YZ$. The *First Main Theorem of Invariant Theory* for this action says that $\Bbbk[Y,Z]^{\mathfrak{G}} = \Bbbk[YZ]$. We have a surjective $\Bbbk$-algebra map:

$$\phi : S^{\mathrm{gen}} \to \Bbbk[Y,Z]^{\mathfrak{G}} = \Bbbk[YZ]$$

sending $X$ to $YZ$, that is $x_{ij}$ to $(YZ)_{ij} = \sum_{\ell=1}^{d} y_{i\,\ell} z_{\ell\, j}$. Clearly the product matrix $YZ$ has rank $d$ and hence we have $I^{\Bbbk}_{d+1}(X^{\mathrm{gen}}_{m,n}) \subseteq \operatorname{Ker}\phi$. The *Second Main Theorem of Invariant Theory* says that $I^{\Bbbk}_{d+1}(X^{\mathrm{gen}}_{m,n}) = \operatorname{Ker}\phi$. Hence

$$S/I^{\Bbbk}_{d+1}(X^{\mathrm{gen}}_{m,n}) \quad \simeq \quad \Bbbk[YZ] \tag{8}$$

6.2. **Generic symmetric determinantal rings as rings of invariants** (sym)**.** We take an $n \times n$ symmetric matrix of variables

$$X^{\mathrm{sym}}_n = \begin{pmatrix} x_{11} & x_{12} & x_{13} & \cdots & x_{1\,n-1} & x_{1n} \\ x_{12} & x_{22} & x_{23} & \cdots & x_{2\,n-1} & x_{2n} \\ x_{13} & x_{23} & x_{33} & \cdots & x_{3\,n-1} & x_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ x_{1n-1} & x_{2n-1} & x_{3n-1} & \cdots & x_{n-1\,n-1} & x_{n-1\,n} \\ x_{1n} & x_{2n} & x_{3n} & \cdots & x_{n-1\,n} & x_{nn} \end{pmatrix}$$

and consider the ideal $I^{\Bbbk}_{d+1}(X^{\mathrm{sym}}_n)$ generated by the $(d+1)$-minors of $X^{\mathrm{sym}}_n$ in the polynomial ring $S^{\mathrm{sym}} = \Bbbk[x_{ij} \,:\, 1 \le i \le j \le n]$. Consider a matrix of variables $Y$ of size $n \times d$ and the following action of the orthogonal group $\mathfrak{G} = \mathcal{O}_d(\Bbbk) = \{A \in \mathrm{GL}_t(\Bbbk) : A^{-1} = A^T\}$ on the polynomial ring $\Bbbk[Y] = \Bbbk[y_{ij}|Y_{ij} \;:\; (i,j) \in [n] \times [t]]$: Any $A \in \mathfrak{G}$ acts by the $\Bbbk$-algebra automorphism of $\Bbbk[Y]$ that sends $Y$ to $YA$. The entries of the product matrix $YY^T$ are invariant under this action and hence the ring of invariants contains the subalgebra $\Bbbk[YY^T]$ generated by the entries of $YY^T$. The *First Main Theorem of Invariant Theory* for this action asserts that $\Bbbk[Y]^G = \Bbbk[YY^T]$. Then we have a surjective presentation:

$$\phi : S^{\mathrm{sym}} \to \Bbbk[YY^T]$$

sending $X$ to $YY^T$. Since the product matrix $YY^T$ has rank $t$ we have $I_{d+1}(X) \subseteq \operatorname{Ker}\phi$. The *Second Main Theorem of Invariant Theory* then says that $I_{d+1}(X) = \operatorname{Ker}\phi$. Hence

$$S^{\mathrm{sym}}/I^{\Bbbk}_{d+1}(X^{\mathrm{sym}}_n) \quad \simeq \quad \Bbbk[YY^T]. \tag{9}$$

6.3. **Generic Pfaffian rings as rings of invariants** (skew)**.** We take an $n \times n$ skew-symmetric matrix of variables

$$X^{\mathrm{skew}}_n = \begin{pmatrix} 0 & x_{12} & x_{13} & \cdots & x_{1\,n-1} & x_{1n} \\ -x_{12} & 0 & x_{23} & \cdots & x_{2\,n-1} & x_{2n} \\ -x_{13} & -x_{23} & 0 & \cdots & x_{3\,n-1} & x_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -x_{1n-1} & -x_{2n-1} & -x_{3n-1} & \cdots & 0 & -x_{n-1\,n} \\ -x_{1n} & -x_{2n} & -x_{3n} & \cdots & -x_{n-1\,n} & 0 \end{pmatrix}$$

and consider the ideal $\mathrm{Pf}^{\Bbbk}_{2t+2}(X)$ generated by the Pfaffians of size $(2d+2)$ in $\Bbbk[x_{ij} \mid 1 \le i < j \le n]$ of $X_n^{\mathrm{skew}}$ in the polynomial ring $S^{\mathrm{skew}} = \Bbbk[x_{ij} \ : \ 1 \le i \le j \le n]$. Consider a matrix of variables $Y$ of size $n \times 2d$ and let $J_{2t}$ be the $2d \times 2d$ block matrix with $d$ blocks

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

on the diagonal and 0 in the other positions. The sympletic group $\mathfrak{G} = \mathrm{Sp}_{2d}(\Bbbk) = \{A \in \mathrm{GL}_{2t}(\Bbbk) : AJA^T = J\}$ acts on the polynomial ring $\Bbbk[Y] = \Bbbk[y_{ij}]$ as follows: an $A \in \mathfrak{G}$ acts on $\Bbbk[Y]$ by the automorphism that sends $Y \to YA$. The entries of the product matrix $YJ_{2d}Y^T$ are invariant under this action and hence the ring of invariants contains the subalgebra $\Bbbk[YJ_{2t}Y^T]$ generated by the entries of $YJ_{2d}Y^T$. The *First Main Theorem of Invariant Theory* for the current action says that $\Bbbk[Y]^G = \Bbbk[YJ_{2t}Y^T]$. Then we have a surjective presentation: $\phi : S^{\mathrm{skew}} \to \Bbbk[YY^T]$ sending $X$ to $YJY^T$. The product matrix $YJY^T$ has rank $2d$ and hence we have $\mathrm{Pf}^{\Bbbk}_{2d+2}(X) \subseteq \mathrm{Ker}\,\phi$. The *Second Main Theorem of Invariant Theory* for this action says that $\mathrm{Pf}^{\Bbbk}_{2d+2}(X) = \mathrm{Ker}\,\phi$. Hence

$$S^{\mathrm{skew}} / \mathrm{Pf}^{\Bbbk}_{2d+2}(X_n^{\mathrm{skew}}) \quad \simeq \quad \Bbbk[YJY^T]. \tag{10}$$

## 7. Determinantal ideals of matrices with 0's and their relation to LSS-ideals

The classical invariant theory point of view described in Section 6 shows that the generic determinantal and Pfaffian ideals are prime as they are kernels of ring maps whose codomains are integral domains. Their height is also well know (see for example [3] and the references given there):

- (gen) The height of the ideal $I^{\Bbbk}_d(X^{\mathrm{gen}}_{m,n})$ of $d$-minors of a $m \times n$ matrix of variables is $(n+1-d)(m+1-d)$.
- (sym) The height of the ideal $I^{\Bbbk}_d(X^{\mathrm{sym}}_n)$ of $d$-minors of a symmetric $n \times n$ matrix of variables is $\binom{n-d+2}{2}$.
- (skew) The height of the ideal of Pfaffians $\mathrm{Pf}^{\Bbbk}_{2d}(X^{\mathrm{skew}}_n)$ of size $2d$ (and degree $d$) of an $n \times n$ skew-symmetric matrix of variables is $\binom{n-2d+2}{2}$.

If one replaces the entries of the matrices with general linear forms in, say, $u$ variables, then Bertini's theorem in combination with the fact that the generic determinantal/Pfaffian rings are Cohen-Macaulay implies that the determinantal/Pfaffian ideals remain prime as long as $u \geq$ 2+height and radical if $u \geq$ 1+height.

But what about the case of special linear sections of determinantal ideals of matrices? And what about the case of coordinate sections? Are the corresponding ideals prime or radical? To describe coordinate sections we employ the following notation.

- (gen) In the generic case we take – as described in the introduction – a bipartite graph $G = ([m] \cup [\tilde{n}], E)$ where $E \subseteq [m] \times [\tilde{n}]$ and denote by $X^{\mathrm{gen}}_G$ the matrix obtained from the $m \times n$ matrix of variables $X = (x_{ij})_{(i,j) \in [m] \times [n]}$ by replacing the entries in position $(i,j)$ with 0 for $\{i, \tilde{j}\} \in E$.
- (sym) In the generic symmetric case we take a subgraph $G = ([n], E)$ of $K_n$ and denote by $X^{\mathrm{sym}}_G$ the matrix obtained from the generic symmetric matrix $X^{\mathrm{sym}}_n$ by replacing for all $\{i,j\} \in E$ the entries in row $i$, column $j$ and row $j$, column $i$ with 0.
- (skew) In the generic skewsymmetric case where we take a subgraph $G = ([n], E)$ of $K_n$ and denote by $X^{\mathrm{skew}}_G$ the matrix obtained from the generic skewsymmetric matrix $X^{\mathrm{skew}}_n$ by replacing for all $\{i,j\} \in E$ the entries in row $i$, column $j$ and row $j$, column $i$ with 0.

In this terminology $I^{\Bbbk}_d(X^{\mathrm{gen}}_G)$ (resp. $I^{\Bbbk}_d(X^{\mathrm{sym}}_G)$) is the ideal of $d$-minors of $X^{\mathrm{gen}}_G$ resp. $X^{\mathrm{sym}}_G$) in $S^{\mathrm{gen}}$ (resp. $S^{\mathrm{sym}}$). We write $\mathrm{Pf}^{\Bbbk}_{2d}(X^{\mathrm{skew}}_G)$ for the ideal of Pfaffians of size $2d$ of $X^{\mathrm{skew}}_G$ in $S^{\mathrm{skew}}$. We ask for conditions on $G$ that imply that $I^{\Bbbk}_d(X^{\mathrm{gen}}_G)$ (resp. $I^{\Bbbk}_d(X^{\mathrm{sym}}_G)$ or $\mathrm{Pf}^{\Bbbk}_{2d}(X^{\mathrm{skew}}_G)$) is radical or prime.

Simple examples show that special linear sections of relatively small height of generic determinantal ideals can give non-prime and non-radical ideals. On the positive side, for maximal minors, we have the following results:

**Remark 7.1.**
1. Eisenbud [17] proved that the ideal of maximal minors of a 1-generic $m \times n$ matrix of linear forms is prime and remains prime even after modding out any set of $\leq m-2$ linear forms. In particular, the ideal of maximal minors of an $m \times n$ matrix of linear forms is prime provided the ideal generated by the entries of the matrix has at least $m(n-1)+2$ generators.
2. Giusti and Merle in [18] studied the ideal of maximal minors of coordinate sections in the generic case. One of their main results, [18, Thm.1.6.1] characterizes, in combinatorial terms, the subgraph graphs $G$ of $K_{m,n}$, $m \leq n$, such that the variety associated to $I_m^{\Bbbk}(X_G^{\text{gen}})$ is irreducible, i.e. the radical of $I_m^{\Bbbk}(X_G^{\text{gen}})$ is prime.
3. Boocher proved in [5] that for any subgraph $G$ of $K_{m,n}$, $m \leq n$, the ideal $I_m^{\Bbbk}(X_G^{\text{gen}})$ is radical. Combining his result with the result of Giusti and Merle, one obtains a characterization of the graphs $G$ such that $I_m^{\Bbbk}(X_G^{\text{gen}})$ is prime.
4. Generalizing the result of Boocher, in [8] and [9] it is proved that ideals of maximal minors of a matrix of linear forms that is either row or column multigraded is radical.

In the generic case every non-zero minor of a matrix of type $X_G^{\text{gen}}$ has no multiple factors because its multidegree is square-free. This suggests that the determinantal ideals of $X_G^{\text{gen}}$ might always be radical. The following example shows that this is not the case:

**Example 7.2.** Let $X_G^{\text{gen}}$ be the $6 \times 6$ matrix associated to the graph from Example 2.2(3). That is, in the $6 \times 6$ generic matrix we set to 0 the entries in positions

$$(1,1),(1,2),(1,3),(1,4),(2,1),(2,2),(3,2),(3,3),(4,3),(4,4),(5,1),(5,4).$$

Then $I_4^{\Bbbk}(X_G^{\text{gen}})$ is not radical over a field of characteristic 0 and very likely over any field. Here the "witness" is $g = x_{1,5}$, i.e. $I_4^{\Bbbk}(X_G^{\text{gen}}) : g \neq I_4^{\Bbbk}(X_G^{\text{gen}}) : g^2$. Since $G$ is contained in $K_{5,4}$ one can consider as well $I_4^{\Bbbk}(X_G^{\text{gen}})$ in the $5 \times 5$ matrix but that ideal turns out to be radical.

Similarly for symmetric matrices we have:

**Example 7.3.** Let $X_G^{\text{sym}}$ be the $7 \times 7$ generic symmetric matrix associated to the graph from Example 2.2(1). That is, in the $7 \times 7$ generic symmetric matrix we set to 0 the entries in positions

$$\{1,2\},\{1,3\},\{1,4\},\{1,5\},\{2,3\},\{2,4\},\{2,6\},\{3,5\},\{4,6\}$$

as well as in the symmetric positions. Then $I_4^{\Bbbk}(X_G^{\text{sym}})$ is not radical over a field of characteristic 0 The witness here is $g = x_{1,6}$. Since $G$ is contained in $K_6$ one can consider as well $I_4^{\Bbbk}(X_G^{\text{sym}})$ in the $6 \times 6$ matrix but that ideal turns out to be radical.

The examples Example 2.2, Example 7.2 and Example 7.3 are indeed closely related as we now explain.

Let $G = ([m] \cup [\tilde{n}], E)$ be a subgraph of the complete bipartite graph $K_{m,n}$. In view of the isomorphism (8) we have that

$$S^{\text{gen}} / \Big( I_{d+1}^{\Bbbk}(X_G^{\text{gen}}) + (x_{ij} \mid \{i, \tilde{j}\} \in E) \Big) \simeq \Bbbk[YZ]/J_G(d)$$

where $Y = (y_{ij}), Z = (z_{ij})$ are respectively $m \times d$ and $d \times n$ matrices of variables and $J_G(d)$ is the ideal of $\Bbbk[YZ]$ generated by $(YZ)_{i,j}$ with $\{i, \tilde{j}\} \in E$. The LSS-ideal $L_G^{\Bbbk}(d) \subset \Bbbk[Y,Z]$ is indeed equal to $J_G(d)\Bbbk[Y,Z]$. Now it is a classical result in invariant theory, (derived from the fact that linear groups are reductive in characteristic 0) that $\Bbbk[YZ]$ is a direct summand of $\Bbbk[Y,Z]$ in characteristic 0. This implies that

$$J_G(d) \quad = \quad L_G^{\Bbbk}(d) \cap \Bbbk[YZ].$$

The next proposition is an immediate consequence.

**Proposition 7.4.** *Let $\Bbbk$ be a field of characteristic* $0$, $d \geq 1$ *and* $G = ([m] \cup [\tilde{n}], E)$ *be a subgraph of* $K_{m,n}$. *If the LSS-ideal* $L_G^{\Bbbk}(d)$ *is radical (resp. is a complete intersection, resp. is prime) then the coordinate section* $I_{d+1}^{\Bbbk}(X_G^{\mathrm{gen}})$ *of the generic determinantal ideal is radical (resp. has maximal height, resp. is prime).*

Now we start from a subgraph $E \subseteq \{\{i,j\} : 1 \leq i < j \leq n\}$ of $K_n$. For $d+1 \leq n$ we may consider the coordinate section $I_{d+1}^{\Bbbk}(X_G^{\mathrm{sym}})$ of $I_{d+1}^{\Bbbk}(X_n^{\mathrm{sym}})$. Using the isomorphism (9) we obtain:

**Proposition 7.5.** *Let $\Bbbk$ be a field of characteristic* $0$ *and* $G = ([n], E)$ *a subgraph of* $K_n$. *If the LSS-ideal* $L_G^{\Bbbk}(d)$ *is radical (resp. is a complete intersection, resp. is prime) then the coordinate section* $I_{d+1}^{\Bbbk}(X_G^{\mathrm{sym}})$ *of the generic determinantal ideal is radical (resp. has maximal height, resp. is prime).*

Now we can go back to LSS-ideals.

**Proposition 7.6.** *Let $\Bbbk$ be a field of characteristic* $0$ *and* $n \in \mathbb{N}$. *Let* $w_n$ *be the largest positive integer such that* $\binom{w_n}{2} \leq n$. *Then:*

(a) $L_{K_n}^{\Bbbk}(d)$ *is not prime for* $d = n + \binom{w_n - 2}{2} - 1$.
(b) $L_{K_n}^{\Bbbk}(d)$ *is not a complete intersection for* $d = n + \binom{w_{n+1} - 2}{2} - 2$.

*Proof.* (a) We set $h_n = \binom{w_n}{2}$ and $m_n = w_n + d - 1$. The numbers are chosen so that, using the formulas mentioned at the beginning of the section, the ideal $I_{d+1}(X)$ of $(d+1)$-minors of a generic symmetric $m_n \times m_n$ matrix $X$ has height $h_n$. Consider $K_n$ as the graph $([m_n], \binom{[n]}{2})$ on $m_n$ vertices where the vertices $n+1, \ldots, m_n$ do not appear in an edge. If, by contradiction, the ideal $L_{K_n}^{\Bbbk}(d)$ is prime then by Proposition 7.5 the ideal $I_{d+1}^{\Bbbk}(X_{K_n}^{\mathrm{sym}})$ is prime and of height $h_n$. But one has

$$I_{d+1}^{\Bbbk}(X_{K_n}^{\mathrm{sym}}) \subset (x_{11}, x_{22}, \ldots, x_{h_n h_n}) \tag{11}$$

which is a contradiction. To check (11) it is enough to prove that the rank of the matrix

$$X_{K_n}^{\mathrm{sym}} \mod (x_{11}, x_{22}, \ldots, x_{h_n h_n})$$

is at most $d$. That is, we have to check that the rank of an $(m_n \times m_n)$-matrix with block decomposition

$$\begin{pmatrix} 0 & A \\ B & C \end{pmatrix}$$

where $0$ is the zero matrix of size $(h_n \times n)$, is at most $d$. Since $d = m_n - n + m_n - h_n$ the latter is obvious.

(b) We set $h_n = \binom{w_{n+1}}{2}$ and $m_n = w_{n+1} + d - 1$. As above, the numbers are chosen so that the ideal $I_{d+1}(X)$ of $(d+1)$-minors of a generic symmetric $m_n \times m_n$ matrix $X$ has height $h_n$.

Assume, by contradiction, that $L_{K_n}^{\Bbbk}(d)$ is a complete intersection. From Proposition 7.5 it follows that $I_{d+1}^{\Bbbk}(X_{K_n}^{\mathrm{sym}})$ has height $h_n$. But

$$I_{d+1}^{\Bbbk}(X_{K_n}^{\mathrm{sym}}) \subset (x_{11}, x_{22}, \ldots, x_{h_n - 1, h_n - 1}) \tag{12}$$

which is a contradiction. As above (12) boils down to an obvious statement about the rank of a matrix with a zero submatrix of a certain size. □

Using this result we can now analyze the asymptotic behavior of both $\mathrm{asym}_{\Bbbk}(\mathrm{c.i.}, K_n)$ and $\mathrm{asym}_{\Bbbk}(\mathrm{prime}, K_n)$.

**Corollary 7.7.** *Let $\Bbbk$ be a field of characteristic* $0$. *Then*

$$\lim_{n \to \infty} \frac{\mathrm{asym}_{\Bbbk}(\mathrm{c.i.}, K_n)}{n} = \lim_{n \to \infty} \frac{\mathrm{asym}_{\Bbbk}(\mathrm{prime}, K_n)}{n} = 2. \tag{13}$$

*Proof.* By Corollary 5.5(i) we have $\text{asym}_{\Bbbk}(\text{prime}, K_n) \le 2n-2$. By Proposition 7.6 we have

$$n + \binom{w_{n+1}-2}{2} - 1 \quad \le \quad \text{asym}_{\Bbbk}(\text{c.i.}, K_n) \le \text{asym}_{\Bbbk}(\text{prime}, K_n) \tag{14}$$

Hence the equalities in (13) follow from the fact that

$$\lim_{n\to\infty} \frac{\binom{w_{n+1}-2}{2}}{n} = 1.$$

$\square$

Using Proposition 7.6 and Theorem 2.3 we obtain further obstructions.

**Corollary 7.8.** *Let $G$ be a graph on $n$ vertices and $\Bbbk$ a field of characteristic $0$. Then $L_G^{\Bbbk}(d)$ is not a complete intersection for $d \le \omega(G) + \binom{w_{\omega(G)+1}}{2} - 2$ and $L_G^{\Bbbk}(d)$ is not prime for $d \le \omega(G) + \binom{w_{\omega(G)}}{2} - 1$ where $w_{\omega(G)}$ is defined as in Proposition 7.6.*

To get an actual feeling of the obstruction. We list a few explicit example of new obstructions derived from Corollary 7.8.

| $d$ | obstruction to complete intersection | obstruction to primeness |
|---|---|---|
| $d \le 2$ | $K_4$ | $K_3$ |
| $d \le 3$ | | $K_4$ |
| $d \le 4$ | $K_5$ | |
| $d \le 5$ | $K_6$ | $K_5$ |
| $d \le 6$ | $K_7$ | $K_6$ |
| $d \le 7$ | $K_8$ | $K_7$ |
| $d \le 8$ | | $K_8$ |
| $d \le 10$ | $K_9$ | |
| $d \le 11$ | $K_{11}$ | $K_{12}$ |
| $d \le 12$ | $K_{12}$ | $K_{13}$ |

For $2d+2 \le n$ we may consider the coordinate section $\text{Pf}_{2d+2}^{\Bbbk}(X_G^{\text{skew}})$ of $\text{Pf}_{2d+2}^{\Bbbk}(X_n^{\text{skew}})$. We may as well consider the same graph $G = ([n], E)$ and the associated twisted LSS-ideal $\hat{L}_G^{\Bbbk}(d)$ defined as follows. For every $i \in [n]$ we consider $2d$ indeterminates $y_{i\,1}, \dots, y_{i\,2d}$. For $e = \{i, j\}$, $1 \le i < j \le n$ we set $\hat{f}_e^{(d)}$ to be the entry of the matrix $YJY^T$ in row $i$ and column $j$, i.e.

$$\hat{f}_e^{(d)} = \sum_{k=1}^{d} \Big( y_{i\,2k-1} y_{j\,2k} - y_{i\,2k} y_{j\,2k-1} \Big).$$

Then we call

$$\hat{L}_G^{\Bbbk}(d) = (\hat{f}_e^{(d)} \ : \ e \in E).$$

the twisted LSS-ideal associated to $G$. For $d = 1$ the twisted LSS-ideal coincides with the so-called binomial edge ideal defined and studied in [21, 25, 29, 32].

**Remark 7.9.** Assume $\sqrt{-1} \in \Bbbk$ and $G$ is bipartite with bipartition $[n] = V_1 \cup V_2$ then the coordinate transformation

- $y_{i\,2k-1} \mapsto y_{i\,2k-1}$ and $y_{i\,2k} \mapsto \sqrt{-1} y_{i\,2k-1}$ for $i \in V_1$,
- $y_{j\,2k} \mapsto \sqrt{-1} \cdot y_{i\,2k-1}$ and $y_{j\,2k-1} \mapsto y_{i\,2k}$ for $j \in V_2$,

sends $\hat{L}_G^{\Bbbk}(d)$ to $L_G^{\Bbbk}(2d)$. In particular, for a bipartite graph $G$ we have that $\hat{L}_G^{\Bbbk}(d)$ is radical (resp. prime) if and only if $L_G^{\Bbbk}(2d)$ radical (resp. prime).

Using the isomorphism (10) we obtain:

**Proposition 7.10.** *Let $\Bbbk$ be a field of characteristic $0$ and $G = ([n], E)$ a subgraph of $K_n$. If the twisted LSS-ideal $\hat{L}_G^{\Bbbk}(d)$ is radical (resp. is c.i., resp. is prime) then the coordinate section $\text{Pf}_{2d+2}^{\Bbbk}(X_G^{\text{skew}})$ of the generic Pfaffian ideal is radical (resp. has maximal height, resp. is prime).*

We have:

**Theorem 7.11.** *Let $\Bbbk$ be a field of characteristic* 0*.*

(gen) *If $G$ is a subgraph of $K_{m,n}$ then the ideals $I_2^{\Bbbk}(X_G^{\mathrm{gen}})$ and $I_3^{\Bbbk}(X_G^{\mathrm{gen}})$ are radical.*
(sym) *If $G$ is a subgraph of $K_n$ then the ideals $I_2^{\Bbbk}(X_G^{\mathrm{sym}})$ and $I_3^{\Bbbk}(X_G^{\mathrm{sym}})$ are radical.*
(skew) *If $G$ is a subgraph of $K_n$ then the ideal $\mathrm{Pf}_4^{\Bbbk}(X_G^{\mathrm{skew}})$ is radical.*

*Furthermore if $G$ is a forest then*

1. *$I_d^{\Bbbk}(X_G^{\mathrm{gen}}), I_d^{\Bbbk}(X_G^{\mathrm{sym}})$ and $\mathrm{Pf}_{2d}^{\Bbbk}(X_G^{\mathrm{skew}})$ are radical for all $d$.*
2. *$I_d^{\Bbbk}(X_G^{\mathrm{gen}})$ and $I_d^{\Bbbk}(X_G^{\mathrm{sym}})$ have maximal height if $d \geq \Delta(G)+1$.*
3. *$I_d^{\Bbbk}(X_G^{\mathrm{gen}})$ and $I_d^{\Bbbk}(X_G^{\mathrm{sym}})$ are prime if $d \geq \Delta(G)+2$.*

*Proof.* The statements for ideals of 2-minors in the cases (gen) and (sym) follow from Proposition 7.4 and Proposition 7.5 using the fact that the edge ideal of a graph is radical. Indeed these results hold over a field of arbitrary characteristic as the corresponding ideals are "toric."

By [22, Thm. 1.1] the ideal $L_G^{\Bbbk}(2)$ is radical for all graphs $G$. Using Proposition 7.4 and Proposition 7.5 this implies that $I_3^{\Bbbk}(X_G^{\mathrm{gen}})$ is radical for bipartite graphs $G$ and $I_3^{\Bbbk}(X_G^{\mathrm{sym}})$ is radical for arbitrary graphs.

By [21, Cor. 2.2] the ideal $\hat{L}_G^{\Bbbk}(1)$ is radical for all graphs $G$. Using Proposition 7.10 this implies the $\mathrm{Pf}_4^{\Bbbk}(X_G^{\mathrm{skew}})$ is radical for arbitrary graphs.

Finally, for a forest $G$ the results in the case of minors are derived from Proposition 7.4, Proposition 7.5 and Theorem 2.10. In the Pfaffian case they follows using Theorem 4.1 and Proposition 7.10. $\square$

The following corollary is an immediate consequence of the assertion (skew) in Theorem 7.11.

**Corollary 7.12.** *Let $G(2,n)$ be the coordinate ring of the Graßmannian of 2-dimension subspaces in $\Bbbk^n$ equipped in its standard Plücker coordinates. Then any subset of the Plücker coordinates generates a radical ideal in $G(2,n)$.*

We note that there are subsets of 2-minors of a generic matrix that define non-radical ideals. For example the ideal generated by the four 2-minors $[12|12], [12|23], [23|12], [23|23]$ of a generic $3\times 3$ is not radical.

A statement analogous to Corollary 7.12 for higher order Graßmannians is not true. Indeed, the point is that a set of $m$-minors of a generic matrix $m\times n$ does not generate a radical ideal in general (as it does for $m=2$). For example in the Graßmannian $G(3,6)$ modulo $[123],[124],[135],[236]$ the class of $[125][346]$ is a non-zero nilpotent.

Next we look into necessary conditions for $I_d^{\Bbbk}(X_G^{\mathrm{gen}})$ and $I_d^{\Bbbk}(X_G^{\mathrm{sym}})$ to be prime. The condition tie in with Proposition 2.8.

**Lemma 7.13.** *Let $G=([n],G)$ be a graph.*

(i) *If $I_{d+1}^{\Bbbk}(X_G^{\mathrm{sym}})$ is prime then $G$ does not contain a subgraph isomorphic to $K_{a,b}$ for $a+b>d$ (i.e. $\bar{G}$ is $(n-d)$-connected).*
(ii) *Assume $G$ is bipartite with bipartition $[n]=V_1\cup V_2$ and $d+1\leq |V_1|,|V_2|$. If $I_{d+1}^{\Bbbk}(X_G^{\mathrm{gen}})$ is prime then deleting any $|V_1|-d-1$ vertices from $V_1$ and $|V_2|-d-1$ vertices from $V_2$ yields a connected graph.*
(iii) *If $G=B_d$ with $d\geq 4$ and $X$ is the generic $(d+2)\times(d+2)$ matrix then $I_{d+1}^{\Bbbk}(X_G^{\mathrm{gen}})$ is not prime.*

*Proof.* (i) Assume $\bar{G}$ is not $(n-d)$-connected. Then there are $n-d-1$ vertices such that the graph obtained from $\bar{G}$ by deleting the vertices is disconnected. This implies that selecting in $X_G^{\mathrm{sym}}$ the rows and columns corresponding to the remaining $d+1$ vertices yields a matrix which after reordering the vertices is block-diagonal with at least two blocks. Hence its determinant is non-zero and reducible. Since the determinant is among a minimal generating set, it follows that $I_{d+1}^{\Bbbk}(X_G^{\mathrm{sym}})$ cannot be prime.

(ii) One easily checks that similar arguments as for the proof of the first part of (i) verify the assertion.

(iii) Set $Y_d = X^{\text{gen}}_{B_d}$, i.e.,

$$Y_d = \begin{pmatrix} x_{11} & 0 & \cdots & 0 & x_{1,d+1} & x_{1,d+2} \\ 0 & x_{22} & \cdots & 0 & x_{2,d+1} & x_{2,d+2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & x_{dd} & \vdots & \vdots \\ x_{d+1,1} & x_{d+1,2} & \cdots & \cdots & x_{d+1,d+1} & x_{d+1,d+2} \\ x_{d+2,1} & x_{d+2,2} & \cdots & \cdots & x_{d+2,d+1} & x_{d+2,d+2} \end{pmatrix}.$$

and $J = I_{d+1}(Y_d)$ and let $S$ be the polynomial ring whose indeterminates are the non-zero entries of $Y_d$. First, we prove that for every $d \geq 1$ the ideal $J$ has the expected height, i.e. height $J = 4$. For $d = 1, 2, 3$ the ideal $J$ is indeed prime of height 4: for $d = 1$ this is obvious because $Y_1$ is the generic $3 \times 3$ matrix while for $d = 2$ and $d = 3$ it follows from the fact that the corresponding LSS-ideal is prime by virtue of Proposition 7.4. For $d > 3$ let $P$ be a prime containing $J$. If $P$ contains $(x_{11}, x_{22}, x_{33}, x_{44})$ then height $P \geq 4$. If $P$ does not contain $(x_{11}, x_{22}, x_{33}, x_{44})$ we may assume $x_{11} \notin P$. Inverting $x_{11}$ and using the standard localization trick for determinantal ideals one sees that $PS_{x_{11}}$ contains, up to a change of variables, $I_d(Y_{d-1})$. Hence height $P =$ height $PS_x \geq 4$. Now that we know that $J$ has height 4 to prove that $J$ is not prime for $d \geq 4$ it is enough to observe that $J \subset (x_{11}, x_{22}, x_{33}, x_{44})$. The latter is straightforward since mod $(x_{11}, x_{22}, x_{33}, x_{44})$ the submatrix of $Y$ consisting of the first 4-rows as rank 2.

□

## 8. Questions and open problems

In Corollary 3.4 and Corollary 3.4 we have seen that for the properties c.i. and prime of $L^{\Bbbk}_G(d)$ there is persistence along the parameter $d$ but Example 2.2 shows persistence need not to hold for the property of being radical.

**Question 8.1.** What patterns

$$\{d < \text{asym}_{\Bbbk}(\text{radical}, G) \ : \ L^{\Bbbk}_G(d) \text{ radical } \}$$

can occur for graphs $G$?

We expect that erratic behavior can occur. For example we believe that there exists a graph $G$ and a number $d \geq 3$ such that $L^{\Bbbk}_G(d)$ and $L^{\Bbbk}_G(d+2)$ are non-radical while $L^{\Bbbk}_G(d+1)$ is radical.

We have seen in Theorem 2.9, [22, Cor. 1.4] and [22, Thm 1.1] that for certain fixed $d$ we can combinatorially classify the graphs $G$ for which $L^{\Bbbk}_G(d)$ is radical, complete intersection or prime. These classifications are based on rather simple graph theoretic properties of $G$.

**Question 8.2.** Fix a number $d$. Are there a (simple) combinatorially defined classes $\mathcal{G}_{\text{radical},d}$, $\mathcal{G}_{\text{c.i.},d}$ and $\mathcal{G}_{\text{prime},d}$ such that (say for fields $\Bbbk$ of characteristic 0):

$$L^{\Bbbk}_G(d) \text{ is radical } \Leftrightarrow G \in \mathcal{G}_{\text{radical},d}$$

and

$$L^{\Bbbk}_G(d) \text{ is a complete intersection } \Leftrightarrow G \in \mathcal{G}_{\text{c.i.},d}$$

and

$$L^{\Bbbk}_G(d) \text{ is prime } \Leftrightarrow G \in \mathcal{G}_{\text{prime},d}?$$

As said above, we expect that the pattern of numbers $d$ for which $L^{\Bbbk}_G(d)$ is radical can be quite erratic. Therefore, let us concentrate on the properties prime and complete intersection. Here the fact that the property is inherited by subgraphs supports hope for the classifications asked for in Question 8.2. After Theorem 2.9 $d = 3$ for complete intersection and $d = 4$ for primeness are the first parameters for which the classification is open.

For $d = 3$ we do not even have a conjecture when $L^{\Bbbk}_G(d)$ is a complete intersection. The graph $G$ from Figure 1(1) gives a graph $G$ for which it can be checked that $L^{\Bbbk}_G(3)$ is not a complete intersection for $\operatorname{char}\Bbbk = 0$ while $G$ still satisfies the necessary conditions from Proposition 2.8, i.e. has no subgraphs $K_{a,b}$ with $a + b = 5$ and $B_4$.

For $d = 4$ geometric results from Lovász's [27, Ch 9.4] book indicate that Proposition 2.8 still carries the essential obstacles for $L^{\Bbbk}_G(4)$ being prime.

**Question 8.3.** Is it true that $L^{\Bbbk}_G(4)$ is prime if and only if $G$ does not contain a subgraph isomorphic to $K_{a,b}$ for $a + b = 5$ and no subgraph isomorphic to $B_4$?

Via the fact that primeness of $L^{\Bbbk}_G(d)$ implies primeness of $I^{\Bbbk}_{d+1}(X_G)$ a result by Giusti and Merle [18, Thm. 1.6.1] guides the intuition behind the following question.

**Question 8.4.** Let $G$ be a subgraph of $K_{m,n}$ graph and assume $m \le n$. Is it true that $L^{\Bbbk}_G(m-1)$ is prime if and only if $G$ does not contain a subgraph isomorphic to $K_{a,b}$ for $a + b \ge m$?

By Proposition 7.4 and Proposition 7.5 we know if $L^{\Bbbk}_G(d)$ is radical or prime then so are $I^{\Bbbk}_{d+1}(X^{\mathrm{gen}}_G)$ and $I^{\Bbbk}_{d+1}(X^{\mathrm{sym}}_G)$ respectively. But our general bounds for $\mathrm{asym}_{\Bbbk}(\mathrm{radical}, G)$ and $\mathrm{asym}_{\Bbbk}(\mathrm{prime}, G)$ from Corollary 5.5 are not good enough to make use of this implication. Indeed, Corollary 7.7 shows that for the properties complete intersection and prime and $n$ large enough there are graphs $G$ for which Proposition 7.5 does not prove primality of an interesting ideal. On the other hand the use of Theorem 2.10 in Theorem 7.11 shows that one can take advantage of this connection in some cases. It would be interesting to exhibit classes different from forests where this is possible.

**Question 8.5.** Are there more interesting classes of graphs $G = ([n], E)$ for which $\mathrm{asym}_{\Bbbk}(\mathrm{c.i.}, G) \le n$ or $\mathrm{asym}_{\Bbbk}(\mathrm{prime}, G) \le n - 1$ ?

Despite the fact that Proposition 7.6 destroys the hope for using Theorem 7.11 for general graphs, it would be interesting replace the asymptotic result by an actual value. By Corollary 7.7 for $n$ large we have $\mathrm{asym}_{\Bbbk}(\mathrm{prime}, K_n) = 2n - c_n$ for some numbers $c_n \in o(n)$ which using the notation of Proposition 7.6 satisfy $n - \binom{w_n-2}{2} + 1 \ge c_n \ge 2$. But we have no conjecture for an actual formula for $c_n$.

**Question 8.6.** What is the value of $\mathrm{asym}_{\Bbbk}(\mathrm{prime}, K_n)$?

For radicality we have a concrete conjecture in the case $G = K_n$.

**Conjecture 8.7.** *We conjecture that* $\mathrm{asym}_{\Bbbk}(\mathrm{radical}, K_n) = 1$ *(at least if* $\operatorname{char}\Bbbk = 0$*). In other words, given a matrix of variables $X$ of size $n \times d$ we conjecture the ideal of the off-diagonal entries of $XX^T$ is radical for all $n, d$.*

It would also be interesting to study the ideal generated by all the entries of $XX^T$. We note that the symplectic version of this problem has been investigated by De Concini in [12].

Next we turn to open problems about hypergraph LSS-ideals. We know from Theorem 2.5(2) that for a hypergraph $H = (V, E)$ for which $E$ is a clutter the ideal $L^{\Bbbk}_H(d)$ is a radical complete intersection for $d \ge \mathrm{pmd}(G)$. But we prove in Theorem 2.5(3) that $L_H(^{\Bbbk})$ is prime for $d \ge \mathrm{pmd}(H) + 1$ only in the case that $H$ is a graph.

**Question 8.8.** Is it true that for a hypergraph $H = (V, E)$, where $E$ is a clutter, we have $L^{\Bbbk}_H(d)$ is prime for $d \ge \mathrm{pmd}(H) + 1$?

Similarly, the persistence results from Theorem 2.3 ask for generalizations.

**Question 8.9.** Let $H = (V, E)$ be a hypergraph, where $E$ is a clutter. Is it true that if $L^{\Bbbk}_H(d)$ is a complete intersection (resp. prime) then so is $L^{\Bbbk}_H(d + 1)$?

For a number $r \ge 1$ we call a hypergraph $H = (V, E)$ an $r$-uniform graph if every element of $E$ has cardinality $r$. In particular, $E$ then it is a clutter. For example, graphs are 2-uniform graphs. We say that an $r$-uniform graph $H = (V, E)$ is $r$-partite if there is a partition $V = V_1 \cup \cdots \cup V_k$ such that $\#(A \cap V_i) = 1$ for all $i \in [r]$. For $r = 2$ a 2-uniform hypergraph is 2-partite if and only if the hypergraph considered as a graph is bipartite. Now we connect the study of ideal $L^{\Bbbk}_H(d)$ for $r$-uniform ($r$-partite) graphs with the study of coordinate sections of the space of tensors with a given rank. We consider two mappings:

($\phi$) Let $\mathfrak{e}_1, \ldots, \mathfrak{e}_n$ be the standard basis vectors of $\Bbbk^n$. For vectors $v_i = (v_{i,j})_{j\in[d]} \in \Bbbk^d$, $i \in [r]$, consider the map $\phi$ that sends $(v_1, \ldots, v_r) \in (\Bbbk^d)^n$ to the tensor

$$\sum_{j=1}^{d} \sum_{\sigma\in S_r} v_{\sigma(i),j} \cdots v_{\sigma(r),j}\, \mathfrak{e}_{\sigma(1)} \otimes \cdots \otimes \mathfrak{e}_{\sigma(r)} \in \underbrace{\Bbbk^n \otimes \cdots \otimes \Bbbk^n}_{r}.$$

We take the sums over the different tensors arising from $\mathfrak{e}_{i_1} \otimes \cdots \otimes \mathfrak{e}_{i_r}$, for numbers $1 \leq i_1 \leq \cdots \leq i_r \leq n$, by permuting the positions as standard basis of the space of symmetric tensors.

($\psi$) Let $n = n_1 + \cdots + n_r$ for natural numbers $n_1, \ldots, n_r \geq 1$. Let $\mathfrak{e}_i^{(j)} \in \Bbbk^{n_j}$ be the $i$-th standard basis vector of $\Bbbk^{n_j}$, $i \in [n_j]$ and $j \in [r]$. For vectors $v_i^{(j)} = (v_{i,j})_{j\in[d]} \in \Bbbk^d$ for $i \in [n_j]$ and $j \in [r]$ consider the map $\psi$ that sends $(v_i^{(j)})_{(i,j)\in[n_j]\times[r]}$ to

$$\sum_{(i_1,\ldots,i_r)\in[n_1]\times\cdots\times[n_r]} v_{i_1}^{(1)} \cdots v_{i_r}^{(r)}\, e_{i_1}^{(1)} \otimes \cdots \otimes \mathfrak{e}_{i_r}^{(r)} \in \Bbbk^{n_1} \otimes \cdots \otimes \Bbbk^{n_r}.$$

We take the tensors $\mathfrak{e}_{i_1}^{(1)} \otimes \cdots \otimes \mathfrak{e}_{i_r}^{(r)}$ for numbers $i_j \in [n_j]$, $j \in [r]$ as the standard basis of $\Bbbk^{n_1} \otimes \cdots \otimes \Bbbk^{n_r}$.

Recall that a (symmetric) tensor has (symmetric) rank $\leq d$ it can written as a sum of $\leq d$ decomposable (symmetric) tensors. For more details on tensor rank and the geometry of bounded rank tensors we refer the reader to [26]. Let $H = (V, E)$ be a hypergraph. We write $\mathcal{V}(L_H^{\Bbbk}(d))$ for the vanishing locus of $L_H^{\Bbbk}(d)$. The the definition of the maps $\phi$ and $\psi$ immediately implies the following proposition.

**Proposition 8.10.** *Let $H = ([n], E)$ be an $r$-uniform hypergraph and $\Bbbk$ an algebraically closed field.*

(i) *Then the restriction of the map $\phi$ to $\mathcal{V}(L_H^{\Bbbk}(d))$ is a parametrization of the space of symmetric tensors in $\underbrace{\Bbbk^n \otimes \cdots \otimes \Bbbk^n}_{r}$ of rank $\leq d$ which when expanded in the standard basis has zero coefficient for the basis elements indexed by $1 \leq i_1 < \cdots < i_r \leq n$ and $\{i_1, \ldots, i_r\} \in E$. In particular, the Zariski-closure of the image of the restriction is irreducible if $L_H^{\Bbbk}(d)$ is prime.*

(ii) *If $H$ is $r$-partite with respect to the partition $V = V_1 \cup \cdots \cup V_r$ where $|V_i| = n_i$, $i \in [r]$. Then the restriction of the map $\psi$ to $\mathcal{V}(L_H^{\Bbbk}(d))$ is a parametrization of the space of tensors in $\Bbbk^{n_1} \otimes \cdots \otimes \Bbbk^{n_r}$ of rank $\leq d$ which when expanded in the standard basis have zero coefficient for the basis elements indexed by $i_1, \ldots, i_r$ where $\{i_1, \ldots, i_r\} \in E$. In particular, the Zariski-closure of the image of the restriction is irreducible if $L_H^{\Bbbk}(d)$ is prime.*

Proposition 8.10 gives further motivation to Question 8.8. Indeed, it suggests to strengthen Question 8.5.

**Question 8.11.** Let $\Bbbk$ be an algebraically closed field. Can one describe classes of $r$-uniform hypergraphs $H$ for which $L_H^{\Bbbk}(d)$ is prime for some $d$ bounded from above by the maximal symmetric rank of a symmetric sensor in $\underbrace{\Bbbk^n \otimes \cdots \otimes \Bbbk^n}_{r}$.

An analogous question can be asked for $r$-partite $r$-uniform hypergraphs and tensors of bounded rank.

## References


[1] J.Abbott, A.M.Bigatti, G.Lagorio, *CoCoA-5: a system for doing Computations in Commutative Algebra*, Available at `http://cocoa.dima.unige.it`.

[2] L.L.Avramov, *Complete intersections and symmetric algebras*, J. Algebra **73** (1981) 248–263.

[3] W.Bruns, U.Vetter, *Determinantal rings*, Lecture Notes in Math. **1327**, Springer-Verlag, Berlin, 1988.

[4] W.Bruns, A.Conca, *Gröbner bases and determinantal ideals*, in: Commutative Algebra, Singularities and Computer Algebra (Sinaia, 2002), 9–66, NATO Sci. Ser. II Math. Phys. Chem. **115**, Kluwer Acad. Publ., Dordrecht, 2003.

[5] A.Boocher, *Free resolutions and sparse determinantal ideals*, Math. Res. Lett. **19** (2012) 805–821.
[6] Bourbaki, *Algebra II*, Chapters 4–7, Springer-Verlag, Berlin, 2003.
[7] D. Cartwright, B.Sturmfels, *The Hilbert scheme of the diagonal in a product of projective spaces*, Int. Math. Res. Not., no. 9, (2010) 1741–1771.
[8] A.Conca, E. De Negri, E.Gorla, *Universal Gröbner bases for maximal minors*, Int. Math. Res. Not. IMRN, no. 11, (2015) 3245–3262.
[9] A.Conca, E.De Negri, E.Gorla, *Universal Gröbner bases and Cartwright-Sturmfels ideals*, arXiv:1608.08942.
[10] A.Conca, E.De Negri, E.Gorla, *Multigraded generic initial ideals of determinantal ideals*, in: Homological and Computational Methods in Commutative Algebra, A. Conca, J. Gubeladze, Joseph, T. Römer (Eds.), 81–96, INdAM Series **20**, Springer-Verlag, Heidelberg, 2017.
[11] A.Conca, E.De Negri, E.Gorla, *Cartwright-Sturmfels ideals associated to graphs and linear spaces*, arXiv:1705.00575.
[12] C.De Concini, *Symplectic standard tableaux*, Adv. Math. **34** (1979) 1–27.
[13] C.De Concini, E.Strickland, *On the variety of complexes*, Adv. Math. **41** (1981) 57–77.
[14] C.De Concini, C.Procesi, *A characteristic free approach to invariant theory*, Adv. Math. **21** (1976) 330–354.
[15] P.Diaconis, D.Eisenbud, B.Sturmfels, *Lattice walks and primary decomposition*, in: Mathematical Essays in Honor of Gian-Carlo Rota, 173–93, Prog. Math. **161**, Birkhäuser Boston, Boston, MA, 1998.
[16] R.Diestel, *Graph Theory*, 3rd Ed., Graduate Texts in Math. **173**, Springer-Verlag, Heidelberg, 2000.
[17] D.Eisenbud, *Linear sections of determinantal varieties*, Amer. J. Math. **110** (1988) 541–575.
[18] M.Giusti, M.Merle, *Singularités isolées et sections planes de variétés déterminantielles. II. Sections de variétés déterminantielles par les plans de coordonnées.* Algebraic geometry (La Rábida, 1981), 103118, Lecture Notes in Math. **961**, Springer-Verlag, Berlin, 1982.
[19] D.R.Grayson, M.E.Stillman, *Macaulay2, a software system for research in algebraic geometry*, Available at `http://www.math.uiuc.edu/Macaulay2/`.
[20] C.Greene, T.Zaslavsky, *On the interpretation of Whitney numbers through arrangements of hyperplanes, zonotopes, non-Radon partitions, and orientations of graphs*, Trans. Amer. Math. Soc. **280** (1983) 97–126.
[21] J.Herzog, T.Hibi, F.Hreinsdóttir, T.Kahle, J.Rauh, *Binomial edge ideals and conditional independence statements*, Adv. in Appl. Math. **45** (2010) 317–333.
[22] J.Herzog, A. Macchia, S.Madani, V.Welker, *On the ideal of the orthogonal representations of a graph in* $\mathbb{R}^2$, Adv. in Appl. Math. **71** (2015) 146–173.
[23] S.Hoşten, S.Sullivant, *Ideals of adjacent minors*. J. Algebra **277** (2004) 615–642.
[24] C.Huneke, *On the symmetric algebra of a module*, J. Algebra **69** (1981) 113-119.
[25] D.Kiani, S.Saeedi Madani, *The Castelnuovo-Mumford regularity of binomial edge ideals*, J. Combin. Theory Ser. A **139** (2016) 80–86.
[26] J.M. Landsberg, *Tensors: geometry and applications*, Graduate Studies in Mathematics **128**, American Mathematical Society, Providence, RI, 2012
[27] L.Lovász, *Geometric Representations of Graphs*, `http://www.cs.elte.hu/~lovasz/geomrep.pdf`, 2016.
[28] L.Lovász, M.Saks, A.Schrijver, *Orthogonal representations and connectivity of graphs*, Linear Alg. Appl. **114/115** (1989) 439–454.
[29] K.Matsuda, S.Murai, *Regularity bounds for binomial edge ideals*, J. Commut. Algebra **5** (2013) 141–149.
[30] J.S. Milne, *Algebraic Geometry*, Version 6.02, `http://www.jmilne.org/math/CourseNotes/AG.pdf`, 2017.
[31] C.Musili, C.S.Seshadri, *Schubert varieties and the variety of complexes*, in: Arithmetic and Geometry, Vol. II, 329–359, Progr. Math. **36**, Birkhäuser Boston, Boston, MA, 1983.
[32] M.Ohtani, *Graphs and ideals generated by some 2-minors*, Comm. Alg. **39** (2011) 905–917.
[33] A.Tchernev, *Universal complexes and the generic structure of free resolutions*, Michigan Math. J. **49** (2001) 65–96.
[34] H.Weyl, *The Classical Groups, their Invariants and Representations*, Princeton Univ. Press, Princeton, N.J., 1939.

Dipartimento di Matematica, Università di Genova, Via Dodecaneso 35, I-16146 Genova, Italy
*E-mail address*: `conca@dima.unige.it`

Philipps-Universität Marburg, Fachbereich Mathematik und Informatik, D-35032 Marburg, Germany
*E-mail address*: `welker@mathematik.uni-marburg.de`