ENUMERATIVE g-THEOREMS FOR THE VERONESE CONSTRUCTION FOR FORMAL POWER SERIES AND GRADED ALGEBRAS

MARTINA KUBITZKE AND VOLKMAR WELKER

Abstract. Let $(a_n)_{n\geq 0}$ be a sequence of integers such that its generating series satisfies $\sum_{n\geq 0} a_n t^n = \frac{h(t)}{(1-t)^d}$ for some polynomial $h(t)$. For any $r \geq 1$ we study the coefficient sequence of the numerator polynomial $h_0(a^{\langle r\rangle}) + \cdots + h_{\lambda'}(a^{\langle r\rangle})t^{\lambda'}$ of the r^{th} Veronese series $a^{\langle r\rangle}(t) = \sum_{n\geq 0} a_{nr}t^n$. Under mild hypothesis we show that the vector of successive differences of this sequence up to the $\lfloor \frac{d}{2}\rfloor^{\mathrm{th}}$ entry is the f-vector of a simplicial complex for large r. In particular, the sequence satisfies the consequences of the unimodality part of the g-conjecture. We give applications of the main result to Hilbert series of Veronese algebras of standard graded algebras and the f-vectors of edgewise subdivisions of simplicial complexes.

1. Introduction

We consider rational formal power series

$$a(t) = \sum_{n\geq 0} a_n t^n \quad = \quad \frac{h_0(a) + \cdots + h_\lambda(a)t^\lambda}{(1-t)^d}, \tag{1}$$

with integer coefficients $(a_n)_{n\geq 0}$, where $h_\lambda(a) \neq 0$. We will always use the convention that $h_i(a) = 0$ if $i > \lambda$. For an integer $r \geq 1$ we are interested in the r^{th} *Veronese series*

$$a^{\langle r\rangle}(t) := \sum_{n\geq 0} a_{nr} t^n \quad = \quad \frac{h_0(a^{\langle r\rangle}) + \cdots + h_{\lambda'}(a^{\langle r\rangle})t^{\lambda'}}{(1-t)^d},$$

where $h_{\lambda'}(a^{\langle r\rangle}) \neq 0$. We call $h(a) := (h_0(a), \ldots, h_\lambda(a))$ the *h-vector* of the rational series $a(t)$ and $g(a) := (g_0(a), \ldots, g_{\lfloor \frac{\lambda}{2}\rfloor}(a))$ the *g-vector* of $a(t)$, where $g_0(a) := h_0(a)$ and $g_i(a) := h_i(a) - h_{i-1}(a)$ for $1 \leq i \leq \lfloor \frac{\lambda}{2}\rfloor$. If we write $a(t)$ as $b_1(t) + \frac{b_2(t)}{(1-t)^d}$ for polynomials $b_1(t)$, $b_2(t)$ where $b_2(t)$ is of degree $< d$, then we call $\chi(a) := b_1(0)$ the *characteristic* of $a(t)$. We are interested in enumerative properties of $h(a^{\langle r\rangle})$ and $g(a^{\langle r\rangle})$ for large r. Our approach builds on results from [5] where the transformation $h(a) \mapsto h(a^{\langle r\rangle})$ was described as a linear transformation and uses ideas from [12] where results similar in spirit to ours were derived for barycentric subdivisions of simplicial complexes. Algebraic, analytic, enumerative and probabilistic aspects of the transformation $h(a) \mapsto h(a^{\langle r\rangle})$ have recently been studied in a

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Martina Kubitzke was supported by the Austrian Science Foundation (FWF) through grant Y463-N13.

series of papers connecting this transformation to numerous other mathematical objects (see for example [2, 4, 5, 8, 11]).

The following is our main result. In the formulation of the result we denote by $f(\Delta) := (f_{-1}(\Delta), f_0(\Delta), \ldots, f_{d-1}(\Delta))$ the *f-vector* of a $(d-1)$-dimensional simplicial complex Δ, where

$$f_i(\Delta) := \Big|\{F \in \Delta \mid \dim F = i\}\Big|$$

for $-1 \leq i \leq d-1$. Recall, that $\dim F := |F| - 1$ for $F \in \Delta$ and that the *dimension* $\dim \Delta$ of Δ equals $\max_{F \in \Delta} \dim F$.

Theorem 1.1. *Let $a(t) = \sum_{n \geq 0} a_n t^n = \frac{h_0(a) + \cdots + h_\lambda(a) t^\lambda}{(1-t)^d}$ be a rational formal power series with integer coefficient sequence $(a_n)_{n \geq 0}$, where $h_\lambda(a) \neq 0$ and $h_0(a) = 1$.*

(i) *If $h_i(a) \geq 0$ for $1 \leq i \leq \lambda$ and $r \geq \max(d, \lambda)$, then there exists a simplicial complex Δ_r such that*

$$g_i(a^{\langle r \rangle}) = f_{i-1}(\Delta_r) \qquad \text{for } 0 \leq i \leq \left\lfloor \frac{d}{2} \right\rfloor.$$

(ii) *If there is an $N > 0$ such that $a_n > 0$ for $n > N$ and $\chi(a) \geq 0$, then there is an $R > N$ such that for each $r \geq R$ and $s \geq d$, there exists a simplicial complex $\Delta_{r \cdot s}$ such that*

$$g_i(a^{\langle r \cdot s \rangle}) = f_{i-1}(\Delta_{r \cdot s}) \qquad \text{for } 0 \leq i \leq \left\lfloor \frac{d}{2} \right\rfloor.$$

Our main motivation and application for the preceding result lies in the study of Hilbert series of Veronese algebras of standard graded algebras. Let k be a field and $A = \bigoplus_{n \geq 0} A_n$ be a standard graded k-algebra of dimension d. For $r \geq 1$ the r^{th} *Veronese algebra* of A is the k-algebra $A^{\langle r \rangle} := \bigoplus_{n \geq 0} A_{nr}$, which again is standard graded of dimension d. The *Hilbert series* $\text{Hilb}(A, t) := \sum_{n \geq 0} \dim_k A_n t^n$ is a rational formal power series of the form (1). By definition we have $\text{Hilb}(A^{\langle r \rangle}, t) = \text{Hilb}(A^{\langle r \rangle}, t)$. We call $h(A) := h(\text{Hilb}(A, t))$ the *h-vector* of A and $g(A) := g(\text{Hilb}(A, t))$ the *g-vector* of A. Using the fact that Cohen-Macaulayness of a standard graded k-algebra A implies that $h(A) > 0$ [7, Cor. 4.1.10] the following corollary is a direct consequence of Theorem 1.1:

Corollary 1.2. *Let A be a d-dimensional standard graded k-algebra with Hilbert series $\text{Hilb}(A, t) = \frac{h_0(a) + \cdots + h_\lambda(a) t^\lambda}{(1-t)^d}$.*

(i) *If A is Cohen-Macaulay and $r \geq \max(\lambda, d)$, then there exists a simplicial complex Δ_r such that*

$$g_i(A^{\langle r \rangle}) = f_{i-1}(\Delta_r) \qquad \text{for } 0 \leq i \leq \left\lfloor \frac{d}{2} \right\rfloor.$$

(ii) *If $\chi(A) \geq 0$, then there is an $R > 0$ such that for $r \geq R$ and $s \geq d$, there exists a simplicial complex $\Delta_{r \cdot s}$ such that*

$$g_i(A^{\langle r \cdot s \rangle}) = f_{i-1}(\Delta_{r \cdot s}) \qquad \text{for } 0 \leq i \leq \left\lfloor \frac{d}{2} \right\rfloor.$$

As a second application of our results, we can deduce properties of h- and g-vectors of the edgewise subdivision of a simplicial complex. Since the definition of edgewise subdivision requires some preparation we postpone the formulation of these results till Section 4.2. In Section 4.1 we formulate further consequences of Theorem 1.1 and some questions. Section 3 contains the proof of Theorem 1.1 which will be based on a detailed analysis of the transformation of the h-vector under the Veronese transformation given in Section 2.

2. Analysis of the h-vector transformation

In [5] it is shown that the map $h(a) \mapsto h(a^{\langle r\rangle})$ is given by a linear transformation with positive integer coefficients. The coefficients are defined as follows. For integers $r \geq 0$, $d \geq 1$ and i, let

$$\mathfrak{C}(r,d,i) := \left\{(u_1,\ldots,u_d) \in \mathbb{Z}^d \;\middle|\; {\scriptstyle u_1+\cdots+u_d=i, \atop 0\leq u_l\leq r \text{ for } 1\leq l\leq d}\right\}.$$

Then set $C(r,d,i) := |\mathfrak{C}(r,d,i)|$ for $d \geq 1$ and $C(r,0,i) := \delta_{0,i}$.

Theorem 2.1. [5, Cor. 1.2] *Let* $a(t) = \sum_{n\geq 0} a_n t^n = \frac{h_0(a)+h_1(a)t+\cdots+h_\lambda(a)t^\lambda}{(1-t)^d}$ *and* $h_\lambda(a) \neq 0$. *Then for any* $r \geq 1$ *we have*

$$a^{<r>}(t) = \frac{h_0(a^{<r>}) + h_1(a^{<r>})t + \cdots + h_m(a^{<r>})t^m}{(1-t)^d},$$

where $m := \max(\lambda, d)$ *and*

$$h_i^{<r>} = \sum_{j=0}^{s} C(r-1, d, ir-j)h_j(a)$$

for $i = 0, \ldots, m$.

In the following we focus on properties of the numbers $C(r,d,i)$. As a first result, we show that they exhibit a certain symmetry and satisfy a recurrence relation.

Lemma 2.2. (i) *For* $r \geq 1$, $d \geq 0$ *and* $0 \leq i, j \leq d$ *it holds that*

$$C(r,d,i) = C(r,d,dr-i).$$

(ii) *For* $r \geq 1$, $d \geq 1$, $i \geq 0$ *it holds that*

$$C(r,d,i) = \sum_{m=0}^{r} C(r,d-1,i-m).$$

Proof. (i) Consider the map

$$\Phi : \left\{ \begin{array}{ccc} \mathfrak{C}(r,d,i) & \to & \mathfrak{C}(r,d,dr-i) \\ (u_1,\ldots,u_d) & \mapsto & (r-u_1,\ldots,r-u_d). \end{array} \right.$$

This map is easily seen to be a bijection between the two given sets. Now the claim follows from the definition of $C(r,d,i)$.

(ii) Let $(u_1,\ldots,u_d) \in \mathfrak{C}(r,d,i)$. Then $u_1 = m$ for some $0 \leq m \leq r$ and $(u_2,\ldots,u_d) \in \mathfrak{C}(r,d-1,i-m)$. This implies the claimed recursion.

□

The following example illustrates the symmetry stated in the last lemma.

Example 2.3. For $r \geq d \geq 1$ let $C^{r,d} = (C(r-1,d,ir-j))_{\substack{0\leq i\leq d\\ 0\leq j\leq r-1}}$ be the matrix which describes the h-vector transformation from Theorem 2.1. For $r=9$ and $d=4$, we obtain

$$C^{9,4} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 216 & 165 & 120 & 84 & 56 & 35 & 20 & 10 & 4 \\ 456 & 480 & 489 & 480 & 456 & 420 & 375 & 324 & 270 \\ 56 & 84 & 120 & 165 & 216 & 270 & 324 & 375 & 420 \\ 0 & 0 & 0 & 0 & 1 & 4 & 10 & 20 & 35 \end{pmatrix}.$$

Note that the submatrix of $C^{9,4}$ which consists of the last four rows and columns is symmetric. More generally, a similar argument as in the proof of Lemma 2.2 (i) shows that we have

$$C(r-1,d,ir-j) = C(r-1,d,(d+1-i)r-(r-(j-d))) \tag{2}$$

for $j \geq d+1$.

To simplify notation we need a few definitions.

Definition 2.4. Let $d \geq 1$, $r \geq 1$ and $k \geq 0$ be integers.

(i) For $0 \leq k \leq r-1$ let $C_k^{r,d} \in \mathbb{Z}^{d+1}$ be the vector whose $(i+1)^{\text{st}}$ entry equals $C(r-1,d,ir-k)$ for $0 \leq i \leq d$. Equivalently, $C_k^{r,d}$ is the transpose of the $(k+1)^{\text{st}}$ column of $C^{r,d}$. For $k \geq r$ we set $C_k^{r,d} \in \mathbb{Z}^{d+1}$ to be the all zero vector.

(ii) For $0 \leq k \leq r-1$ let $\hat{g}_k^{r,d} = (g_0, g_1, \ldots, g_{\lfloor \frac{d}{2} \rfloor +1}) \in \mathbb{Z}^{\lfloor \frac{d}{2} \rfloor +2}$ be the vector defined by $g_0 := C(r-1,d,0\cdot r-k) = \delta_{0,k}$ and $g_i := C(r-1,d,i\cdot r-k) - C(r-1,d,(i-1)\cdot r-k)$ for $1 \leq i \leq \lfloor \frac{d}{2} \rfloor + 1$. For $k \geq r$ we set $\hat{g}_k^{r,d} \in \mathbb{Z}^{\lfloor \frac{d}{2} \rfloor +2}$ to be the all zero vector.

(iii) Let $g_k^{r,d} \in \mathbb{Z}^{\lfloor \frac{d}{2} \rfloor +1}$ be the vector obtained from $\hat{g}_k^{r,d}$ by deleting its last entry.

For example in Example 2.3 we have $C_1^{9,4} = (0,165,480,84,0)$. We can rewrite the h-vector and induced g-vector transformation for power series of the form (1) in terms of the vectors $C_k^{r,d}$ and $g_k^{r,d}$.

Remark 2.5. Let

$$a(t) = \frac{h_0 + h_1(a)t + \cdots + h_\lambda(a)t^\lambda}{(1-t)^d}.$$

be a rational series with $h_\lambda(a) \neq 0$. For $r \geq 1$ we have

$$h(a^{\langle r \rangle}) = \sum_{k=0}^{\lambda} h_k(a) C_k^{r,d} \quad \text{and} \quad g(a^{\langle r \rangle}) = \sum_{k=0}^{\lambda} h_k(a) g_k^{r,d}.$$

For $b \in \mathbb{Z}$ and a vector $v = (v_1, \ldots, v_d) \in \mathbb{Z}^d$ we use the notation

$$(b,v) := (b, v_1, \ldots, v_d) \qquad \text{and} \qquad (v,b) := (v_1, \ldots, v_d, b).$$

Moreover, we denote by $\operatorname{last}(v)$ the rightmost entry of v. We record some first observations about the vectors $g_k^{r,d}$ and $\hat{g}_k^{r,d}$.

Lemma 2.6. *Let* $1 \leq d \leq r$ *and* $0 \leq k \leq d$ *be integers.*

(i) *If* d *is odd, then* $\mathrm{last}(\hat{g}_k^{r,d}) + \mathrm{last}(\hat{g}_{d-k}^{r,d}) = 0$.

(ii) *If* d *is even, then* $\mathrm{last}(\hat{g}_k^{r,d}) + \mathrm{last}(g_{d-k}^{r,d}) = 0$.

Proof. The proof follows exactly the same steps as the proof of Lemma 2.7 in [12] except of the use of Lemma 2.2 instead of [12, Lem. 2.5 (ii)]. □

The next lemma provides some helpful recursions for the vectors $C_k^{r,d}$, $g_k^{r,d}$ and $\hat{g}_k^{r,d}$.

Lemma 2.7. *Let* $1 \leq d \leq r$ *and* $0 \leq k \leq d$ *be integers. Then:*

(i)

$$C_k^{r,d} = \sum_{j=0}^{k-1} (0, C_j^{r,d-1}) + \sum_{j=k}^{r-1} (C_j^{r,d-1}, 0).$$

(ii) *If* d *is even, then*

$$g_k^{r,d} = \sum_{l=0}^{k-1} (0, g_l^{r,d-1}) + \sum_{l=k}^{r-1} \hat{g}_l^{r,d-1}. \tag{3}$$

In particular,

$$\mathrm{last}(g_k^{r,d}) = \sum_{l=0}^{k-1} \mathrm{last}(g_l^{r,d-1}) + \sum_{l=k}^{r-1} \mathrm{last}(\hat{g}_l^{r,d-1}). \tag{4}$$

(iii) *If* d *is odd, then*

$$\hat{g}_k^{r,d} = \sum_{l=0}^{k-1} (0, g_l^{r,d-1}) + \sum_{l=k}^{r-1} \hat{g}_l^{r,d-1}. \tag{5}$$

In particular,

$$\mathrm{last}(\hat{g}_k^{r,d}) = \sum_{l=0}^{k-1} \mathrm{last}(g_l^{r,d-1}) + \sum_{l=k}^{r-1} \mathrm{last}(\hat{g}_l^{r,d-1}). \tag{6}$$

Proof. (i) is a rephrasing of the recursion given in Lemma 2.2 (ii). Part (ii) and (iii) follow by a straightforward computation from the definition of the vectors $g_k^{r,d}$ and $\hat{g}_k^{r,d}$ and Lemma 2.2 (ii). □

We will further need the following technical lemma, which analyzes the behavior of the last entries of the vectors $\hat{g}_k^{r,d}$ for $k \geq d+1$ (as opposed to Lemma 2.6 where this is achieved for $0 \leq k \leq d$).

Lemma 2.8. *Let* $1 \leq d \leq r$ *and* $d+1 \leq k \leq r-1$ *be integers.*

(i) *If* d *is even, then* $\mathrm{last}(\hat{g}_k^{r,d}) + \mathrm{last}(\hat{g}_{r-(k-d)}^{r,d}) = 0$.

(ii) *If* d *is odd, then*

$$\mathrm{last}(\hat{g}_k^{r,d}) + C(r-1, d, \frac{d+1}{2}r + k - d) \quad = \quad C(r-1, d, \frac{d-1}{2}r + k - d)$$

Proof. (i) Let d be even and $d+1 \leq k \leq r-1$. By definition it holds that

$$\begin{aligned}
\operatorname{last}(\hat{g}_k^{r,d}) &= C\left(r-1, d, (\frac{d}{2}+1)r-k\right) - C\left(r-1, d, \frac{d}{2}r-k\right) \\
&= C\left(r-1, d, \frac{d}{2}r-(r-(k-d))\right) - C\left(r-1, d, (\frac{d}{2}+1)r-(r-(k-d))\right) \\
&= -\operatorname{last}(\hat{g}_{r-(k-d)}^{r,d}),
\end{aligned}$$

where the second equality follows from the symmetry in (2). This shows (i).

(ii) Let d be odd and $d+1 \leq k \leq r-1$. Since d is odd, it holds that $\lfloor \frac{d}{2} \rfloor = \frac{d-1}{2}$ and thus by definition

$$\begin{aligned}
\operatorname{last}(\hat{g}_k^{r,d}) &= C\left(r-1, d, (\frac{d-1}{2}+1)r-k\right) - C\left(r-1, d, \frac{d-1}{2}r-k\right) \\
&\overset{(2)}{=} C\left(r-1, d, \frac{d+1}{2}r-(r-(k-d))\right) - C\left(r-1, d, \frac{d+3}{2}r-(r-(k-d))\right).
\end{aligned}$$

This finishes the proof of (ii).

□

The following example provides a list of the vectors $C_k^{r,d}$, $g_k^{r,d}$ and $\hat{g}_k^{r,d}$ for small values of d.

Example 2.9.

		$C_k^{r,d}$	$g_k^{r,d}$	$\hat{g}_k^{r,d}$
$d=1$	$k=0$	$(1,0)$	(1)	$(1,-1)$
	$k=1$	$(0,1)$	(0)	$(0,1)$
	$k=2,\dots,r-1$	$(0,1)$	(0)	$(0,1)$
$d=2$	$k=0$	$(1,r-1,0)$	$(1,r-2)$	$(1,r-2,-r+1)$
	$k=1$	$(0,r,0)$	$(0,r)$	$(0,r,-r)$
	$k=2,\dots,r-1$	$(0,r-k+1,k-1)$	$(0,r-k+1)$	$(0,r-k+1,2k-r-2)$
$d=3$	$k=0$	$(1,7,1,0)$	$(1,6)$	$(1,6,-6)$
$r=3$	$k=1$	$(0,6,3,0)$	$(0,6)$	$(0,6,-3)$
	$k=2$	$(0,3,6,0)$	$(0,3)$	$(0,3,3)$
$d=4$	$k=0$	$(1,31,31,1,0)$	$(1,30,0)$	$(1,30,0,-30)$
$r=4$	$k=1$	$(0,20,40,4,0)$	$(0,20,20)$	$(0,20,20,-36)$
	$k=2$	$(0,10,44,10,0)$	$(0,10,34)$	$(0,10,34,-34)$
	$k=3$	$(0,4,40,20,0)$	$(0,4,36)$	$(0,4,36,-20)$

Note that in the above example, the entries of the vectors $g_k^{r,d}$ are exclusively non-negative whereas the last entry of $\hat{g}_k^{r,d}$ can also be negative. Our next aim is to prove that this is true in general.

Lemma 2.10. *Let $1 \leq d \leq r$ and $0 \leq k \leq r-1$ be integers. Then:*

(i) $g_k^{r,d}$ *is non-negative and* $\operatorname{last}(g_k^{r,d}) = 0$ *if and only if d is even, $d=r$ and $k=0$, or $d=1$ and $k>0$.*

(ii) *If d is odd, then* $\operatorname{last}(\hat{g}_k^{r,d}) > 0$ *for* $k \geq \frac{d}{2}$ *and* $\operatorname{last}(\hat{g}_k^{r,d}) < 0$ *for* $k < \frac{d}{2}$.

(iii) *If d is even, then* $\operatorname{last}(\hat{g}_k^{r,d}) > 0$ *for* $k > \frac{d+r}{2}$ *and* $\operatorname{last}(\hat{g}_k^{r,d}) < 0$ *for* $k < \frac{d+r}{2}$. *If r is even, then* $\operatorname{last}(\hat{g}_{\frac{d+r}{2}}^{r,d}) = 0$.

Proof. For $d \leq 2$ the statements follow from Example 2.9. Let $d > 2$.

Case: d even.

(i) By Equation (3) we can express $g_k^{r,d}$ as

$$g_k^{r,d} = \sum_{l=k}^{r-1} \hat{g}_l^{r,d-1} + \sum_{l=0}^{k-1} (0, g_l^{r,d-1}).$$

From part (i) of the induction hypothesis we infer that all but the last entry of $g_k^{r,d}$ are non-negative. The last entry is given as

$$\operatorname{last}(g_k^{r,d}) = \sum_{l=k}^{r-1} \operatorname{last}(\hat{g}_l^{r,d-1}) + \sum_{l=0}^{k-1} \operatorname{last}(g_l^{r,d-1}). \tag{7}$$

We now distinguish the cases $k \leq \frac{d}{2}$ and $k < \frac{d}{2}$.

▷ $k \geq \frac{d}{2}$. The induction hypothesis (ii) implies that $\operatorname{last}(\hat{g}_k^{r,d-1}) > 0$. Since the first sum on the right-hand side of (7) contains at least one summand and since $\operatorname{last}(g_l^{r,d-1}) \geq 0$ by the induction hypothesis (i) we conclude that $\operatorname{last}(g_k^{r,d}) > 0$.

▷ $k < \frac{d}{2}$. Lemma 2.6 (i) implies that $\sum_{l=k}^{d-1-k} \operatorname{last}(\hat{g}_k^{r,d-1}) = 0$. We therefore obtain

$$\operatorname{last}(g_k^{r,d}) = \sum_{l=d-k}^{r-1} \operatorname{last}(\hat{g}_l^{r,d-1}) + \sum_{l=0}^{k-1} \operatorname{last}(g_l^{r,d-1}).$$

Since $d-1 > 1$ and $l \geq \frac{d}{2}$ for $d-k \leq l$ and $k < \frac{d}{2}$, all summands in the first and the second sum on the right-hand side of Equation (7) are strictly positive by the induction hypothesis (ii) and (i), respectively. Hence, $\operatorname{last}(g_k^{r,d}) \geq 0$. Furthermore, $\operatorname{last}(g_k^{r,d}) = 0$ if and only if $d-k > r-1$ and $k = 0$, equivalently $d > r-1$ and $k = 0$, equivalently $d = r$ and $k = 0$.

(iii) Let $k > \frac{d+r}{2}$. Using Lemma 2.2 (ii) and the definition of $\hat{g}_k^{r,d}$ we obtain that the last entry of $\hat{g}_k^{r,d}$ can be written as

$$\begin{aligned}
\mathrm{last}(\hat{g}_k^{r,d}) &= \sum_{l=0}^{k-1} \mathrm{last}(\hat{g}_l^{r,d-1}) + \\
&\quad \sum_{l=k}^{r-1} \Big(C(r-1, d-1, \frac{d+2}{2}r - l) - C(r-1, d-1, \frac{d}{2}r - l) \Big) \\
&= \sum_{l=0}^{d-1} \mathrm{last}(\hat{g}_l^{r,d-1}) + \sum_{l=d}^{d+r-k-1} \mathrm{last}(\hat{g}_l^{r,d-1}) + \sum_{l=d+r-k}^{k-1} \mathrm{last}(\hat{g}_l^{r,d-1}) + \\
&\quad \sum_{l=k}^{r-1} \Big(C(r-1, d-1, \frac{d+2}{2}r - l) - C(r-1, d-1, \frac{d}{2}r - l) \Big).
\end{aligned}$$

By Lemma 2.6 (i) it holds that $\sum_{l=0}^{d-1} \mathrm{last}(\hat{g}_l^{r,d-1}) = 0$. Lemma 2.8 (ii) further implies

$$\sum_{l=k}^{r-1} \Big(C(r-1, d-1, \frac{d+2}{2}r - l) - C(r-1, d-1, \frac{d}{2}r - l) \Big) = - \sum_{l=d}^{d+r-k-1} \mathrm{last}(\hat{g}_l^{r,d-1}).$$

The expression for $\mathrm{last}(\hat{g}_k^{r,d})$ thus simplifies to

$$\mathrm{last}(\hat{g}_k^{r,d}) = \sum_{l=d+r-k}^{k-1} \mathrm{last}(\hat{g}_l^{r,d-1}).$$

From the induction hypothesis (ii) we finally conclude that $\mathrm{last}(\hat{g}_k^{r,d}) \geq 0$. Furthermore, we have $\mathrm{last}(\hat{g}_k^{r,d}) = 0$ if and only if $k-1 < d+r-k$, i.e., $k < \frac{d+r+1}{2}$. Since we also have $k \geq \frac{d+r}{2}$ the last condition is true if and only if $k = \frac{d+r}{2}$ and r is even. From the symmetry in Lemma 2.8 (i) it further follows that $\mathrm{last}(\hat{g}_k^{r,d}) < 0$ for $d+1 \leq k < \frac{d+r}{2}$. Since, by Lemma 2.6 (ii), $\mathrm{last}(\hat{g}_k^{r,d}) = -\mathrm{last}(g_{d-k}^{r,d})$ for $0 \leq k \leq d$ we deduce from (i) that $\mathrm{last}(\hat{g}_k^{r,d}) < 0$ for $0 \leq k \leq d$.

Case: d odd.

(i) Using (5) we can write the vector $\hat{g}_k^{r,d}$ as

$$\hat{g}_k^{r,d} = \sum_{l=k}^{r-1} \hat{g}_l^{r,d-1} + \sum_{l=0}^{k-1} (0, g_l^{r,d-1}).$$

It follows from part (i) of the induction hypothesis that $g_k^{r,d}$ is non-negative. Since the first sum on the right-hand side of the last equation contains the term $\hat{g}_{r-1}^{r,d-1}$ which by the induction hypothesis (iii) has a strictly positive last entry, we see that $\mathrm{last}(g_k^{r,d}) > 0$.

(ii) By (6) we have

$$\mathrm{last}(\hat{g}_k^{r,d}) = \sum_{l=k}^{r-1} \mathrm{last}(\hat{g}_l^{r,d-1}) + \sum_{l=0}^{k-1} \mathrm{last}(g_l^{r,d-1}). \tag{8}$$

Assume first that $k < d$. By Lemma 2.8 (i) it holds that $\sum_{l=d}^{r-1} \mathrm{last}(\hat{g}_l^{r,d-1}) = 0$. This yields

$$\begin{aligned} \mathrm{last}(\hat{g}_k^{r,d}) &= \sum_{l=k}^{d-1} \mathrm{last}(\hat{g}_l^{r,d-1}) + \sum_{l=0}^{d-1-k} \mathrm{last}(g_l^{r,d-1}) + \sum_{l=d-k}^{k-1} \mathrm{last}(g_l^{r,d-1}) \\ &= \sum_{l=d-k}^{k-1} \mathrm{last}(g_l^{r,d-1}), \end{aligned}$$

where the last equality holds by Lemma 2.6 (ii). For $k \geq \frac{d}{2}$ the last sum consists at least of the summand $\mathrm{last}(g_{\frac{d-1}{2}}^{r,d-1})$ and the induction hypothesis (i) finally implies $\mathrm{last}(\hat{g}_k^{r,d}) > 0$. This combined with Lemma 2.6 (i) also shows that $\mathrm{last}(\hat{g}_k^{r,d}) < 0$ for $0 \leq k \leq \frac{d}{2}$.

Now let $k \geq d$. The induction hypothesis (i) implies that $\sum_{l=0}^{k-1} \mathrm{last}(g_l^{r,d-1}) > 0$. If $k > \frac{d-1+r}{2}$, then we infer from part (iii) of the induction hypothesis that $\sum_{l=k}^{r-1} \mathrm{last}(\hat{g}_l^{r,d-1}) > 0$ and the claim follows from (8). Assume that $k \leq \frac{d-1+r}{2}$. In this case we can write

$$\sum_{l=k}^{r-1} \mathrm{last}(\hat{g}_l^{r,d-1}) = \sum_{l=k}^{r-k+d} \mathrm{last}(\hat{g}_l^{r,d-1}) + \sum_{l=r-k+d+1}^{r-1} \mathrm{last}(\hat{g}_l^{r,d-1}).$$

By Lemma 2.8 (i) it holds that $\sum_{l=k}^{r-k+d} \mathrm{last}(\hat{g}_l^{r,d-1}) = 0$ and from the induction hypothesis (iii) we know that $\sum_{l=r-k+d+1}^{r-1} \mathrm{last}(\hat{g}_l^{r,d-1}) \geq 0$. The above reasoning together with (8) finally completes the proof.

□

The next lemma is concerned with the behavior of the vectors $g_k^{r,d}$ when r grows.

Proposition 2.11. *Let $1 \leq d \leq r$ and $0 \leq k \leq r$ be integers. Then:*

(i) $g_k^{r,d} \leq g_k^{r+1,d}$ *(componentwise).*

(ii) *If d is odd, then* $\mathrm{last}(\hat{g}_k^{r+1,d}) \geq \mathrm{last}(\hat{g}_k^{r,d})$ *for* $k \geq \frac{d}{2}$.

(iii) *If d is even, then* $\mathrm{last}(\hat{g}_{k+1}^{r+1,d}) \geq \mathrm{last}(\hat{g}_k^{r,d})$ *for* $k \geq d$.

Proof. We proceed by induction on d and show all three parts of the lemma simultaneously. For $d \leq 2$, Example 2.9 verifies the assertion. Now let $d > 2$.

Case: d even.

(i) By Equation (3) the vector $g_k^{r+1,d}$ can be computed in the following way:

$$g_k^{r+1,d} = \sum_{l=0}^{k-1}(0, g_l^{r+1,d-1}) + \sum_{l=k}^{r} \hat{g}_l^{r+1,d-1}.$$

In the sequel, let $\bar{v}$ denote the vector which is obtained from a vector v by deleting its last entry. It follows from the induction hypothesis (i) that

$$\begin{aligned}\bar{g}_k^{r+1,d} &\geq \sum_{l=0}^{k-1} \overline{(0, g_l^{r,d-1})} + \sum_{l=k}^{r-1} g_l^{r,d-1} + g_r^{r+1,d-1} \\ &= \bar{g}_k^{r,d} + g_r^{r+1,d-1} \geq \bar{g}_k^{r,d},\end{aligned}$$

where the last inequality holds by Lemma 2.10 (i). It remains to show the desired inequality for the last entries of $g_k^{r+1,d}$ and $g_k^{r,d}$. First assume that $k \geq \frac{d}{2}$. In this case, it holds that

$$\begin{aligned}\mathrm{last}(g_k^{r+1,d}) &= \sum_{l=0}^{k-1} \mathrm{last}(g_l^{r+1,d-1}) + \sum_{l=k}^{r} \mathrm{last}(\hat{g}_l^{r+1,d-1}) \\ &\geq \sum_{l=0}^{k-1} \mathrm{last}(g_l^{r,d-1}) + \sum_{l=k}^{r-1} \mathrm{last}(\hat{g}_l^{r+1,d-1}) + \mathrm{last}(\hat{g}_r^{r+1,d-1}) \\ &\geq \sum_{l=0}^{k-1} \mathrm{last}(g_l^{r,d-1}) + \sum_{l=k}^{r-1} \mathrm{last}(\hat{g}_l^{r,d-1}) + \mathrm{last}(\hat{g}_r^{r+1,d-1}) \\ &\geq \sum_{l=0}^{k-1} \mathrm{last}(g_l^{r,d-1}) + \sum_{l=k}^{r-1} \mathrm{last}(\hat{g}_l^{r,d-1})\end{aligned}$$

Here the first and the second inequality follow from the induction hypothesis (i) and (ii), respectively. The third inequality is a consequence of Lemma 2.10 (ii). Next, let $k < \frac{d}{2}$. In this case, we infer from Lemma 2.6 (i) that $\sum_{l=k}^{d-1-k} \mathrm{last}(\hat{g}_l^{r+1,d-1}) = 0$. Combining this with the previous reasoning we can therefore conclude that

$$\begin{aligned}\mathrm{last}(g_k^{r+1,d}) &\geq \sum_{l=0}^{k-1} \mathrm{last}(g_l^{r,d-1}) + \sum_{l=d-k}^{r-1} \mathrm{last}(\hat{g}_l^{r+1,d-1}) + \mathrm{last}(\hat{g}_r^{r+1,d-1}) \\ &\geq \sum_{l=0}^{k-1} \mathrm{last}(g_l^{r,d-1}) + \sum_{l=d-k}^{r-1} \mathrm{last}(\hat{g}_l^{r+1,d-1}),\end{aligned}$$

where the last inequality follows from Lemma 2.10 (ii). For $l \geq d-k$ and $k < \frac{d}{2}$ it holds that $l > \frac{d}{2}$. Hence, part (ii) of the induction hypothesis (ii) implies

$\operatorname{last}(\hat{g}_l^{r+1,d-1}) \geq \operatorname{last}(\hat{g}_l^{r,d-1})$ for $d-k \leq l \leq r-1$. We finally obtain

$$\begin{aligned}\operatorname{last}(g_k^{r+1,d}) &\geq \sum_{l=0}^{k-1} \operatorname{last}(g_l^{r,d-1}) + \sum_{l=d-k}^{r-1} \operatorname{last}(\hat{g}_l^{r,d-1})\\ &= \sum_{l=0}^{k-1} \operatorname{last}(g_l^{r,d-1}) + \sum_{l=k}^{r-1} \operatorname{last}(\hat{g}_l^{r,d-1})\\ &= \operatorname{last}(g_k^{r,d-1}).\end{aligned}$$

For the second equality we use that by Lemma 2.6 (i) we have $\sum_{l=k}^{d-k-1} \operatorname{last}(\hat{g}_l^{r,d-1}) = 0$.

(iii) We have shown in the proof of Lemma 2.10 (iii) that for $k \geq d$ the last entry of $\hat{g}_k^{r,d}$ can be expressed as

$$\operatorname{last}(\hat{g}_k^{r,d}) = \sum_{l=d+r-k}^{k-1} \operatorname{last}(\hat{g}_l^{r,d-1}). \tag{9}$$

Using part (ii) of the induction hypothesis and Lemma 2.10 (ii) we deduce

$$\begin{aligned}\operatorname{last}(\hat{g}_k^{r,d}) &\leq \sum_{l=d+r-k}^{k-1} \operatorname{last}(\hat{g}_l^{r+1,d-1})\\ &\leq \sum_{l=d+r-k}^{k-1} \operatorname{last}(\hat{g}_l^{r+1,d-1}) + \operatorname{last}(\hat{g}_k^{r+1,d-1})\\ &= \operatorname{last}(\hat{g}_{k+1}^{r+1,d}),\end{aligned}$$

where the last equality holds by (9).

Case: d odd.

(i) It follows from (5) that the vector $g_k^{r+1,d}$ can be computed as

$$\begin{aligned}g_k^{r+1,d} &= \sum_{l=0}^{k-1} \overline{(0, g_l^{r+1,d-1})} + \sum_{l=k}^{r} g_l^{r+1,d-1}\\ &\geq \sum_{l=0}^{k-1} \overline{(0, g_l^{r,d-1})} + \sum_{l=k}^{r-1} g_l^{r,d-1} + g_r^{r+1,d-1}\\ &\overset{(5)}{=} g_k^{r,d-1} + g_r^{r+1,d-1}\\ &\geq g_k^{r,d}.\end{aligned}$$

Here, the first and the last inequality follow from the induction hypothesis (i) and Lemma 2.10 (i), respectively.

(ii) First, suppose that $\frac{d}{2} \leq k < d$. We have shown in the proof of Lemma 2.10 (ii) that the last entry of $\hat{g}_k^{r,d}$ can be written as

$$\mathrm{last}(\hat{g}_k^{r,d}) = \sum_{l=d-k}^{k-1} \mathrm{last}(g_l^{r,d-1}).$$

Using this expression and part (i) of the induction hypothesis we obtain

$$\mathrm{last}(\hat{g}_k^{r,d}) \leq \sum_{l=d-k}^{k-1} \mathrm{last}(g_l^{r+1,d-1}) = \mathrm{last}(\hat{g}_k^{r+1,d}).$$

Next, assume that $k \geq d$. The last entry of $\hat{g}_k^{r+1,d}$ is given by

$$\mathrm{last}(\hat{g}_k^{r+1,d}) = \sum_{l=0}^{k-1} \mathrm{last}(g_l^{r+1,d-1}) + \sum_{l=k}^{r} \mathrm{last}(\hat{g}_l^{r+1,d-1}).$$

Moreover, Lemma 2.8 (i) implies that $\sum_{l=k}^{r+1-k+d} \mathrm{last}(\hat{g}_l^{r+1,d-1}) = 0$. Using this and part (i) of the induction hypothesis we conclude

$$\begin{aligned}\mathrm{last}(\hat{g}_k^{r+1,d}) &= \sum_{l=0}^{k-1} \mathrm{last}(g_l^{r+1,d-1}) + \sum_{l=r+2-k+d}^{r} \mathrm{last}(\hat{g}_l^{r+1,d-1}) \\ &\geq \sum_{l=0}^{k-1} \mathrm{last}(g_l^{r,d-1}) + \sum_{l=r+2-k+d}^{r} \mathrm{last}(\hat{g}_l^{r+1,d-1}).\end{aligned}$$

Applying part (iii) of the induction hypothesis to the last sum, it follows that

$$\begin{aligned}\mathrm{last}(\hat{g}_k^{r+1,d}) &\geq \sum_{l=0}^{k-1} \mathrm{last}(g_l^{r,d-1}) + \sum_{l=r+2-k+d}^{r} \mathrm{last}(\hat{g}_{l-1}^{r,d-1}) \\ &= \sum_{l=0}^{k-1} \mathrm{last}(g_l^{r,d-1}) + \sum_{l=k}^{r-k+d} \mathrm{last}(\hat{g}_l^{r,d-1}) + \sum_{l=r+1-k+d}^{r-1} \mathrm{last}(\hat{g}_l^{r,d-1}) \\ &= \sum_{l=0}^{k-1} \mathrm{last}(g_l^{r,d-1}) + \sum_{l=k}^{r-1} \mathrm{last}(\hat{g}_l^{r,d-1}) \\ &= \mathrm{last}(\hat{g}_k^{r,d}).\end{aligned}$$

For the first equality we have used that $\sum_{l=k}^{r-k+d} \mathrm{last}(\hat{g}_l^{r,d-1}) = 0$ by Lemma 2.8 (i).

□

As a consequence of the last lemma and of Lemma 2.10 we obtain the following relation between the vectors $g_k^{1,d}$ and $g_k^{r,d}$ for $r \geq d$.

Corollary 2.12. *Let $1 \leq d \leq r$ and $0 \leq k \leq d-1$ be integers. Then:*

$$g_k^{1,d} \leq g_k^{d,d} \leq \cdots \leq g_k^{r,d} \leq g_k^{r+1,d}.$$

Proof. All inequalities except for the leftmost follow from Proposition 2.11. It is easily seen that

$$g_k^{1,d} = \begin{cases} (0, \quad 0, 0, \dots, 0) & \text{if } k \neq 0 \\ (1, -1, 0, \dots, 0) & \text{if } k = 0 \end{cases}$$

The claim now follows directly from Lemma 2.10 (i). □

3. Proof of Theorem 1.1

We now recall the definition and important properties of admissible vectors introduced by Murai in [12].

Definition 3.1. Let $f = (1, f_0, f_1, \dots, f_d) \in \mathbb{Z}^{d+2}$ be the f-vector of a simplicial complex (We do not assume $f_d \neq 0$) and let $\alpha = (0, 1, \alpha_0, \dots, \alpha_{d-1})$, $\beta \in \mathbb{Z}^{d+2}$.

- (i) The vector α is called a *basic admissible vector* for f if $(1, \alpha_0, \dots, \alpha_{d-1})$ is the f-vector of a simplicial complex and $f_i \geq \alpha_i$ for $0 \leq i \leq d-1$.
- (ii) The vector β is called *admissible* for f if there exist vectors $\beta^{(1)}, \dots, \beta^{(t)} \in \mathbb{Z}^{d+2}$ such that $\beta = \beta^{(1)} + \cdots + \beta^{(t)}$ and $\beta^{(k)}$ is a basic admissible vector for $f + \beta^{(1)} + \cdots + \beta^{(k-1)}$ for $1 \leq k \leq t$.

The following lemma states some properties of admissible vectors which will be crucial for the proof of Theorem 1.1 (i).

Lemma 3.2. [12, Lem. 3.1] *Let $f = (1, f_0, f_1, \dots, f_d) \in \mathbb{Z}^{d+2}$ be the f-vector of a simplicial complex and let $\alpha = (0, \alpha_{-1}, \alpha_0, \dots, \alpha_{d-1}) \in \mathbb{Z}^{d+2}$ be admissible for f. Then:*

- (i) *$f + \alpha$ is the f-vector of a simplicial complex.*
- (ii) *If $g \in \mathbb{Z}^{d+2}$ is the f-vector of a simplicial complex and $g \geq f$ (componentwise), then α is admissible for g.*
- (iii) *If $\beta \in \mathbb{Z}^{d+2}$ is admissible for f, then $\alpha + \beta$ is admissible for f.*
- (iv) *For any integer $0 \leq b \leq \alpha_{d-1}$, the vector $(0, 1, \alpha_0, \dots, \alpha_{d-2}, \alpha_{d-1} - b)$ is admissible for f.*

Now Theorem 1.1 (i) follows from the next proposition, Lemma 3.2 and Remark 2.5. Note that Proposition 3.3 and Lemma 3.2 require $r \geq d$. If $r \geq \lambda$ then Proposition 3.3 (ii) implies that $g_k^{r,d}$ is admissible for $g_0^{r,d}$ for $1 \leq k \leq \lambda$. Hence by Remark 2.5 and Lemma 3.2 the assertion follows for $r \geq \max(d, \lambda)$.

Proposition 3.3. *Let $r \geq d > 0$ be positive integers. Then:*

- (i) *$g_0^{r,d}$ is the f-vector of a simplicial complex.*
- (ii) *$g_k^{r,d}$ is admissible for $g_0^{r,d}$ for $1 \leq k \leq r$.*
- (iii) *If d is odd and $k \geq \frac{d}{2}$, then $\hat{g}_k^{r,d}$ is admissible for $(g_0^{r,d}, 0)$.*

Note, that Theorem 1.1 (i) already follows from Proposition 3.3 (i) and (ii). However, we will use assertion (iii) to show (i) and (ii) by induction.

Proof. Throughout the proof we denote for a vector v by $\bar{v}$ the vector obtained from v by deleting its last entry.

We proceed by induction on d and prove (i), (ii) and (iii) simultaneously. For $d \leq 2$ the claim follows from Example 2.9. Now let $d > 2$.

Case: d odd.

(i) We obtain from (5)

$$g_0^{r,d} = \sum_{l=0}^{r-1} g_l^{r,d-1}. \tag{10}$$

It follows from the induction hypothesis (i) that $g_0^{r,d-1}$ is the f-vector of a simplicial complex and from (ii) it follows that $g_l^{r,d-1}$ is admissible for $g_0^{r,d-1}$ for $1 \leq l \leq r-1$. Lemma 3.2 (iii) implies that $\sum_{l=1}^{r-1} g_l^{r,d-1}$ is admissible for $g_0^{r,d-1}$. From part (i) of the same lemma and (10) we infer that $g_0^{r,d}$ is the f-vector of a simplicial complex. This shows (i).

(ii) From (5) we deduce that

$$g_k^{r,d} = \sum_{l=k}^{r-1} g_l^{r,d-1} + \sum_{l=0}^{k-1} \overline{(0, g_l^{r,d-1})}.$$

By part (ii) of the induction hypothesis $g_l^{r,d-1}$ is admissible for $g_0^{r,d-1}$ for $k \leq l \leq r-1$. Thus, from Lemma 3.2 we infer that $\sum_{l=k}^{r-1} g_l^{r,d-1}$ is admissible for $g_0^{r,d-1}$. Lemma 2.10 (i) combined with (10) yields that $g_0^{r,d} \geq g_0^{r,d-1}$. Since we have already shown that $g_0^{r,d}$ is the f-vector of a simplicial complex it now follows from Lemma 3.2 (ii) that $\sum_{l=k}^{r-1} g_l^{r,d-1}$ is also admissible for $g_0^{r,d}$. It remains to prove that $\sum_{l=0}^{k-1} \overline{(0, g_l^{r,d-1})}$ is admissible for $g_0^{r,d}$. The claim then follows from Lemma 3.2 (iii). We know from the induction hypothesis and Lemma 3.2 (i) that $\sum_{l=0}^{k-1} g_l^{r,d-1}$ is the f-vector of a simplicial complex. Removal of the last entry preserves this property, thus $\sum_{l=0}^{k-1} \bar{g}_l^{r,d-1}$ is the f-vector of a simplicial complex. Since by (10) and Lemma 2.10 it further holds that $\sum_{l=0}^{k-1} \bar{g}_l^{r,d-1} \leq \bar{g}_0^{r,d}$ we conclude that $(0, \sum_{l=0}^{k-1} \bar{g}_l^{r,d-1})$ is a basic admissible vector for $\bar{g}_0^{r,d}$ which finishes the proof of (ii).

(iii) By (5) we have that

$$\begin{aligned} \hat{g}_k^{r,d} &= \sum_{l=k}^{r-1} \hat{g}_l^{r,d-1} + \sum_{l=0}^{k-1} (0, g_l^{r,d-1}) \\ &= \sum_{l=k}^{d-1} \hat{g}_l^{r,d-1} + \sum_{l=d}^{r-1} \hat{g}_l^{r,d-1} + \sum_{l=0}^{k-1} (0, g_l^{r,d-1}) \end{aligned}$$

Since by Lemma 2.8 (i) $\sum_{l=d}^{r-1} \mathrm{last}(\hat{g}_l^{r,d-1}) = 0$ it follows that $\sum_{l=d}^{r-1} \hat{g}_l^{r,d-1} = \sum_{l=d}^{r-1} (g_l^{r,d-1}, 0)$. Lemma 2.10 states that $\mathrm{last}(\hat{g}_l^{r,d-1}) < 0$ for $k \leq l \leq d-1$, i.e., $\hat{g}_l^{r,d-1} \leq (g_l^{r,d-1}, 0)$ (componentwise). We thus obtain

$$\begin{aligned}\hat{g}_k^{r,d} &\le \sum_{l=k}^{d-1}(g_l^{r,d-1},0)+\sum_{l=d}^{r-1}(g_l^{r,d-1},0)+\sum_{l=0}^{k-1}(0,g_l^{r,d-1})\\ &=\sum_{l=k}^{r-1}(g_l^{r,d-1},0)+\sum_{l=0}^{k-1}(0,g_l^{r,d-1}).\end{aligned}$$

The proof of (ii) shows that the right-hand side of the above inequality is admissible for $(g_0^{r,d},0)$. Since by Lemma 2.10 (i) and (ii) $\hat{g}_k^{r,d}$ is non-negative and since the inequality above is an equality for all but the last entry it follows from Lemma 3.2 (iv) that $\hat{g}_k^{r,d}$ is admissible for $(g_0^{r,d},0)$.

Case: d even.

(i) Equation (3) implies

$$\begin{aligned}g_0^{r,d} &= \sum_{l=0}^{r-1}\hat{g}_l^{r,d-1}\\ &=\sum_{l=0}^{d-1}(g_l^{r,d-1},\mathrm{last}(\hat{g}_l^{r,d-1}))+\sum_{l=d}^{r-1}\hat{g}_l^{r,d-1}.\end{aligned}\tag{11}$$

By Lemma 2.6 (i) it holds that $\sum_{l=0}^{d-1}\mathrm{last}(\hat{g}_l^{r,d-1})=0$. By the induction hypothesis (i) and (ii) and Lemma 3.2 $\sum_{l=1}^{d-1}g_l^{r,d-1}$ is admissible for $g_0^{r,d-1}$. Hence, $\sum_{l=1}^{d-1}(g_l^{r,d-1},0)$ is admissible for $(g_0^{r,d-1},0)$. By part (iii) of the induction hypothesis $\hat{g}_l^{r,d-1}$ is admissible for $(g_0^{r,d-1},0)$ as well. Lemma 3.2 (i) together with the above reasoning implies that $g_0^{r,d}$ is the f-vector of a simplicial complex.

(ii) Equation (5) yields

$$\begin{aligned}\hat{g}_k^{r,d} &= \sum_{l=k}^{r-1}\hat{g}_l^{r,d-1}+\sum_{l=0}^{k-1}(0,g_l^{r,d-1})\\ &=\sum_{l=0}^{k-1}(0,g_l^{r,d-1})+\sum_{l=k}^{\frac{d}{2}-1}\hat{g}_l^{r,d-1}+\sum_{l=\frac{d}{2}}^{r-1}\hat{g}_l^{r,d-1}\\ &\le\sum_{l=0}^{k-1}(0,g_l^{r,d-1})+\sum_{l=k}^{\frac{d}{2}-1}(g_l^{r,d-1},0)+\sum_{l=\frac{d}{2}}^{r-1}\hat{g}_l^{r,d-1},\end{aligned}$$

where the last inequality follows from Lemma 2.10 (ii). Since this inequality is an equality for all entries but the last one it follows from Lemma 3.2 (iv) that it suffices to show that the right-hand side of the above inequality is admissible for $g_0^{r,d}$. By the induction hypothesis $g_l^{r,d-1}$ is admissible for $g_0^{r,d-1}$ which by Lemma 3.2

(i) and (iii) implies that $\sum_{l=0}^{k-1} g_l^{r,d-1}$ is the f-vector of a simplicial complex. From (11) and Lemma 2.10 (i) we deduce that $\sum_{l=0}^{k-1} g_l^{r,d-1} \leq \bar{g}_0^{r,d}$. This finally shows that $\sum_{l=0}^{k-1}(0, g_l^{r,d-1})$ is admissible for $g_0^{r,d}$. Furthermore, by the induction hypothesis, $g_l^{r,d-1}$ is admissible for $g_0^{r,d-1}$ for $k \leq l \leq \frac{d}{2} - 1$ and thus, by Lemma 3.2 (iii) $\sum_{l=k}^{\frac{d}{2}-1} g_l^{r,d-1}$ is admissible for $g_0^{r,d-1}$. In particular, $\sum_{l=k}^{\frac{d}{2}-1}(g_l^{r,d-1}, 0)$ is admissible for $(g_0^{r,d-1}, 0)$. Since by Lemma 2.10 (i) and (11) it holds that $g_0^{r,d} \geq (g_0^{r,d-1}, 0)$ Lemma 3.2 (ii) implies that $\sum_{l=k}^{\frac{d}{2}-1}(g_l^{r,d-1}, 0)$ is admissible for $g_0^{r,d}$. By (iii) of the induction hypothesis $\hat{g}_l^{r,d-1}$ is admissible for $(g_0^{r,d-1}, 0)$ for $\frac{d}{2} \leq l \leq r - 1$. This finishes the proof of (ii).

$\square$

It remains to verify Theorem 1.1 (ii).

Proof of Theorem 1.1 (ii). The assumptions imply that after passing to any Veronese higher than the $(N+1)^{\text{st}}$ all coefficients of the power series are non-negative. In addition for sufficiently high Veronese the series will be of the form $\chi(A) + \frac{b_2(t)}{(1-t)^d}$ for a polynomial $b_2(t)$ of degree $< d$. By Theorem 1.2 from [2] it then follows that there is an $R > N$ such that for $r \geq R$ the series $\left(\frac{b_2(t)}{(1-t)^d}\right)^{\langle r \rangle}$ has all coefficients of its numerator polynomial positive and is for degree $\leq d$. Theorem 1.1 [5] then implies that the coefficients of the numerator polynomial go to infinity when taking higher Veronese series, except for the constant coefficient. Hence by $\chi(A) \geq 0$ high enough Veronese series of $a(t)$ will have a numerator polynomial with positive coefficients. Now the assertion follows from Theorem 1.1 (i). $\square$

4. Further results, questions and applications

4.1. **Further results and questions.** We first prove a monotonicity result on the entries of the g-vectors.

Proposition 4.1. *Let* $a(t) = \sum_{n \geq 0} a_n t^n = \frac{h_0(a) + \cdots + h_\lambda(a) t^\lambda}{(1-t)^d}$ *with* $h_\lambda(a) \neq 0$. *If* $h_i(a) \geq 0$ *for* $0 \leq i \leq \lambda$, *then*

$$g(a) \leq g(a^{\langle d \rangle}) \leq g(a^{\langle d+1 \rangle}) \leq g(a^{\langle d+2 \rangle}) \leq \cdots \qquad .$$

Proof. The claim follows directly from the g-vector transformation stated in Remark 2.5 combined with Corollary 2.12. $\square$

Recall that a sequence $(a_0, \ldots, a_t) \in \mathbb{N}^{t+1}$ is called an *M-sequence* if it is the Hilbert function of a 0-dimensional standard graded Artinian k-algebra. Macaulay gave a characterization of such sequences by means of numerical conditions (see e.g., [7, Thm. 4.2.10]). In particular, it is well-known that f-vectors of simplicial complexes are M-sequences. As a consequence of Corollary 1.2 we thus obtain.

Remark 4.2. Let A be a d-dimensional standard graded k-algebra.

(i) If A is Cohen-Macaulay and $r \geq d$, then $g(A^{\langle r \rangle})$ is an M-sequence.

(ii) If $\chi(A) \geq 0$, then there is an $R > 0$ such that for $r \geq R$ and $s \geq d$, the vector $g(A^{\langle r \cdot s \rangle})$ is an M-sequence.

Satoshi Murai has explained to us an algebraic proof of part (i) of the preceding remark. It is based on a suitable linear system of parameters and a linear form that multiplies injectively up to the middle degree.

In the case A is Cohen-Macaulay the h-vector of A and all its Veronese algebras is an M-sequence. The latter follows since Veronese algebras of Cohen-Macaulay algebras are again Cohen-Macaulay by a result from [10]. For general standard graded algebras A with $\chi(A) \geq 0$ the numerator polynomial of the Hilbert series of high Veronese algebras of A will have positive coefficients but already here it is not clear if it finally will become an M-sequence. Indeed the following is true.

Proposition 4.3. *Let* $a(t) = \sum_{n \geq 0} a_n t^n = \frac{h_0(a) + \cdots + h_\lambda(a) t^\lambda}{(1-t)^d}$ *be a rational formal power series with integer coefficient sequence* $(a_n)_{n \geq 0}$, *where* $h_\lambda(a) \neq 0$ *and* $h_0(a) = 1$. *If there is an* $N > 0$ *such that* $a_n > 0$ *for* $n > N$ *and* $\chi(a) \geq 0$, *then there is an* $R > N$ *such that for each* $r \geq R$ *the coefficient sequence of the numerator polynomial of* $a^{\langle r \rangle}(t)$ *is an* M*-sequence.*

Proof. As in the proof of Theorem 1.1 we deduce that the numerator polynomial of $a^{\langle r \rangle}(t)$ will have positive coefficients for large enough r. Now by [5, Thm. 1.4] this polynomial will also be real rooted for large enough r. Then the result follows from [3, Thm. 3.6] where it is shown that the coefficient sequence of a real rooted polynomial $1 + c_1 t + \cdots + c_d t^d$ with positive integer coefficients is an M-sequence. $\square$

The preceding proposition raises the question if real rootedness of a polynomial $1 + c_1 t + \cdots + c_d t^d$ with positive integer coefficients implies more than the coefficient sequence being an M-sequence. Indeed, already in [3] Bell and Skandera conjecture that the assumptions imply that the coefficient sequence is the f-vector of a simplicial complex. Here we would like to ask the question if indeed real rootedness already implies the consequences of Theorem 1.1 (i).

Question 4.4. Let $1 + c_1 t + \cdots + c_d t^d$ be a real rooted polynomial with positive integer coefficients. Assume that $1 < c_1 < \cdots < c_l$. Is $(1, c_1 - 1, c_2 - c_1, \ldots, c_l - c_{l-1})$ a f-vector of a simplicial complex?

Another interesting question arises from Theorem 1.1 (ii). First the limiting behavior does not give a single bound R such that for $r > R$ the assertions are valid for r^{th} Veronese series, rather the bound depends on a 'starting' parameter. Second it is not clear to us if the assumption $\chi(A) \geq 0$ is really needed. Indeed, under the remaining assumptions high Veronese series will have a numerator polynomial with positive coefficients except for the highest coefficient which is $\chi(A)$. These two observations motivate the following questions.

Question 4.5. Assume there is an $N > 0$ such that $a_n > 0$ for $n > N$. Does there exist an $R > N$ such that for each $r \geq R$, there exists a simplicial complex Δ_r such that

$$g_i(a^{\langle r \rangle}) = f_{i-1}(\Delta_r) \qquad \text{for } 0 \leq i \leq \left\lfloor \frac{d}{2} \right\rfloor ?$$

The main result of [13] suggests another possible strengthening of Theorem 1.1. Indeed in [13] the authors conclude that the γ-vector of the barycentric subdivision of homology sphere is the f-vector of a balanced simplicial complex. But the conclusion of Theorem 1.1 (i) cannot be modified in this direction, indeed the g-vector of the 9^{th} of the polynomial ring in 8 variables is not the f-vector of a balanced simplicial complex.

4.2. **Application to edgewise subdivisions.** In this section we review the edgewise subdivision of a simplicial complex Δ (see [9]) and its relation to the Veronese algebras of the Stanley-Reisner ring of Δ (see [6]). This allows us to apply the results obtained in Section 3 to the g-vectors of those complexes.

Before we proceed to edgewise subdivision we recall some basic definitions. Let k be a field. For an abstract simplicial complex Δ on vertex set $[n] := \{1, \dots, n\}$ the *Stanley-Reisner ring* $k[\Delta]$ of Δ is the quotient of the polynomial ring $k[x_1, \dots, x_n]$ by the Stanley-Reisner ideal $I_\Delta := \langle \prod_{i \in F} x_i \mid F \notin \Delta \rangle$ generated by the squarefree monomials whose support does not lie in Δ. The *h-vector* $h(\Delta) = h(k[\Delta])$ of Δ is defined as the h-vector of $k[\Delta]$. We call $g(\Delta) := g(k[\Delta])$ the *g-vector* of Δ.

We are now ready to describe the construction of the r^{th} edgewise subdivision of a simplicial complex. Let Δ be a simplicial complex on vertex set $[n]$ and let $r \geq 1$ be a positive integer. Set $\Omega_r := \{(i_1, \dots, i_n) \in \mathbb{N}^n \mid i_1 + \cdots + i_n = r\}$. Denote by $\mathfrak{e}_i$ the i^{th} unit vector of $\mathbb{R}^n$. By the obvious identification, we can consider Δ as a simplicial complex over the vertex set $\Omega_1 = \{\mathfrak{e}_1, \dots, \mathfrak{e}_n\}$. For $i \in [n]$ set $\mathfrak{u}_i := \mathfrak{e}_i + \cdots + \mathfrak{e}_n$ and for $a = (a_1, \dots, a_n) \in \mathbb{Z}^n$, $a_l \geq 0$ for $1 \leq l \leq n$, let $\mathfrak{i}(a) := \sum_{l=1}^{n} a_l \cdot \mathfrak{u}_l$. The *$r^{th}$ edgewise subdivision* of Δ is the simplicial complex $\Delta(r)$ on ground set Ω_r such that $F \subseteq \Omega_r$ is a simplex in $\Delta(r)$ if and only if

(i) $\bigcup_{a \in F} \operatorname{supp}(a) \in \Delta$
(ii) For all $a, \tilde{a} \in F$ either $\mathfrak{i}(a - \tilde{a}) \in \{0,1\}^n$ or $\mathfrak{i}(\tilde{a} - a) \in \{0,1\}^n$.

The following result by Brun and Römer [6] links the r^{th} edgewise subdivision of a simplicial complexe to the r^{th} Veronese algebra of its Stanley-Reisner ring. The formulation of the result requires some familiarity with the basic theory of Gröbner bases; see for example [1].

Proposition 4.6. *Let Δ be a simplicial complex on ground set $[n]$ and let $r \geq 1$. Set $S(r) := k[y_{i_1,\dots,i_n} \mid (i_1, \dots, i_n) \in \Omega_r]$ and let $I(r)$ be such that $k[\Delta]^{<r>} = S(r)/I(r)$. Then there is a term order $\preceq$ for which $I_{\Delta(r)}$ is the initial ideal of $I(r)$.*

By basic facts on initial ideals and Hilbert functions it follows that

$$\operatorname{Hilb}(S(r)/I(r), t) = \operatorname{Hilb}(k[\Delta(r)], t).$$

Thus, the r^{th} Veronese algebra of the Stanley-Reisner ring of a simplicial complex Δ and the r^{th} edgewise subdivision of this complex have the same h- and g-vector, respectively. From this, we infer that the h- and g-vectors of edgewise subdivisions of simplicial complexes satisfy the same conditions as the h- and g-vectors of Veronese algebras of Stanley-Reisner rings. Also recall that a simplicial complex Δ is called *Cohen-Macaulay over a field k* if

$k[\Delta]$ is a Cohen-Macaulay ring. Therefore, by the fact that the numerator polynomial of $\mathrm{Hilb}(k[\Delta], t)$ has degree $\leq d$ the Corollary 1.2 immediately implies the following.

Corollary 4.7. *Let Δ be a $(d-1)$-dimensional simplicial complex and let $\Delta(r)$ be the r^{th} edgewise subdivision of Δ.*

(i) *If Δ is Cohen-Macaulay over some field and $r \geq d$, then there exists a simplicial complex Δ_r such that*

$$g_i(\Delta^{\langle r\rangle}) = f_{i-1}(\Delta_r) \qquad \text{for } 0 \leq i \leq \left\lfloor \frac{d}{2} \right\rfloor.$$

(ii) *Then there exists an $R > 0$ such that for $r \geq R$ and $s \geq d$, there exists a simplicial complex $\Gamma_{r\cdot s}$ such that*

$$g_i(\Delta(r\cdot s)) = f_{i-1}(\Gamma_{r\cdot s}) \qquad \text{for } 0 \leq i \leq \left\lfloor \frac{d}{2} \right\rfloor.$$

In particular, $g(\Delta(r\cdot s))$ is an M-sequence.

5. Acknowledgment

We thank Satoshi Murai for pointing out to us a mistake in the formulation of Theorem 1.1 (i) in a preliminary version of the paper and explaining to us an algebraic proof of Remark 4.2 (i). We are grateful to Eran Nevo for helpful comments and suggestions.

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Fakultät für Mathematik, Universität Wien, Garnisongasse 3, A-1090 Wien, Austria
E-mail address: `martina.kubitzke@univie.ac.at`

Fachbereich Mathematik und Informatik, Philipps-Universität Marburg, D-35032 Marburg, Germany
E-mail address: `welker@mathematik.uni-marburg.de`