HOMOLOGY OF POWERS OF IDEALS: ARTIN–REES NUMBERS OF SYZYGIES AND THE GOLOD PROPERTY

JÜRGEN HERZOG, VOLKMAR WELKER, AND SIAMAK YASSEMI

Abstract. For an ideal I in a regular local ring $(R, \mathfrak{m})$ with residue class field $K = R/\mathfrak{m}$ or a standard graded K-algebra R we show that for $k \gg 0$

- the Artin–Rees number of the syzygy modules of I^k as submodules of the free modules from a free resolution is constant, and thereby present the Artin-Rees number as a proper replacement of regularity in the local situation,
- the ring R/I^k is Golod, its Poinceré-Betti series is rational and the Betti numbers of the free resolution of K over R/I^k are polynomials in k of a specific degree.

The first result is an extension of work of Kodiyalam and Cutkosky, Herzog & Trung on the regularity of I^k for $k \gg 0$ from the graded situation to the local situation. The polynomiality consequence of the second result is an analog of work by Kodiyalam on the behavior of Betti numbers of the minimal free resolution of R/I^k over R.

1. Introduction

Over the last 20 years the study of algebraic, homological and combinatorial properties of powers of ideals has been one of the major topics in Commutative Algebra. In this paper we extend this in two so far unexplored directions.

Artin-Rees numbers : The most important invariants of a graded ideal I in a polynomial ring provided by the Betti diagram are the projective dimension and the regularity. A result by Brodmann [3] shows that depth R/I^k and hence proj dim I^k are constant for $k \gg 0$. It was shown in [11] and [4] that the regularity reg I^k of I^k is a linear function for $k \gg 0$ (see also [7] and [8]) for structural results on the point of stabilization and constant term of the linear function). While the regularity can only be defined for graded ideals, the projective dimension is defined and is a finite number for any ideal in a regular local ring. Thus it is natural to ask: Which numerical invariant of an ideal in a regular local ring corresponds to the regularity of a graded ideal in a polynomial ring? Does one obtain stability in the sense of [11] and [4] for high powers of I ? We approach this question by observing that for a graded ideal, the linearity of reg I^k for $k \gg 0$ implies that there is an upper bound independent of k for the degrees of the entries of the matrices describing the syzygies of I^k. Thus in order to find a suitable replacement for the regularity in the

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local case we need a measure that bounds the "size" of the entries of the syzygies. We show that the Artin–Rees number (see (1)) of syzygy modules of I^k inside the corresponding free module is constant for large k. We further support the choice of the Artin–Rees number as a substitute for regularity by comparing regularity and Artin–Rees number in Theorem 3.4 for finitely generated graded modules over polynomial rings which implies a linear upper bound on the regularity of I^k.

Golod property : In [10] Kodiyalam showed that the total Betti numbers $\beta_i^R(I^k)$ of an ideal I in a Noetherian local ring are polynomial functions for large k. In the case that R is a polynomial ring and I is a graded ideal, a certain refinement of this statement can be found in the more recent paper [9]. For an ideal I in a regular local ring $(R, \mathfrak{m})$ with $R/\mathfrak{m} = K$ or a polynomial ring R over K we study the Betti numbers $\beta_i^{R/I^k}(K)$ of the free resolution of K over R/I^k. We show in Theorem 4.1 that R/I^k is Golod for $k \gg 0$, a homological property that by definition implies trivial multiplication in the Tor algebra $\mathrm{Tor}_\bullet^R(R/I, K)$, rationality of the Poincaré-Betti series of R/I^k by a result of Golod and thereby connects the Betti numbers $\beta_i^{R/I^k}(K)$ and $\beta_i^R(R/I^k)$. As a consequence we show in Proposition 4.4 that the Betti numbers $\beta_i^{R/I^k}(K)$ and the deviations $\varepsilon_i(R/I^k)$ are polynomial functions of a specific degree. We were inspired by a result of G. Levin [12, Thm. 3.15] saying that for any Noetherian local ring $(R, \mathfrak{m})$ the canonical epimorphism $R \to R/\mathfrak{m}^k$ is a Golod homomorphism for all $k \gg 0$.

For the convenience of the reader and due to the lack of suitable references we recall in Section 2 some basic facts about Artin–Rees numbers, and show in Corollary 2.2 how the degrees of the entries of a matrix describing a graded submodule N of a free module F are bounded by the Artin–Rees number $\rho(N, F)$. In Section 3 we consider the asymptotic behaviour of the Artin–Rees numbers of the syzygies of powers of ideals, prove Theorem 3.3 and Theorem 3.4. In Section 4 we study the Golod property of R/I^k and prove Theorem 4.1 and Proposition 4.4.

For all unexplained concepts from commutative algebra we refer the reader to the book [6].

2. Preliminaries regarding Artin–Rees numbers

Let $(R, \mathfrak{m})$ denote a Noetherian local ring or standard graded K-algebra with graded maximal ideal $\mathfrak{m}$, M a finitely generated R-module and $N \subset M$ a submodule of M. In the graded case we assume that M is graded and N is a graded submodule of M.

By the Artin–Rees Lemma [6, Lem. 5.1], there exists an integer r such that

$$N \cap \mathfrak{m}^i M = \mathfrak{m}^{i-r}(N \cap \mathfrak{m}^r M) \qquad \text{for all } i \geq r. \tag{1}$$

The smallest such number r is called the *Artin-Rees number* and will be denoted $\rho(N, M)$.

We denote by S the associated graded ring $\mathrm{gr}_{\mathfrak{m}}(R)$ of R, and by $\mathfrak{n}$ the graded maximal ideal of S. Of course, if R is standard graded, then $R = S$ and $\mathfrak{m} = \mathfrak{n}$. For an R-module M and $x \in M$ with $x \neq 0$ we set $\ell(x) = x + \mathfrak{m}^{k+1}M$ where k is the largest integer such that $x \in \mathfrak{m}^k M$. The element $\ell(x) \in \mathrm{gr}_{\mathfrak{m}}(M)$ is called the leading form of x.

Proposition 2.1. *Let N^* be the kernel of the natural epimorphism $\mathrm{gr}_{\mathfrak{m}}(M) \to \mathrm{gr}_{\mathfrak{m}}(M/N)$ of graded S-modules. Then*

(a) $N^* = \sum_{x \in N} S\ell(x)$,

$$(N^*)_k = (N \cap \mathfrak{m}^k M)/(N \cap \mathfrak{m}^{k+1}M) \quad \textit{for all} \quad k,$$

and

$$\rho(N, M) = \max\{k \colon (N^*/\mathfrak{n}N^*)_k \neq 0\}.$$

In addition, in the graded case $N^* = \sum_{x \in \mathrm{homogen}(N)} S\ell(x)$.

(b) *for $x_1, \ldots, x_r \in N$ such that $\ell(x_1), \ldots, \ell(x_r)$ generate N^*, the elements $x_1, \ldots, x_r$ generate N.*

Proof. (a) It is clear that $\ell(x) \in N^*$ for all $x \in N$. Conversely, suppose that $\ell(x) = x + \mathfrak{m}^{k+1}M$ belongs to N^*. Then $\varkappa(x) + \mathfrak{m}^{k+1}W = 0$, where $W = M/N$ and $\varkappa \colon M \to W$ is the canonical epimorphism. Hence there exists $y \in \mathfrak{m}^{k+1}W$ such that $\varkappa(x) = y$. Let $z \in \mathfrak{m}^{k+1}M$ with $\varkappa(z) = y$ and set $x' = x - z$. Then $x' \in N$ and $\ell(x') = \ell(x)$.

Next observe that the exact sequence

$$0 \to N \to M \to W \to 0$$

induces for all k the exact sequence

$$0 \to N \cap \mathfrak{m}^k M \to \mathfrak{m}^k M \to \mathfrak{m}^k W \to 0.$$

This shows that $(N^*)_k = (N \cap \mathfrak{m}^k M)/(N \cap \mathfrak{m}^{k+1}M)$ for all k, and this implies immediately that $\rho(N, M) = \max\{k \colon (N^*/\mathfrak{n}N^*)_k \neq 0\}$.

(b) Let $U = \sum_{i=1}^r Rx_i$. We want to show that $U = N$. Let $x \in N$ with $\ell(x) = x + \mathfrak{m}^{k+1}M$. By assumption there exist $a_i \in R$ such that $\ell(x) = \sum_i \ell(a_i)\ell(x_i)$. It follows that $x' = x - y \in \mathfrak{m}^{k+1}M$, where $y = \sum_i a_i x_i \in U$. Let $\deg \ell(x') = t$. Then $t \geq k+1$, and as before we find $z \in U$ such that $x'' = x' - z \in \mathfrak{m}^{t+1}M$. In other words, $x - w \in \mathfrak{m}^{t+1}M$ where $w = y + z \in U$. Proceeding in this way we can find for any given number s an element $u \in U$ such that $x - u \in \mathfrak{m}^{s+1}M$. Choosing $s = \rho(N, M)$, we see that for any $x \in N$ there exists $u \in U$ such that $x - u \in \mathfrak{m}N$. Thus we have shown that $N = U + \mathfrak{m}N$. Nakayama's lemma implies that $U = N$, as desired. □

Corollary 2.2. *Suppose R is standard graded and F is a finitely generated graded free R-module with homogeneous basis $e_1,\dots,e_s$. Let $N\subset F$ be a graded submodule of F. Then for any minimal set of homogeneous generators of $x_1,\dots,x_t$ of N with*

$$x_i=\sum_{j=1}^{s}a_{ij}e_j,\quad i=1,\dots,t,$$

such that

$$\rho(N,F)\geq \min_{i=1,\dots,t}\{\min_{j=1,\dots,s}\{\deg a_{ij}\}\},\tag{2}$$

and

$$\rho(N,F)\geq \max_{i=1,\dots,t}\{\min_{j=1,\dots,s}\{\deg a_{ij}\}\}\tag{3}$$

for a suitable choice of $x_1,\dots,x_t$.

Here we use the convention that $\deg a=\infty$ if $a=0$.

Proof. We first prove (3). By Proposition 2.1(a) there exist homogeneous elements $x_1,\dots,x_r\in N$ such that the set of leading forms

$$\ell(x_1),\dots,\ell(x_r)$$

is a minimal set of generators of N^*. By Proposition 2.1(b), $x_1,\dots,x_r$ is a system of generators of N. Since R is standard graded, a suitable subset of $x_1,\dots,x_r$, say, $x_1,\dots,x_t$ is a minimal set of generators of N. Therefore, $\ell(x_1),\dots,\ell(x_t)$ is part of a minimal set of generators of N^*. Hence, if $\ell(x_i)=x_i+\mathfrak{m}^{k_i+1}F$ for $i=1,\dots,t$, then it follows from Proposition 2.1(a) that $k_i\leq\rho(N,F)$ for $i=1,\dots,t$. Now if $x_i=\sum_{j=1}^{s}a_{ij}e_j$, then $k_i=\min\{\deg a_{ij}\colon\ j=1,\dots,s\}$. Thus the assertion follows.

Inequality (3) is stronger than Inequality (2). Moreover, the right hand side of Inequality (2) is independent of the chosen minimal system of generators. From this Inequality (2) follows for all minimal systems of generators of N. □

3. Asymptotic behavior of Artin-Rees numbers

In this section we are going to consider the behavior of Artin-Rees numbers of the syzygy modules of powers of ideals and compare them in the graded case with their regularity. For a graded R-module M over a polynomial ring R of projective dimension p we denote by $\operatorname{reg}_j:=\max\{k-j\ :\ \beta^R_{jk}(M)\neq 0\}$, $0\leq j\leq p$, the regularity of its j^{th} syzygy module, here $\beta^R_{jk}(M)$ are the graded Betti numbers of M over R. In particular, $\operatorname{reg}_0(M)$ is the maximal degree of a homogeneous generator of M. We write $\operatorname{reg}(M):=\max_{0\leq j\leq p}\operatorname{reg}_j(M)$ for the regularity of M. For the sake of later use we also introduce here for an module M over a standard graded or local ring R the notation $\beta^R_i(M)$ for the i^{th} Betti number of M over R.

Since powers of ideals are best studied as the graded components of Rees algebras, which for graded ideals have a natural bigraded structure, one is led to consider bigraded modules.

Lemma 3.1. *Let K be a field, $T = K[x_1, \dots, x_n, y_1, \dots, y_m]$ the standard bigraded polynomial ring over K, $A = K[x_1, \dots, x_n]$ and W a finitely generated bigraded T-module. For each integer k consider the finitely generated graded A-module*

$$W_k = \bigoplus_j W_{jk}.$$

Then $\operatorname{reg}_0(W_k)$ is constant for $k \gg 0$.

Proof. Let $w_1, \dots, w_r$ be a set of homogeneous generators of W with $\deg w_i = (a_i, b_i)$. We may assume that $a_1 \leq a_2 \leq \cdots \leq a_r$. Let $B = K[y_1, \dots, y_m]$ be the polynomial ring over K in the variables $y_1, \dots, y_m$. Then

$$W_k = \sum_{i=1}^{r} AB_{k-b_i} w_i$$

for all k. It follows that $\operatorname{reg}_0(W_k) \leq a_r$.

Assume that $B_{k-b_r} w_r \not\subset \sum_{i=1}^{r-1} AB_{k-b_i} w_i$ for all $k \geq b$, where $b = \max\{b_i \colon i = 1, \dots, r\}$. Then $\operatorname{reg}_0(W_k) = a_r$ for all $k \geq b$.

On the other hand, if $B_{k_1-b_r} w_r \subset \sum_{i=1}^{r-1} AB_{k_1-b_i} w_i$ for some $k_1 \geq b$, then $B_{k-b_r} w_r \subset \sum_{i=1}^{r-1} AB_{k-b_i} w_i$ for all $k \geq k_1$. It follows that $W_k = W'_k$ for all $k \geq k_1$, where $W' = \sum_{i=1}^{r-1} T w_i$. Applying induction on the number of generators of W, the desired conclusion follows. □

Let $S = R[y_1, \dots, y_m]$ be the polynomial ring over $(R, \mathfrak{m})$ in the variables $y_1, \dots, y_m$, where according to our general assumption R is either a Noetherian local ring or a standard graded K-algebra. We consider S to be a graded ring by setting $\deg a = 0$ for all $a \in R \setminus \{0\}$ and $\deg y_j = 1$ for $j = 1, \dots, m$. In particular, for a finitely generated graded S-module M each graded component M_k is a finitely generated R-module.

Proposition 3.2. *Let $N \subset M$ be graded S-modules. Then $\rho(N_k, M_k)$ is constant for $k \gg 0$.*

Proof. By Proposition 2.1 we have $\rho(N_k, M_k) = \operatorname{reg}_0(N_k)^*$. We let $W = M/N$ and N^* be the kernel of the canonical epimorphism $\operatorname{gr}_{\mathfrak{m}}(M) \to \operatorname{gr}_{\mathfrak{m}}(W)$. The ring $\operatorname{gr}_{\mathfrak{m}}(S) = \operatorname{gr}_{\mathfrak{m}}(R)[y_1, \dots, y_m]$ is naturally bigraded and N^* is a bigraded $\operatorname{gr}_{\mathfrak{m}}(S)$-module with

$$(N^*)_{jk} = \operatorname{Ker}(\mathfrak{m}^j M_k / \mathfrak{m}^{j+1} M_k \longrightarrow \mathfrak{m}^j W_k / \mathfrak{m}^{j+1} W_k).$$

We set $(N^*)_k = \bigoplus_j (N^*)_{jk}$. Then we see that $(N^*)_k = (N_k)^*$ for all k, and hence $\rho(N_k, M_k) = \operatorname{reg}_0(N^*)_k$ for all k.

If $\operatorname{emb\,dim} R = n$, then there exists an epimorphism $T = K[x_1, \dots, x_n, y_1, \dots, y_m] \to \operatorname{gr}_{\mathfrak{m}}(S)$ of standard bigraded K-algebras. Thus we may view N^* to be a bigraded T-module. Applying Lemma 3.1, we conclude that $\rho(N_k, M_k)$ is constant for $k \gg 0$. □

Next we will apply this result to study the Artin-Rees numbers of the syzygies of powers of an ideal. Let $(\mathbb{F}, \varphi)$ be a graded free R-resolution of M. The j^{th} syzygy module of M with respect to $\mathbb{F}$ is defined to be the module $N_j = \operatorname{Im}(\varphi_j) \subset F_{j-1}$. In the case that $(\mathbb{F}, \varphi)$ is a graded minimal free R-resolution of M, we set $\rho_j(M) := \rho(N_j, F_{j-1})$ for all $j \geq 1$ and $\rho_0(M) := \operatorname{reg}_0(M)$.

Theorem 3.3. *Let $(R, \mathfrak{m})$ be a Noetherian local ring or a standard graded K-algebra with graded maximal ideal $\mathfrak{m}$, and $I \subset \mathfrak{m}$ an ideal. In the case that R is graded, we assume that I is a graded ideal. Then for all $j \geq 1$ there exists an integer c_j such that $\rho_j(I^k) = c_j$ for all $k \gg 0$. In particular, if R is regular, then there exists an integer c such that $\rho_j(I^k) \leq c$ for all $j \geq 1$ and all k.*

Proof. The Rees ring $R(I) = \bigoplus_{k \geq 0} I^k t^k$ of $I = (f_1, \dots, f_m)$ is a standard graded R-algebra with generators $f_i t$ for $i = 1, \dots, m$. Let $S = R[y_1, \dots, y_m]$ be the polynomial ring over R in the indeterminates $y_1, \dots, y_m$. We define the surjective R-algebra homomorphism $\varkappa\colon S \to R(I)$ with $\varkappa(y_i) = f_i t$ for $i = 1, \dots, m$. Then $R(I) = S/J$, where $J = \operatorname{Ker} \varkappa$. Let $(\mathbb{F}, \varphi)$ be a graded minimal free S-resolution of $R(I)$, minimal in the sense that $\varphi(\mathbb{F}) \subset \mathfrak{n}\mathbb{F}$, where $\mathfrak{n} = (\mathfrak{m}, y_1, \dots, y_m)$.

We denote by $\mathbb{F}^{(k)}$ the k^{th} homogeneous component of the resolution $\mathbb{F}$. Then $\mathbb{F}^{(k)}$ is a (not necessarily minimal) free R-resolution of I^k. Let N_j be the j^{th} syzygy module of $R(I)$ viewed as an S-module. Then the k^{th} component $N_j^{(k)}$ of N_j is the j^{th} syzygy module of I^k with respect to the resolution $\mathbb{F}^{(k)}$.

By Proposition 3.2, there exists an integer c_j such that $\rho(N_j^{(k)}, F_{j-1}^{(k)}) = c_j$ for $k \gg 0$. Let $\mathbb{G}^{(k)}$ be a minimal graded free R-resolution of I^k, and $W_j^{(k)}$ be the j^{th} syzygy module of I^k with respect to $\mathbb{G}$. Then for each k there exists a free R-module $H_{j-1}^{(k)}$ such that $F_{j-1}^{(k)} = G_{j-1}^{(k)} \oplus H_{j-1}^{(k)}$ and $N_j^{(k)} = W_j^{(k)} \oplus H_{j-1}^{(k)}$. We deduce that $\rho(N_j^{(k)}, F_{j-1}^{(k)}) = \rho(W_j^{(k)}, G_{j-1}^{(k)})$ for all k. This yields the desired conclusion. ☐

The following result provides a comparison of the numbers $\rho_j(M)$ with the regularity $\operatorname{reg}(M)$ of M in the case that M is a graded module over the polynomial ring.

Theorem 3.4. *Let $R = K[x_1, \dots, x_n]$ be the polynomial ring, and M a finitely generated graded R-module of projective dimension p. Then*

$$\operatorname{reg}_j(M) \leq \sum_{k=0}^{j} \rho_k(M) - j \quad \textit{for} \quad j = 0, \dots, p.$$

In particular, $\operatorname{reg}(M) \leq \max\{\sum_{k=0}^{j} \rho_k(M) - j\colon\ j = 0, \dots, p\}$.

Proof. Within the proof we write β_j for $\beta_j^R(M)$. Let $(\mathbb{F}, \partial)$ be a graded minimal free R-resolution of M with

$$F_j = \bigoplus_{k=1}^{\beta_j} S(-b_{jk}) \qquad \text{and} \qquad \alpha = (a_{rs})_{\substack{r=1,\dots,\beta_j \\ s=1,\dots,\beta_{j-1}}}$$

the matrix representing the differential $\partial_j\colon\ F_j \to F_{j-1}$ with respect to homogeneous bases of F_{j-1} and F_j. The assertion of the theorem will follow once we have shown that

$$\sum_{k=0}^{j} \rho_j(M) \geq \max\{b_{jk}\colon\ k = 1, \dots, \beta_j\}.$$

This inequality will be shown by induction on j. For $j = 0$, the inequality is an equality by the definition of $\rho_0(M)$. Now assume that $j > 0$. By a suitable choice of a basis of F_j we may assume that the matrix α satisfies

$$\rho_j(M) \geq \max_{r=1,\dots,\beta_j}\{\min_{s=1,\dots,\beta_{j-1}}\{\deg a_{rs}\}\},$$

see Corollary 2.2.

Observe that $\deg a_{rs} = b_{jr} - b_{j-1,s}$ if $a_{rs} \neq 0$ and $\deg a_{rs} = \infty$, otherwise. Let r_0 be the index with the property that $b_{jr_0} = \max\{b_{jr}\colon\ r = 0, \dots, \beta_j\}$, s_0 the index with the property that $b_{j-1,s_0} = \max\{b_{j-1,s}\colon\ s = 0, \dots, \beta_{j-1}\}$ and s_1 the index with $\deg a_{r_0 s_1} = \min\{\deg a_{r_0 s}\colon\ s = 1, \dots, \beta_{j-1}\}$. Then, by using induction on j, we obtain

$$\rho_j(M) \geq \deg a_{r_0 s_1} = b_{jr_0} - b_{j-1,s_1} \geq b_{jr_0} - b_{j-1,s_0} \geq b_{jr_0} - \sum_{k=0}^{j-1} \rho_k(M),$$

from which it follows that $\sum_{k=0}^{j} \rho_k(M) \geq b_{jr_0}$, as desired. $\square$

By [10] and [4] it is known that for a graded ideal $I \subset K[x_1, \dots, x_n]$ there exist integers a and b such that $\operatorname{reg}(I^k) = ak + b$ for all k. By using Theorem 3.4 and Theorem 3.3 we obtain

Corollary 3.5. *Let $I \subset K[x_1, \dots, x_n]$ be a graded ideal, then there exist integers a and b such that $\operatorname{reg}(I^k) \leq ak + b$ for all k.*

4. The Golod property

Let $(R, \mathfrak{m})$ be a Noetherian ring with residue class field K, or a standard graded K-algebra with graded maximal ideal $\mathfrak{m}$, and let $\mathbf{x} = x_1, \dots, x_n$ a minimal (homogeneous) system of generators of $\mathfrak{m}$. We denote by $(K(R), \partial)$ the Koszul complex of R with respect to $\mathbf{x}$. Let $Z(R)$, $B(R)$ and $H(R)$ denote the module of cycles, boundaries and the homology of $K(R)$.

Recall (see [2, Def. 5.5 and 5.6]) that R is said to be *Golod*, if for each subset $\mathcal{S}$ of homogeneous elements of $\bigoplus_{i=1}^{n} H_i(R)$ there exists a function γ, which is defined on the set of finite sequences of elements from $\mathcal{S}$ with values in $\mathfrak{m} \oplus \bigoplus_{i=1}^{n} K_i(R)$, subject to the following conditions:

(G1) if $h \in \mathcal{S}$, then $\gamma(h) \in Z(R)$ and $h = [\gamma(h)]$;

(G2) if $h_1, \ldots, h_m$ is a sequence in $\mathcal{S}$ with $m > 1$, then

$$\partial\gamma(h_1, \ldots, h_m) = \sum_{\ell=1}^{m-1} \overline{\gamma(h_1, \ldots, h_\ell)}\gamma(h_{\ell+1}, \ldots, h_m),$$

where $\bar{a} = (-1)^{i+1}a$ for $a \in K_i(R)$.

A function γ defined on finite sequences from $\mathcal{S}$ with the properties (G1) and (G2) is called a *Massey operation* on $\mathcal{S}$

In this section we want to prove the following

Theorem 4.1. *Let $(R, \mathfrak{m})$ be a regular local ring or the polynomial ring over K with graded maximal ideal $\mathfrak{m}$, and $I \subset \mathfrak{m}$ an ideal. We assume that I is graded if R is the polynomial ring. Then the ring R/I^k is Golod for $k \gg 0$.*

Given integers $i, s \geq 1$ and an integer $j \geq 0$ there is a natural R-module homomorphism

$$*\colon \begin{cases} I^j \otimes K_i(R/I^s) & \to & K_i(R/I^{s+j}) \\ a \otimes c & \mapsto & a * v \end{cases}$$

which is defined as follows: observing that $K_i(R/I^t) = K_i(R)/I^tK_i(R)$ for any integer $t \geq 1$, we let $K_i(R) \to K_i(R/I^t)$ be the canonical epimorphism which assigns to $u \in K_i(R)$ the residue class $u + I^tK_i(R)$. Now if $a \in I^j$ and $v = u + I^sK_i(R) \in K_i(R/I^s)$. Then we set $a * v := au + I^{s+j}K_i(R)$. Obviously this map is well defined, that is, independent of the choice of u. By a simple calculation it follows that for $a \in I^j$, $b \in I^\ell$ and $v \in K_i(R/I^s)$

$$(ab) * v \quad = \quad a * (b * v) \in K_i(R/I^{s+j+\ell}) \tag{4}$$

Notice that if $v \in Z_i(R/I^s)$, then $a * v \in Z_i(R/I^{s+j})$. Indeed, we have $v = u + I^sK_i(R) \in Z_i(R/I^s)$ if and only if $\partial(u) \subset I^sK_{i-1}(R)$. In that case $\partial(au) \in I^{s+j}K_{i-1}(R)$, and hence $\partial(a * v) = \partial(au) + I^{j+s}K_i(R) = 0$. Similarly one shows that $a * v \in B_i(R/I^{s+j})$, if $v \in B_i(R/I^s)$. Thus $*$ induces a map

$$I^j \otimes H_i(R/I^s) \to H_i(R/I^{s+j}), \quad a \otimes [z] \mapsto a * [z] := [a * z].$$

We will denote the image of the map $* : I^j \otimes H_i(R/I^s) \to H_i(R/I^{s+j})$ by $I^j * H_i(R/I^s) \subset H_i(R/I^{s+j})$.

For the proof of the theorem we shall need the following

Lemma 4.2. *There exists an integer s' such that for $s \geq s'$*

$$I^j * H_i(R/I^s) = H_i(R/I^{s+j})$$

for all integers $i \geq 1$ and $j \geq 0$.

Proof. Observe that $H_i(R/I^k) \cong H_{i-1}(I^k)$. By using this isomorphism it follows that $I^j \otimes H_i(R/I^s) \to H_i(R/I^{s+j})$ is surjective if and only if the map

$$I^j \otimes H_{i-1}(I^s) \quad \to \quad H_{i-1}(I^{s+j}) \tag{5}$$

with $a \otimes [z] \mapsto [az]$ is surjective. Thus it amounts to show that there exists an integer s such that $I^j \otimes H_i(I^s) \to H_i(I^{s+j})$ is surjective for all $i = 0, \dots, n-1$ and all $j \geq 0$.

To this end we consider the Koszul complex $K(\mathbf{x}; R(I))$ of the sequence $\mathbf{x}$ with values in the Rees ring $R(I)$ of I. Recall that $H_i(\mathbf{x}; R(I))$ is a finitely generated graded $R(I)$-module with graded pieces

$$H_i(\mathbf{x}; R(I))_k \quad = \quad H_i(I^k), \quad \text{for all} \quad k, \tag{6}$$

where the graded $R(I)$-module structure is given by (5). Thus for each i there exists an integer s_i such that

$$H_i(I^{s_i}) = I^{k-s_i} H_i(\mathbf{x}; R(I))_{s_i} = H_i(\mathbf{x}; R(I)_k = H_i(I^k)$$

for all $k \geq s_i$.

Hence the number $s' = \max\{s_0, \dots, s_{n-1}\}$ satisfies the condition of the lemma. $\square$

Proof of Theorem 4.1. In the case that R is the polynomial ring, we denote by $\hat{R}$ the $\mathfrak{m}$-adic completion of R. Since $H(\hat{R}/I\hat{R}) = H(R/I)$, we may replace R by its completion and may therefore assume that R is local.

We claim the following: for any integer $r \geq 1$, there exists an integer s_r such that for all $k \geq s_r$ and each homogeneous subset $\mathcal{S} \subset \bigoplus_{i=1}^n H_i(R/I^k)$ there exists a function γ, defined on the set of sequences of elements of $\mathcal{S}$ of length $\leq r$, such that (G1) and (G2) hold.

The claim will yield the desired result, since for any such function we have that $\gamma(h_1, \dots, h_r) \in K_a(R/I^k)$ where $a \geq 2r - 1$. Hence if $r \geq n/2 + 1$, we necessarily have $\gamma(h_1, \dots, h_r) = 0$.

We will prove the claim by induction on r. For $r = 1$, we may choose $s_r = 1$ and the assertion is trivial.

Now let $r \geq 1$ and assume that the claim is proved for all integers $\leq r$. By Lemma 4.2 there exists an integer s such that $I^{k-s} H_i(R/I^s) = H_i(R/I^k)$ for all $k \geq s$ and all $i > 0$. We set $s_{r+1} = \max\{s_r, s\} + 1$. Let $\mathcal{G} = \{g_1, \dots, g_t\}$ be a homogeneous K-basis of $\bigoplus_{i=1}^n H_i(R/I^{s_r})$. By induction hypothesis there exists a function γ, defined on the set of sequences of elements of $\mathcal{G}$ of length $\leq r$, such that (G1) and (G2) hold.

Let $k \geq s_{r+1}$ and let $\mathcal{S} \subset \bigoplus_{i=1}^n H_i(R/I^k)$ be any set of homogeneous elements. We are going to define a function γ on sequences from $\mathcal{S}$ of length $r+1$ satisfying (G1) and (G2).

First, let $h_1, \dots, h_m$ be any sequence of elements $h_i \in \mathcal{S}$ of length $m \leq r$. Since $s_{r+1} > s$, each h_i can be written as $h_i = \sum_{j=1}^t a_{ij} g_j$ with $a_{ij} \in I^{k-s_r}$ and all

$g_j \in H_t(R/I^{s_r})$ if $h_i \in H_t(R/I^k)$. We define γ on this sequence by multi-linear extension, that is, we set

$$\gamma(h_1,\dots,h_m) := \sum_{j_1=1}^{t}\sum_{j_2=1}^{t}\cdots\sum_{j_m=1}^{t}(a_{1j_1}a_{2j_2}\cdots a_{mj_m})*\gamma(g_{j_1},\dots,g_{j_m}).$$

Here we consider $a_{1j_1}a_{2j_2}\cdots a_{mj_m}$ as an element of I^{k-s_r} so that $\gamma(h_1,\dots,h_m) \in K(R/I^k)$.

Next we verify that the function γ satisfies (G1) and (G2) on sequences from $\mathcal{S}$ of length $\leq r$:

(G1) Let $h \in \mathcal{S}$ with presentation $h = \sum_{j=1}^{t} a_j g_j$ where $a_j \in I^{k-s_r}$ for all j. Since $\gamma(g_j) \in Z(R/I^{s_r})$ it follows that $a_j * \gamma(g_j) \in Z(R/I^k)$, and hence we see that $\gamma(h) = \sum_{j=1}^{t} a_j * \gamma(g_j)$ belongs to $Z(R/I^k)$ as well. Moreover, we have

$$[\gamma(h)] = \sum_{j=1}^{t}[a_j*\gamma(g_j)] = \sum_{j=1}^{t} a_j*[\gamma(g_j)] = \sum_{j=1}^{t} a_j g_j = h.$$

(G2) Since ∂ is linear with respect to $*$ we have

$$\partial\gamma(h_1,\dots,h_m) = \sum(a_{1j_1}a_{2j_2}\cdots a_{mj_m})*\partial\gamma(g_{j_1},\dots,g_{j_m}),$$

where the sum is taken over all j_k ranging between 1 and t. For each of the summands in $\partial\gamma(h_1,\dots,h_m)$, by using $a*(v+w) = a*v + a*w$ and $(ab)*(vw) = (a*v)(b*w)$, we have

$$\begin{aligned}
& (a_{1j_1}a_{2j_2}\cdots a_{mj_m})*\partial\gamma(g_{j_1},\dots,g_{j_m}) \\
= \; & (a_{1j_1}a_{2j_2}\cdots a_{mj_m})*\sum_{\ell=1}^{m-1}\overline{\gamma(g_{j_1},\dots,g_{j_\ell})}\gamma(g_{j_{\ell+1}},\dots,g_{j_m}) \\
= \; & \sum_{\ell=1}^{m-1}(a_{1j_1}a_{2j_2}\cdots a_{mj_m})*\left(\overline{\gamma(g_{j_1},\dots,g_{j_\ell})}\gamma(g_{j_{\ell+1}},\dots,g_{j_m})\right) \\
= \; & \sum_{\ell=1}^{m-1}\left((a_{1j_1}a_{2j_2}\cdots a_{\ell j_\ell})*\overline{\gamma(g_{j_1},\dots,g_{j_\ell})}\right)\left((a_{\ell+1 j_{\ell+1}}\cdots a_{mj_m})*\gamma(g_{j_{\ell+1}},\dots,g_{j_m})\right).
\end{aligned}$$

Summing over all indices j_t independently yields the desired identity (G2).

In order to define γ on sequences of elements of $\mathcal{S}$ of length $r+1$, it suffices to show that for any sequence $h_1,\dots,h_{r+1}$ of elements from $\mathcal{S}$, the element

$$b = \sum_{\ell=1}^{r}\overline{\gamma(h_1,\dots,h_\ell)}\gamma(h_{\ell+1},\dots,h_{r+1}),$$

is a boundary of $K(R/I^k)$. To this end, observe that b is a linear combination of expressions of the form

$$b' = \sum_{\ell=1}^{r} \overline{\gamma(g_{k_1},\ldots,g_{k_\ell})}\gamma(g_{k_{\ell+1}},\ldots,g_{k_{r+1}})$$

with coefficients in $I^{(r+1)(k-s_r)} \subset I^{k-s_r}$. Let a be the coefficient of b'. Since b' is a boundary in $K(R/I^{s_r})$ and $a \in I^{k-s_r}$, it follows that ab' is a boundary in $K(R/I^k)$. Thus $b \in B(R/I^k)$, as desired. □

In the following we consider for a Noetherian local ring $(R,\mathfrak{m})$ with residue field $K = R/\mathfrak{m}$ or a standard graded K-algebra R with graded maximal ideal $\mathfrak{m}$ its deviations $\varepsilon_i(R)$ and the Betti numbers $\beta_i^R(K)$ of the minimal free resolution of $K = R/\mathfrak{m}$ over R. We refer the reader to [1] and [5] for standard facts about these invariants. The characterization of the Golod property in terms of Poincaré-Betti series (see [5, Cor. 4.2.4] or [1, (5.0.1)] and Theorem 4.1) immediately yield the following result.

Corollary 4.3. *Let $(R,\mathfrak{m})$ be a regular local ring with $K = R/\mathfrak{m}$ or the polynomial ring over K with graded maximal ideal $\mathfrak{m}$, and $I \subset \mathfrak{m}$ an ideal. We assume that R is of dimension d and that I is graded if R is the polynomial ring. Then for $k \gg 0$ the multiplication on $Tor_*^R(R/I^k,K)$ is trivial and*

$$\sum_{i\geq 0} \beta_i^{R/I^k}(K)z^i \quad = \quad \frac{(1+z)^d}{1-z\sum_{i=1}^d \beta_i^R(R/I^k)z^i}$$

The next result will exploit this Corollary in order to obtain specific information about the growth of $\beta_i^{R/I^k}(K)$ and $\varepsilon_i(R/I^k)$ as a function of k. For the formulation of the result we denote for an ideal I by $\ell(I)$ its analytic spread.

Proposition 4.4. *Let I be an ideal in the regular local ring R of dimension d or I a graded ideal in $R = K[x_1,\ldots,x_d]$. If $\ell(I) \geq 2$ then for $i \geq 0$*

- (i) *$\beta_i^{R/I^k}(K)$ is a polynomial of degree $(\ell(I)-1)\cdot\lfloor\frac{i}{2}\rfloor$ in k for $k \gg 0$.*
- (ii) *$\varepsilon_i(R/I^k)$ is a polynomial of degree $(\ell(I)-1)\cdot\lfloor\frac{i+1}{2}\rfloor$ in k for $k \gg 0$.*

Proof of Proposition 4.4 (i). By Theorem 4.1 the ring R/I^k is Golod for $k \gg 0$. Hence by Corollary 4.3 the Equation (7) is valid for $k \gg 0$. Multiplying with the denominator of the right hand side of (7) we obtain for $2 \leq i$:

$$\beta_i^{R/I^k}(K) \quad = \quad \sum_{l=1}^{\min\{i-1,d\}} \beta_{i-1-l}^{R/I^k}(K)\beta_l^R(R/I^k) + \binom{d}{i} \tag{7}$$

Now by [10, Cor. 7] each $\beta_i^R(R/I^k)$ is a polynomial in k for $k \gg 0$. We assume that k is large enough to satisfy the two preceding conclusions. We know that $\beta_0^{R/I^k}(K) = 1$ and $\beta_1^{R/I^k}(K) = d = \dim R$. In addition we know by [9, Prop. 2.2] that $\deg \beta_1^R(R/I^k) = \ell(I)-1 \geq \cdots \geq \deg \beta_n^R(R/I^k)$. Hence we deduce from (7) and induction on i that $\beta_i^R(K)$ is a polynomial of degree $(\ell(I)-1)\cdot\lfloor\frac{i}{2}\rfloor$ in k. $\square$

Before we can proceed to the proof of Proposition 4.4 (ii) we need some simple calculations relating $\varepsilon_i(R/I)$ and $\beta_i^{R/I}(K)$ for general regular local rings $(R,\mathfrak{m})$ or polynomial rings R and (graded) ideals I. Recall that for $i \geq 0$ the deviations $\varepsilon_i(R/I)$ can be defined through

$$(8)\quad \prod_{i\geq 0}\frac{(1+z^{2i+1})^{\varepsilon_i(R/I)}}{(1-z^{2i+2})^{\varepsilon_{i+1}(R/I)}} = \prod_{i\geq 0}(1+(-1)^i z^{i+1})^{(-1)^i\cdot\varepsilon_i(R/I)} = \sum_{i\geq 0}\beta_i^{R/I}(K)z^i$$

We now assume that $k \gg 0$ is large enough so that Equation (7) from Corollary 4.3 is valid for R/I^k. For sake of simple notation we will for a moment abbreviate $\varepsilon_i(R/I^k)$ by ε_i and $\beta_i^R(R/I^k)$ by β_i.

Taking the logarithm on the left hand side of (8) we obtain:

$$(9)\quad \begin{aligned}\sum_{i\geq 0}(-1)^i\varepsilon_i\log(1+(-1)^i z^{i+1}) &= \sum_{i\geq 0}(-1)^i\varepsilon_i\sum_{n\geq 1}\frac{(-1)^{n-1}}{n}(-1)^{ni}z^{n(i+1)}\\ &= \sum_{m\geq 1}\Big(\sum_{n|m}(-1)^{m/n}\frac{\varepsilon_{m/n-1}}{n}\Big)(-z)^m\end{aligned}$$

Taking logarithm on the right hand side of (7) we obtain:

$$(10)\ d\log(1+z)-\log(1-\sum_{i=1}^{d}\beta_i z^{i+1}) = \sum_{i\geq 1}\Big(\frac{(-1)^{i-1}d}{i}z^i+\frac{1}{i}\big(\sum_{j=1}^{d}\beta_j z^{j+1}\big)^i\Big)$$

By comparing coefficients of $z^{i+1}i$ from (9) and (10) we obtain for $i \geq 0$

$$(11)\ (-1)^{i+1}\Big(\sum_{n|i+1}(-1)^{\frac{i+1}{n}}\frac{\varepsilon_{(i+1)/n-1}}{n}\Big) = \frac{(-1)^{i+1}d}{i+1}+\sum_{\substack{j_1+\cdots+j_n+n=i+1\\ 1\leq j_1,\ldots,j_n\leq d}}\frac{1}{n}\beta_{j_1}\cdots\beta_{j_n}$$

Proof of Proposition 4.4 (ii). As in the proof of Proposition 4.4 (i) we assume k is large enough so that R/I^k is Golod and each $\beta_j^R(R/I^k)$ is a polynomial in k. Again we proceed by induction. For $i=0,1$, we have $\varepsilon_0(R/I^k)=d$ and $\varepsilon_1(R/I^k)=\beta_1^R(R/I^k)$. The claim holds since $\beta_1^R(R/I^k)$ is a polynomial of degree $\ell(I)-1$.

Now let $i\geq 2$. By (11) we can express $\varepsilon_i(R/I^k)$ as a linear combination of $\varepsilon_j(R/I^k)$ for $j<\lfloor\frac{i+1}{2}\rfloor-1$ and products of $\beta_{j_1}^R(R/I^k)\cdots\beta_{j_n}^R(R/I^k)$ for $1\leq j_1,\ldots,j_n\leq d$ and $j_1+\cdots+j_n+n=i+1$. We first show that this second summand has the right degree as a polynomial in k.

Claim: The right hand side of (11) is a polynomial of degree $(\ell(I)-1)\lfloor\frac{i+1}{2}\rfloor$ in k.

◁ For $1\leq m\leq n$ the Betti number $\beta_{j_m}^R(R/I^k)$ is a polynomial of degree $\leq \ell(I)-1$ in k by [9, Prop. 2.2]. It follows that $\beta_{j_1}^R(R/I^k)\cdots\beta_{j_n}^R(R/I^k)$ is a polynomial of degree $\leq (\ell(I)-1)\cdot n$ in k. Since $1 \leq j_1, \ldots, j_n$ the maximal degree is achieved for $n = \lfloor \frac{i}{2} \rfloor$. For i even we can choose $j_1 = \cdots = j_n = 1$ and for i odd we can choose $j_1 = \cdots = j_{n-1} = 1$ and $j_n = 2$. Since $\beta_1^R(R/I^k)$ is of degree $\ell(I)-1$ in k it follows from $\ell(I) \geq 2$ and [9, Prop 2.2, Rem. 2.5] that $\beta_2^R(R/I^k)$ is also of degree $\ell(i)-1$ in k. Hence the asserted bound is achieved and the claim follows since the sum on the right hand side of (11) runs over a positive linear combination of products of polynomials with positive leading coefficients. $\triangleright$

By induction hypothesis for $j \leq \lfloor \frac{i+1}{2} \rfloor - 1$ we have that $\varepsilon_j(R/I^k)$ is a polynomial of degree

$$(\ell(I)-1)\lfloor \frac{j+1}{2} \rfloor \leq (\ell(I)-1)\lfloor \frac{i/2}{2} \rfloor < (\ell(I)-1)\lfloor \frac{i+1}{2} \rfloor.$$

Hence there is no contribution in degree $(\ell(I)-1)\lfloor \frac{i+1}{2} \rfloor$ from the $\varepsilon_j(R/I^k)$ for $j < i$ on the right hand side of (11). Thus the assertion follows. $\square$

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Fachbereich Mathematik, Universität Duisburg-Essen, Campus Essen, 45117 Essen, Germany
E-mail address: juergen.herzog@uni-essen.de

Philipps-Universität Marburg, Fachbereich Mathematik und Informatik, 35032 Marburg, Germany
E-mail address: welker@mathematik.uni-marburg.de

School of Mathematics, Statistics and Computer Science, College of Science, University of Tehran, Tehran, Iran, and School of Mathematics, Institute for Research in Fundamental Sciences (IPM), P.O. Box 19395-5746, Tehran, Iran
E-mail address: yassemi@ipm.ir