BUCHSBAUM* COMPLEXES

CHRISTOS A. ATHANASIADIS AND VOLKMAR WELKER

Abstract. A class of finite simplicial complexes, which we call Buchsbaum* over a field, is introduced. Buchsbaum* complexes generalize triangulations of orientable homology manifolds as well as doubly Cohen-Macaulay complexes. By definition, the Buchsbaum* property depends only on the geometric realization and the field. Characterizations in terms of simplicial homology are given. It is proved that Buchsbaum* complexes are doubly Buchsbaum. Various constructions, among them one which generalizes convex ear decompositions, are shown to yield Buchsbaum* simplicial complexes. Graph theoretic and enumerative properties of Buchsbaum* complexes are investigated.

1. Introduction

A major theme in the study of finite simplicial complexes in the past few decades has been the interplay between their algebraic, combinatorial, homological and topological properties. Several classes of simplicial complexes, such as Buchsbaum, Cohen-Macaulay or Gorenstein complexes, have been introduced and studied in order to isolate important features of triangulations of fundamental geometric objects, such as balls, spheres and various other manifolds. We refer the reader to [22] for a comprehensive introduction to the subject. The objective of this paper is to introduce and develop the basic properties of a new class of simplicial complexes, named Buchsbaum* complexes, which generalize triangulations of orientable homology manifolds.

In this introductory section we motivate our main definition and outline the remainder of the paper (we refer the reader to [22, Chapter II] [4, Chapter 5] [23, Chapter II] for any undefined terminology and to [13] for background on algebraic topology). A *simplicial complex* over the ground set Ω is a collection Δ of subsets of Ω such that $\sigma \subseteq \tau \in \Delta$ implies $\sigma \in \Delta$. Throughout this paper, we will always assume Ω to be finite (thus we will only consider finite simplicial complexes). Given a field k, the *face ring* (or *Stanley-Reisner ring*) $k[\Delta]$ of Δ over k is the quotient of the polynomial ring $k[x_\omega : \omega \in \Omega]$ by the ideal generated by the monomials $\prod_{\omega \in N} x_\omega$ for all subsets N of Ω not in Δ. Recall (see [22]) that Δ is called *Buchsbaum* (respectively, *Cohen-Macaulay*, *Gorenstein*) over k if the face ring $k[\Delta]$ is a Buchsbaum (respectively, Cohen-Macaulay, Gorenstein) ring. Such a complex Δ is called *doubly Buchsbaum* [12] (respectively, *doubly Cohen-Macaulay* [2] [22, p. 71]) over k if for every vertex v of Δ, the complex $\Delta \smallsetminus v$, obtained from Δ by removing all faces which contain v, is Buchsbaum (respectively, Cohen-Macaulay) over k of the same dimension as Δ.

Date: November 9, 2010.

It is known that Buchsbaumness [20][22, Theorem 8.1]; see also Theorem 2.1 (respectively, Cohen-Macaulayness; see [14] [22, Proposition 4.3]) of Δ is a topological property, meaning that it depends only on the homeomorphism type of the geometric realization $|\Delta|$ [5, Section 9] of Δ. For instance, all triangulations of manifolds (with or without boundary) are Buchsbaum and all triangulations of balls and spheres are Cohen-Macaulay over all fields. It was conjectured by Baclawski [2] and proved by Walker [25] as an immediate consequence of the following theorem, that double Cohen-Macaulayness is a topological property as well. In the sequel, we will write $\widetilde{H}_i(X;k)$ for the reduced singular homology of the space X and $H_i(X,A;k)$ for the singular homology of the pair of spaces (X,A). We will also write $\widetilde{H}_i(\Delta;k)$ and $H_i(\Delta,\Gamma;k)$ for the (reduced) simplicial homology of the simplicial complex Δ and of the pair of simplicial complexes (Δ,Γ).

Theorem 1.1. (Walker, [25, Theorem 9.8]) *Let Δ be a $(d-1)$-dimensional Cohen-Macaulay simplicial complex over a field k. The following conditions are equivalent:*

(i) *Δ is doubly Cohen-Macaulay over k.*
(ii) *$\widetilde{H}_{d-2}(|\Delta|-p;k)=0$ holds for every $p\in|\Delta|$.*

Examples of complexes which are doubly Cohen-Macaulay over all fields are all triangulations of spheres. In contrast, no triangulation of a ball is doubly Cohen-Macaulay over any field.

Double Buchsbaumness of a simplicial complex is also a topological property [12] and thus doubly Buchsbaum complexes generalize homology manifolds (without boundary) in a way analogous to the way doubly Cohen-Macaulay complexes generalize homology spheres. However, in certain respects double Buchsbaumness turns out to be too weak of an analogue of double Cohen-Macaulayness. For instance, it is known [22, p. 71] that every doubly Cohen-Macaulay complex Δ has non-vanishing top-dimensional homology, whereas this is not true for every doubly Buchsbaum complex (since it is not true for every homology manifold without boundary). These considerations and condition (ii) in Theorem 1.1 motivate the following definition.

Definition 1.2. Let Δ be a $(d-1)$-dimensional Buchsbaum simplicial complex over a field k. The complex Δ is called *Buchsbaum* over k* if

$$\dim_k \widetilde{H}_{d-2}(|\Delta|-p;k)=\dim_k \widetilde{H}_{d-2}(|\Delta|;k) \tag{1.1}$$

holds for every $p\in|\Delta|$.

The results of this paper show that the notion of a Buchsbaum* complex provides a well behaved manifold analogue to that of a doubly Cohen-Macaulay complex in terms of various homological, graph theoretic and enumerative properties. We summarize some of these results as follows.

The class of Buchsbaum* complexes is shown to be included in the class of doubly Buchsbaum complexes (Corollary 2.9) with non-vanishing top-dimensional homology (Corollary 2.4), to include all triangulations of orientable homology manifolds (Proposition 2.7) and to reduce to the class of doubly Cohen-Macaulay complexes, when restricted to the class of all Cohen-Macaulay complexes (Proposition 2.5). Products of Buchsbaum* complexes and proper skeleta of Buchsbaum complexes are shown to be Buchsbaum* (Propositions 3.1 and 3.6). A notion of higher Buchsbaum* connectivity, generalizing that of higher Cohen-Macaulay connectivity [2], is introduced in Section 3.3, where it is shown that passing to proper skeleta increases the degree of connectivity (Theorem 3.7).

Partially extending results of Kalai [11] on homology manifolds and Nevo [16] on doubly Cohen-Macaulay complexes, it is shown that the graph of a connected Buchsbaum* complex of dimension $d-1 \geq 2$ is generically d-rigid (Theorem 4.1). This implies that the face numbers of such a complex satisfy the inequalities of Barnette's lower bound theorem (Proposition 5.5) and part of the conditions predicted by the g-conjecture for triangulations of spheres (Proposition 5.6). An analogue of a recursive formula for the $\mathfrak{h}$-vector of a pure simplicial complex, in terms of that of the deletion and the link of a vertex, is shown to be valid for Buchsbaum* complexes (Propositions 5.1 and 5.2) and an application to the face enumeration of flag Buchsbaum* complexes is given (Corollary 5.4).

This paper is structured as follows. Section 2 gives characterizations of Buchsbaum* complexes, deduces their basic properties and lists examples. Section 3 investigates the behavior of the Buchsbaum* property under standard operations on topological spaces and constructs a large family of Buchsbaum* complexes by suitably gluing orientable homology manifolds with boundary to an orientable homology manifold without boundary (Corollary 3.11). Sections 4 and 5 focus on graph theoretic and enumerative properties. Section 6 briefly discusses some further properties of Buchsbaum* complexes which appeared in the literature after the results of this paper were first publicized.

2. Characterizations and elementary properties

This section provides characterizations and discusses basic properties of Buchsbaum* complexes. Throughout this paper, if not specified otherwise, k is an arbitrary field. We recall the following characterization of Buchsbaum complexes.

Theorem 2.1 (Schenzel [20])**.** *For a $(d-1)$-dimensional simplicial complex Δ, the following conditions are equivalent:*

(i) *Δ is Buchsbaum over k.*
(ii) *Δ is pure and $\mathrm{lk}_\Delta(\sigma)$ is Cohen-Macaulay over k for every $\sigma \in \Delta \smallsetminus \{\varnothing\}$.*
(iii) *$H_i(|\Delta|, |\Delta| - p; k) = 0$ holds for all $i < d-1$ and $p \in |\Delta|$.*

Remark 2.2. Let Δ be a $(d-1)$-dimensional simplicial complex. The following statements are immediate consequences of Theorem 2.1 and Definition 1.2.

(i) Δ is Buchsbaum over k if and only if every connected component of Δ is Buchsbaum over k of dimension $d-1$.
(ii) Assume that $d \geq 2$. Then Δ is Buchsbaum* over k if and only if every connected component of Δ is Buchsbaum* over k of dimension $d-1$.

The following proposition provides equivalent versions of Definition 1.2.

Proposition 2.3. *For a $(d-1)$-dimensional simplicial complex Δ which is Buchsbaum over k, the following conditions are equivalent:*

(i) *Δ is Buchsbaum* over k.*

(ii) *For every $p \in |\Delta|$, the inclusion map $\iota : |\Delta| - p \hookrightarrow |\Delta|$ induces an injection*

$$\iota_* : \widetilde{H}_{d-2}(|\Delta| - p; k) \to \widetilde{H}_{d-2}(|\Delta|; k).$$

(iii) *For every $p \in |\Delta|$, the inclusion map $\iota : |\Delta| - p \hookrightarrow |\Delta|$ induces an isomorphism*

$$\iota_* : \widetilde{H}_{d-2}(|\Delta| - p; k) \to \widetilde{H}_{d-2}(|\Delta|; k).$$

(iv) *For every $p \in |\Delta|$, the canonical map*

$$\rho_* : \widetilde{H}_{d-1}(|\Delta|; k) \to H_{d-1}(|\Delta|, |\Delta| - p; k)$$

is surjective.

Proof. Since Δ is Buchsbaum over k, we have $H_{d-2}(|\Delta|, |\Delta| - p; k) = 0$ by condition (iii) of Theorem 2.1. Hence, the long exact sequence of the pair $(|\Delta|, |\Delta| - p)$ gives the exact sequence

$$0 \longrightarrow \widetilde{H}_{d-1}(|\Delta| - p; k) \longrightarrow \widetilde{H}_{d-1}(|\Delta|; k) \xrightarrow{\rho_*} H_{d-1}(|\Delta|, |\Delta| - p; k) \longrightarrow \widetilde{H}_{d-2}(|\Delta| - p; k) \xrightarrow{\iota_*} \widetilde{H}_{d-2}(|\Delta|; k) \longrightarrow 0.$$

It follows that ι_* is surjective. This proves that (ii) $\Leftrightarrow$ (iii). Assuming that Δ is Buchsbaum* over k, surjectivity of ι_* and (1.1) imply that ι_* is an isomorphism. This proves that (i) $\Rightarrow$ (iii). The reverse implication is trivial. The same exact sequence proves the equivalence (iii) $\Leftrightarrow$ (iv). $\square$

Corollary 2.4. *If Δ is a $(d-1)$-dimensional Buchsbaum* simplicial complex over k, then*

$$\widetilde{H}_{d-1}(\Delta; k) \neq 0.$$

Proof. Let us choose $p \in |\Delta|$ in the relative interior of a $(d-1)$-dimensional face of Δ. Clearly, we have $H_{d-1}(|\Delta|, |\Delta| - p; k) \cong k$. The desired statement follows by applying condition (iv) of Proposition 2.3 to such a point p. $\square$

We note that part (i) of the next proposition fails if Buchsbaum* is replaced by doubly Buchsbaum (see, for instance, part (i) of Example 2.11).

Proposition 2.5. *Let Δ be a simplicial complex.*

(i) *Assume that Δ is Cohen-Macaulay over k. Then Δ is Buchsbaum* over k if and only if Δ is doubly Cohen-Macaulay over k.*

(ii) *Assume that Δ is Gorenstein over k. Then Δ is Buchsbaum* over k if and only if Δ is Gorenstein* over k.*

Proof. The assumption that Δ is Cohen-Macaulay over k implies that Δ is Buchsbaum over k and that $\widetilde{H}_{d-2}(\Delta; k) = 0$, where $d-1$ is the dimension of Δ. Therefore, under this assumption, Definition 1.2 implies that Δ is Buchsbaum* over k if and only if we have $\widetilde{H}_{d-2}(\Delta - p; k) = 0$ for every $p \in |\Delta|$. Thus, part (i) follows from Theorem 1.1.

Assume that Δ is Gorenstein over k. This assumption also implies that Δ is Buchsbaum over k. A Gorenstein simplicial complex Γ of dimension $d-1$ is Gorenstein* if and only if $\widetilde{H}_{d-1}(\Gamma; k) \neq 0$. Thus if Δ is Buchsbaum* over k, then Δ is Gorenstein* over k by Corollary 2.4. Conversely, if Δ is Gorenstein* over k, then Δ is doubly Cohen-Macaulay over k and hence it is Buchsbaum* over k by part (i). This proves part (ii). □

Example 2.6. A zero-dimensional simplicial complex is Buchsbaum* over k if and only if it has at least two vertices. Suppose Δ is one-dimensional, so that Δ is a graph. Then by Remark 2.2 (ii), Δ is Buchsbaum* over k if and only if so is each connected component of Δ. Since a graph regarded as a one-dimensional simplicial complex is Cohen-Macaulay over k if and only if it is connected, we conclude from Proposition 2.5 (i) that Δ is Buchsbaum* over k if and only if each connected component of Δ is doubly connected as a graph. □

By the term *homology manifold* (without further specification) in this paper, we will always mean one without boundary.

Proposition 2.7. *Let Δ be a triangulation of a homology manifold X over k. Then Δ is Buchsbaum* over k if and only if X is orientable over k.*

Proof. In view of Remark 2.2 (ii), we may assume that $|\Delta|$ is connected. Let $d-1$ be the dimension of Δ and let $p \in |\Delta|$. Our assumptions imply that Δ is Buchsbaum over k, that $\widetilde{H}_{d-1}(|\Delta| - p; k) = 0$ and that $H_{d-1}(|\Delta|, |\Delta| - p; k) \cong k$. Thus, the long exact homology sequence considered in the proof of Proposition 2.3 shows that the canonical map $\rho_* : \widetilde{H}_{d-1}(|\Delta|; k) \to H_{d-1}(|\Delta|, |\Delta| - p; k)$ is surjective if and only if $\widetilde{H}_{d-1}(|\Delta|; k) \cong k$. Since the latter holds if and only if X is orientable over k, the proof follows from the equivalence (i) $\Leftrightarrow$ (iv) in Proposition 2.3. □

The following proposition provides another equivalent version of Definition 1.2. The condition in part (ii) of this proposition is a stronger version of one which appeared in [12] (see also the proof of Corollary 2.9 below). Recall that the *contrastar* of a face σ of a simplicial complex Δ is defined as the subcomplex $\mathrm{cost}_\Delta(\sigma) = \{\tau \in \Delta : \sigma \not\subseteq \tau\}$ of Δ.

Proposition 2.8. *For a $(d-1)$-dimensional simplicial complex Δ which is Buchsbaum over k, the following conditions are equivalent:*

(i) *Δ is Buchsbaum* over k.*
(ii) *For every pair of faces $\sigma \subseteq \tau$ of Δ, the map*

$$j_* : H_{d-1}(\Delta, \mathrm{cost}_\Delta(\sigma); k) \to H_{d-1}(\Delta, \mathrm{cost}_\Delta(\tau); k), \tag{2.1}$$

induced by inclusion, is surjective.

Proof. Recall that for $p \in |\Delta|$ there is a deformation retraction of $|\Delta| - p$ onto $|\mathrm{cost}_\Delta(\tau)|$, where τ is the unique face of Δ such that p lies in the relative interior of $|\tau|$. As a result, condition (iv) of Proposition 2.3 is equivalent to the condition that for each $\tau \in \Delta$, the canonical map

$$\rho_*^\tau : \widetilde{H}_{d-1}(\Delta; k) \to H_{d-1}(\Delta, \operatorname{cost}_\Delta(\tau); k)$$

is surjective. The commutative diagram of canonical maps

$$\begin{array}{ccc} & & H_{d-1}(\Delta, \operatorname{cost}_\Delta(\sigma); k) \\ & \overset{\rho_*^\sigma}{\nearrow} & \\ \widetilde{H}_{d-1}(\Delta; k) & & \Big\downarrow j_* \\ & \underset{\rho_*^\tau}{\searrow} & \\ & & H_{d-1}(\Delta, \operatorname{cost}_\Delta(\tau); k) \end{array}$$

for pairs $\sigma \subseteq \tau$ of faces of Δ shows that the latter condition is equivalent to (ii). □

Corollary 2.9. *Every Buchsbaum* complex over k is doubly Buchsbaum over k.*

Proof. This statement follows from the implication (i) ⇒ (ii) of Proposition 2.8 and the fact (see [12, Theorem 4.3]) that a $(d-1)$-dimensional simplicial complex Δ is doubly Buchsbaum over k if and only if Δ is Buchsbaum over k and the map (2.1) is surjective for every pair of nonempty faces $\sigma \subseteq \tau$ of Δ. □

The following property of Buchsbaum* complexes was proved for connected orientable homology manifolds in [19, Theorem 2.1].

Corollary 2.10. *Let Δ be a $(d-1)$-dimensional Buchsbaum* simplicial complex over k. Then the socle of the local cohomology module of $k[\Delta]$ in homological dimension d with respect to the irrelevant ideal satisfies $\left(\operatorname{Soc} H^d(k[\Delta])\right)_i = 0$ for all $i \neq 0$.*

Proof. As noted in the proof of [19, Theorem 2.1], it follows from [10, Theorem 2] that the conclusion of the corollary holds if the map

$$j^* : H^{d-1}(\Delta, \operatorname{cost}_\Delta(\tau); k) \to H^{d-1}(\Delta, \operatorname{cost}_\Delta(\tau \smallsetminus \{l\}); k)$$

in simplicial cohomology, induced by the identity map, is injective for every $\tau \in \Delta \smallsetminus \{\varnothing\}$ and $l \in \tau$. This holds if Δ is Buchsbaum* over k by Proposition 2.8. □

Example 2.11. Some examples of doubly Buchsbaum complexes which are not Buchsbaum* are the following.

(i) The one-dimensional simplicial complex Δ on the vertex set $\{a, b, c, d, p\}$ with facets (edges) $\{p, a\}$, $\{p, b\}$, $\{a, b\}$, $\{p, c\}$, $\{p, d\}$, $\{c, d\}$ is doubly Buchsbaum but not Buchsbaum*, since $\widetilde{H}_0(|\Delta|; k) = 0$ and $\widetilde{H}_0(|\Delta| - p; k) \cong k$ (alternatively, since Δ is not doubly connected as a graph).

(ii) Condition (ii) of Theorem 2.1 and the fact that all homology spheres over k are doubly Cohen-Macaulay over k imply that all homology manifolds over k are doubly Buchsbaum over k. This fact and Proposition 2.7 imply that every non-orientable homology manifold over k is doubly Buchsbaum but not Buchsbaum* over k.

(iii) Let Γ be a triangulation of the two-dimensional torus for which some three edges of Γ of the form $\{a, b\}$, $\{b, c\}$ and $\{c, a\}$ are the support of a 1-cycle which represents an element of a basis of $\widetilde{H}_1(\Gamma; k)$. Let Δ be the simplicial complex obtained from Γ by adding the two-dimensional face $\sigma = \{a, b, c\}$. It is easy to check that Δ is doubly Buchsbaum over all fields k. However, since $\widetilde{H}_1(\Delta; k) \cong k$ and $\widetilde{H}_1(|\Delta| - p; k) \cong k^2$ for every point p in the relative interior of $|\sigma|$, the complex Δ is not Buchsbaum* over k.

Corollary 2.12. *If Δ is a Buchsbaum* simplicial complex over k, then $\mathrm{lk}_\Delta(\sigma)$ is doubly Cohen-Macaulay over k for every nonempty face σ of Δ.*

Proof. This statement follows from Corollary 2.9 and the fact (see, for instance, [12, Lemma 4.2]) that the link of any nonempty face in a doubly Buchsbaum complex is doubly Cohen-Macaulay. □

3. Constructions

This section investigates the behavior of the Buchsbaum* property under taking products, joins and skeleta of simplicial complexes, studies a notion of higher Buchsbaum* connectivity and shows that a large family of Buchsbaum* complexes can be constructed by gluing orientable homology manifolds with boundary to an orientable homology manifold without boundary in a suitable way.

3.1. **Products.** This section shows that the Buchsbaum and Buchsbaum* properties are preserved under direct products of simplicial complexes. We note that the corresponding statement fails for both the Cohen-Macaulay and doubly Cohen-Macaulay properties.

Proposition 3.1. *Let Γ be a $(d-1)$-dimensional simplicial complex and Δ be an $(e-1)$-dimensional simplicial complex.*

(i) *If Γ and Δ are Buchsbaum over k, then every simplicial complex triangulating $|\Gamma| \times |\Delta|$ is Buchsbaum over k.*

(ii) *If Γ and Δ are Buchsbaum* over k, then every simplicial complex triangulating $|\Gamma| \times |\Delta|$ is Buchsbaum* over k.*

Proof. Let $p \in |\Gamma| \times |\Delta|$. There are unique faces $\sigma \in \Gamma$ and $\tau \in \Delta$ such that p lies in the relative interior of $|\sigma| \times |\tau|$. Then $|\mathrm{cost}_\Gamma(\sigma)| \times |\Delta| \cup |\Gamma| \times |\mathrm{cost}_\Delta(\tau)|$ is a deformation retract of $|\Gamma| \times |\Delta| - p$ and hence

$$(|\Gamma|, |\mathrm{cost}_\Gamma(\sigma)|) \times (|\Delta|, |\mathrm{cost}_\Delta(\tau)|) \;\; \simeq \;\; (|\Gamma| \times |\Delta|, |\Gamma| \times |\Delta| - p) \tag{3.1}$$

is a deformation retraction.

(i) By (3.1) and the Künneth formula we have:

$$H_i(|\Gamma| \times |\Delta|, |\Gamma| \times |\Delta| - p; k) \cong \bigoplus_{j=0}^{i} \begin{matrix} H_j(\Gamma, \mathrm{cost}_\Gamma(\sigma); k) \\ \otimes \\ H_{i-j}(\Delta, \mathrm{cost}_\Delta(\tau); k). \end{matrix}$$

For $i < d+e-2$ we have either $j < d-1$ or $i-j < e-1$. Thus by Buchsbaumness of Γ and Δ, either $H_j(\Gamma, \mathrm{cost}_\Gamma(\sigma); k) = 0$ or $H_{i-j}(\Delta, \mathrm{cost}_\Delta(\tau); k) = 0$. Thus

$$H_i(|\Gamma| \times |\Delta|, |\Gamma| \times |\Delta| - p; k) = 0$$

for $i < d+e-2$. Hence every simplicial complex triangulating $|\Gamma|\times|\Delta|$ is Buchsbaum over k.

(ii) By (3.1), the map

$$\widetilde{H}_{d+e-2}(|\Gamma| \times |\Delta|; k) \xrightarrow{\rho_*} H_{d+e-2}(|\Gamma| \times |\Delta|, |\Gamma| \times |\Delta| - p; k)$$

from the exact sequence of the pair $(|\Gamma| \times |\Delta|, |\Gamma| \times |\Delta| - p)$ equals the map

$$\widetilde{H}_{d+e-2}(|\Gamma| \times |\Delta|; k) \xrightarrow{\rho_*} H_{d+e-2}\Big(\begin{matrix} (|\Gamma|, |\mathrm{cost}_\Gamma(\sigma)|) \\ \times \\ (|\Delta|, |\mathrm{cost}_\Delta(\tau)|) \end{matrix} ; k\Big).$$

In turn, by the Künneth formula, this map can be written as

$$\widetilde{H}_{d-1}(\Gamma; k) \otimes \widetilde{H}_{e-1}(\Delta; k) \xrightarrow{\rho_*} H_{d-1}(\Gamma, \mathrm{cost}_\Gamma(\sigma); k) \otimes H_{e-1}(\Delta, \mathrm{cost}_\Delta(\tau); k).$$

From the fact that Γ is Buchsbaum* over k and Proposition 2.3 we deduce that the projection from $\widetilde{H}_{d-1}(\Gamma; k)$ to $H_{d-1}(\Gamma, \mathrm{cost}_\Gamma(\sigma); k)$ is surjective. Analogously, the projection from $\widetilde{H}_{e-1}(\Delta; k)$ to $H_{e-1}(\Delta, \mathrm{cost}_\Delta(\tau); k)$ is surjective. Hence ρ_* is surjective as well. This fact and Proposition 2.3 imply that every triangulation of $|\Gamma| \times |\Delta|$ is Buchsbaum* over k.

□

3.2. **Joins.** We recall that the join $\Gamma * \Delta$ of two simplicial complexes Γ and Δ on disjoint ground sets is the simplicial complex whose faces are the sets of the form $\sigma \cup \tau$, where $\sigma \in \Gamma$ and $\tau \in \Delta$. The following proposition classifies the situations in which $\Gamma * \Delta$ is Buchsbaum*. A similar statement (with a similar proof) holds for the Buchsbaum property.

Proposition 3.2. *Let Γ and Δ be simplicial complexes, each having at least one vertex. The following are equivalent:*

(i) $\Gamma * \Delta$ *is Buchsbaum* over* k.
(ii) $\Gamma * \Delta$ *is doubly Cohen-Macaulay over* k.
(iii) Γ *and* Δ *are doubly Cohen-Macaulay over* k.

Proof. (i) ⇒ (iii): Since Δ contains at least one vertex, there exists a nonempty maximal simplex $\sigma \in \Delta$. Since $\Gamma * \Delta$ is Buchsbaum* over k and the link of σ in $\Gamma * \Delta$ is equal to Γ, it follows from Corollary 2.12 that Γ is doubly Cohen-Macaulay over k. It follows in a similar way that Δ is doubly Cohen-Macaulay over k.

(iii) ⇒ (ii): It is well known that the join of two Cohen-Macaulay simplicial complexes over k is Cohen-Macaulay over k. The implication follows from this statement, the definition of double Cohen-Macaulayness and the fact that for every vertex v, say of Γ, the complex $(\Gamma * \Delta) \smallsetminus v$ is equal to the simplicial join $(\Gamma \smallsetminus v) * \Delta$.
(ii) ⇒ (i): This follows from Proposition 2.5 (i). □

3.3. **Higher Buchsbaum* connectivity and skeleta.** Given a subset τ of the set of vertices of Δ, we denote by $\Delta \smallsetminus \tau$ the subcomplex $\{\sigma \in \Delta : \sigma \cap \tau = \varnothing\}$ of Δ, consisting of all faces of Δ which do not contain any element of τ. We define a notion of higher Buchsbaum* connectivity for simplicial complexes as follows.

Definition 3.3. Let Δ be a simplicial complex and let m be a nonnegative integer. We call Δ *m-Buchsbaum** over k if $m = 0$ and Δ is Buchsbaum over k or $m \geq 1$ and $\Delta \smallsetminus \tau$ is Buchsbaum* over k of the same dimension as Δ for every set τ of vertices of Δ of cardinality less than m.

Thus the class of 0-Buchsbaum* complexes coincides with that of Buchsbaum complexes and the class of 1-Buchsbaum* complexes coincides with that of Buchsbaum* complexes. Our notion of higher connectivity for Buchsbaum* complexes is analogous to that already existing for Buchsbaum and Cohen-Macaulay complexes: Given a positive integer m, a simplicial complex Δ is called *m-Buchsbaum* over k in [12] (respectively, *m-Cohen-Macaulay* over k in [2]) if $\Delta \smallsetminus \tau$ is Buchsbaum over k (respectively, Cohen-Macaulay over k) of the same dimension as Δ for every set τ of vertices of Δ of cardinality less than m.

The following two statements generalize Proposition 2.5 (i) and Corollary 2.9, respectively.

Proposition 3.4. *For a Cohen-Macaulay simplicial complex Δ over k and a nonnegative integer m, the following conditions are equivalent:*

(i) *Δ is m-Buchsbaum* over k.*
(ii) *Δ is $(m+1)$-Cohen-Macaulay over k.*

Proof. (i) ⇒ (ii): The implication is trivial for $m = 0$ and follows from Proposition 2.5 for $m = 1$. We assume that $m \geq 2$ and proceed by induction on m. Suppose that Δ is m-Buchsbaum* over k. To verify (ii), it suffices to show that $\Delta \smallsetminus v$ is m-Cohen-Macaulay over k of the same dimension as Δ for every vertex v of Δ. Indeed, Δ is doubly Cohen-Macaulay over k by the special case $m = 1$ already treated and hence $\Delta \smallsetminus v$ is Cohen-Macaulay over k of the same dimension as Δ. Since $\Delta \smallsetminus v$ is $(m-1)$-Buchsbaum* over k by Definition 3.3, the desired statement follows from the induction hypothesis.
(ii) ⇒ (i): This follows from part (i) of Proposition 2.5 and the relevant definitions. □

Proposition 3.5. *Let m be a nonnegative integer and Δ be a simplicial complex. If Δ is m-Buchsbaum* over k, then Δ is $(m+1)$-Buchsbaum over k.*

Proof. Let $d-1$ be the dimension of Δ. The statement is a tautology for $m = 0$. Assume that $m \geq 1$ and let τ be a set of vertices of Δ of cardinality at most m. We need to show that $\Delta \smallsetminus \tau$ is Buchsbaum over k of dimension $d-1$. This is clear if $\tau = \varnothing$. Otherwise, let

v be an element of τ and let $\sigma = \tau \smallsetminus \{v\}$ and $\Gamma = \Delta \smallsetminus \sigma$. The complex Γ is Buchsbaum* over k by Definition 3.3 and hence it is doubly Buchsbaum over k by Corollary 2.9. This implies that $\Gamma \smallsetminus v$ is Buchsbaum over k of dimension $d-1$. Since $\Gamma \smallsetminus v = \Delta \smallsetminus \tau$, the latter complex is Buchsbaum over k of dimension $d-1$. This completes the proof. □

Next we show that Buchsbaum* connectivity increases when passing to skeleta. Recall that the *i-skeleton* of a simplicial complex Δ is defined as the simplicial complex $\Delta^{\langle i \rangle}$ of all faces of Δ of dimension $\le i$. It is known [12, Corollary 7.6] that if Δ is $(d-1)$-dimensional and Buchsbaum over k, then the i-skeleton of Δ is doubly Buchsbaum over k for every $i \le d-2$. In view of Corollary 2.9, the following is a stronger statement.

Proposition 3.6. *Let Δ be a $(d-1)$-dimensional simplicial complex which is Buchsbaum over k. Then the i-skeleton $\Delta^{\langle i \rangle}$ of Δ is Buchsbaum* over k for every $i \le d-2$.*

Proof. It suffices to prove the assertion for $i = d-2$. We set $\Gamma = \Delta^{\langle d-2 \rangle}$. For $d = 2$, the 0-skeleton of any $(d-1)$-dimensional simplicial complex consists of at least two points and therefore it is Buchsbaum* over all fields. Assume that $d \ge 3$. It is known (and follows, for instance, from condition (ii) of Theorem 2.1) that Γ is Buchsbaum over k. Thus we only need to check that $\widetilde{H}_{d-3}(|\Gamma| - p; k) \cong \widetilde{H}_{d-3}(|\Gamma|; k)$ for every $p \in |\Gamma|$. By condition (iii) of Theorem 2.1 and the long exact homology sequence for the pair $(|\Delta|, |\Delta| - p)$, we know that $\widetilde{H}_{d-3}(|\Delta| - p; k) \cong \widetilde{H}_{d-3}(|\Delta|; k)$ holds for every $p \in |\Delta|$. Since $p \in |\Gamma| \subseteq |\Delta|$, it follows from the fact that the chains groups of Γ and Δ in simplicial homology coincide in dimensions $\le d-2$ that $\widetilde{H}_{d-3}(|\Gamma| - p; k) \cong \widetilde{H}_{d-3}(|\Delta| - p; k)$ and $\widetilde{H}_{d-3}(|\Gamma|; k) \cong \widetilde{H}_{d-3}(|\Delta|; k)$. This completes the proof. □

The following result extends to Buchsbaum* connectivity analogous statements on Buchsbaum [12, Corollary 7.6] and Cohen-Macaulay [9, Corollary 2.7] connectivity.

Theorem 3.7. *Let Δ be a $(d-1)$-dimensional simplicial complex which is m-Buchsbaum* over k. Then the i-skeleton $\Delta^{\langle i \rangle}$ is $(m+d-i-1)$-Buchsbaum* over k for every $0 \le i \le d-1$.*

Proof. The statement is a tautology for $i = d-1$. By induction on $d-i-1$, it suffices to show that $\Delta^{\langle d-2 \rangle}$ is $(m+1)$-Buchsbaum* over k. Let τ be any set of vertices of Δ of cardinality at most m and set $\Gamma = \Delta \smallsetminus \tau$. Since Δ is $(m+1)$-Buchsbaum over k by Proposition 3.5, the complex Γ is Buchsbaum over k of dimension $d-1$. It follows from Proposition 3.6 that $\Gamma^{\langle d-2 \rangle}$ is Buchsbaum* over k. Since this skeleton is equal to $\Delta^{\langle d-2 \rangle} \smallsetminus \tau$, we conclude that the latter complex is Buchsbaum* over k of dimension $d-2$. Since τ was arbitrary of cardinality at most m, the desired statement follows. □

Remark 3.8. It is known by the results of Miyazaki [12] and Walker [25] that double Buchsbaumness and double Cohen-Macaulayness (over a fixed field) are topological properties. It is also known that for $m \ge 3$, neither m-Buchsbaumness nor m-Cohen-Macaulayness is a topological property. In view of Proposition 3.4, it follows that for $m \ge 2$, the m-Buchsbaum* condition is not a topological property either.

Remark 3.9. A different notion of a higher Buchsbaum* property one may try is the following. Let Δ be a $(d-1)$-dimensional simplicial complex and let m be a nonnegative integer. We consider the following condition:

(M) Δ is Buchsbaum over k and $\dim_k \widetilde{H}_{d-2}(|\Delta| - P; k) = \dim_k \widetilde{H}_{d-2}(\Delta; k)$ holds for every set $P \subset |\Delta|$ of cardinality at most m.

This condition reduces to Buchsbaumness for $m = 0$ and to the Buchsbaum* condition for $m = 1$. We claim, however, that no simplicial complex of positive dimension satisfies (M) for $m \geq 2$. Clearly, it suffices to show that no such complex satisfies (M) for $m = 2$. Suppose on the contrary that Δ is such a complex. We choose two points $p, q \in |\Delta|$ which lie in the relative interior of some $(d-1)$-dimensional simplex, say τ, of Δ. We triangulate τ by adding a vertex z and faces $\{z\} \cup \sigma$ for $\sigma \subset \tau$. We realize the new complex in such a way that p and q lie in the relative interior of the realization of two distinct $(d-1)$-dimensional simplices $\tau_p = \{z\} \cup \sigma_p$ and $\tau_q = \{z\} \cup \sigma_q$. We denote by Δ' the simplicial complex whose simplices are those of Δ other than τ and the faces triangulating τ in the way just described. In particular, $|\Delta' \smallsetminus \{\tau_p, \tau_q\}|$ is a deformation retract of $|\Delta| - \{p, q\}$. The assumption that Δ is Buchsbaum over k and excision give

$$0 = H_{d-2}(|\Delta|, |\Delta| - q; k) \cong H_{d-2}(|\Delta| - p, |\Delta| - \{p, q\}; k).$$

The long exact sequence of the triple $(|\Delta|, |\Delta| - p, |\Delta| - \{p, q\})$ then yields that $H_{d-2}(|\Delta|, |\Delta| - \{p, q\}; k) = 0$. Thus, using the same arguments as in the proof of Proposition 2.3, it follows from (M) that the inclusion map $\Delta' \smallsetminus \{\tau_p, \tau_q\} \to \Delta'$ induces an isomorphism

$$\widetilde{H}_{d-2}(\Delta' \smallsetminus \{\tau_p, \tau_q\}; k) \to \widetilde{H}_{d-2}(\Delta'; k). \tag{3.2}$$

Clearly, $\partial_{d-1}(\tau_p)$ is a boundary in Δ'. Since the map (3.2) is an isomorphism, $\partial_{d-1}(\tau_p)$ must be a boundary in $\Delta' \smallsetminus \{\tau_p, \tau_q\}$ as well. However, this is not possible since $\tau_p \cap \tau_q$ is a $(d-2)$-dimensional simplex which lies in the support of $\partial_{d-1}(\tau_p)$ and which is not contained in any $(d-1)$-dimensional simplex of Δ' other than τ_p and τ_q. This yields the desired contradiction. □

3.4. **A generalized convex ear decomposition.** In the sequel we describe a class of Buchsbaum* complexes significantly larger than that provided by Proposition 2.7. The construction is motivated by and generalizes the convex ear decomposition of simplicial complexes, introduced by Chari [8].

Theorem 3.10. *Let Γ and Δ be two simplicial complexes such that:*

(i) *Γ is $(d-1)$-dimensional and Buchsbaum* over k.*
(ii) *Δ is a $(d-1)$-dimensional connected orientable homology manifold over k with boundary $\partial\Delta$ which has the following properties:*
   (a) *$\partial\Delta$ is a $(d-2)$-dimensional connected orientable homology manifold over k.*
   (b) *$\partial\Delta = \Gamma \cap \Delta$.*
   (c) *The inclusion map induces the zero homomorphism $\widetilde{H}_{d-2}(\partial\Delta; k) \to \widetilde{H}_{d-2}(\Gamma; k)$.*

Then $\Gamma \cup \Delta$ is Buchsbaum over k.*

As a corollary we obtain an inductive construction as follows.

Corollary 3.11. *Suppose that Δ is a $(d-1)$-dimensional simplicial complex and that there exist subcomplexes $\Delta_1, \Delta_2, \dots, \Delta_m$ such that:*

(i) $\Delta = \Delta_1 \cup \Delta_2 \cup \cdots \cup \Delta_m$.
(ii) Δ_1 *is a $(d-1)$-dimensional orientable homology manifold over k.*
(iii) *For $2 \le i \le m$, Δ_i is a $(d-1)$-dimensional connected orientable homology manifold over k with boundary $\partial\Delta_i$ which has the following properties:*
  (a) $\partial\Delta_i$ *is a $(d-2)$-dimensional connected orientable homology manifold over k.*
  (b) $\partial\Delta_i = \Delta_i \cap (\Delta_1 \cup \cdots \cup \Delta_{i-1})$.
  (c) *The inclusion map induces the zero homomorphism*

$$\widetilde{H}_{d-2}(\partial\Delta_i; k) \to \widetilde{H}_{d-2}(\Delta_1 \cup \cdots \cup \Delta_{i-1}; k).$$

Then Δ is Buchsbaum over k.*

Proof. The complex Δ_1 is Buchsbaum* over k by Proposition 2.7. The theorem follows from this statement and Theorem 3.10 by induction on m. □

Proof of Theorem 3.10. Since Γ and Δ are Buchsbaum over k of dimension $d-1$ and $\Gamma \cap \Delta$ is Buchsbaum over k of dimension $d-2$, it follows by a standard argument (used, for instance, in the proof of [7, Lemma 1]) that $\Gamma \cup \Delta$ is also Buchsbaum over k. We consider a point $p \in |\Gamma \cup \Delta|$. To show that (1.1) (or the equivalent condition (iv) of Proposition 2.3) holds for $\Gamma \cup \Delta$, we distinguish three cases.

Case 1: $p \in |\Gamma| \smallsetminus |\Delta|$. The naturality of the long exact homology sequence for pairs gives the commutative diagram

$$\begin{array}{ccc} \widetilde{H}_{d-1}(|\Gamma \cup \Delta|; k) & \xrightarrow{\widetilde{\rho}_*} & H_{d-1}(|\Gamma \cup \Delta|, |\Gamma \cup \Delta| - p; k) \\ \uparrow & & \uparrow \epsilon_* \\ \widetilde{H}_{d-1}(|\Gamma|; k) & \xrightarrow{\rho_*} & H_{d-1}(|\Gamma|, |\Gamma| - p; k). \end{array}$$

Since Γ is Buchsbaum* over k, the map ρ_* is surjective. The map ϵ_* is an excision map and hence an isomorphism. The commutativity of the diagram implies that $\widetilde{\rho}_*$ is surjective as well.

Case 2: $p \in |\Delta| \smallsetminus |\Gamma|$. The long exact homology sequences for the triples $(|\Delta|, |\Delta| - p, \partial|\Delta|)$ and $(|\Gamma \cup \Delta|, |\Gamma \cup \Delta| - p, \partial|\Delta|)$ and the naturality of such sequences yield the following commutative diagram:

$$\begin{array}{ccccccc} 0 \to & H_{d-1}(|\Gamma\cup\Delta|-p,\partial|\Delta|;k) & \to & H_{d-1}(|\Gamma\cup\Delta|,\partial|\Delta|;k) & \xrightarrow{\widetilde{\delta}_*} & H_{d-1}(|\Gamma\cup\Delta|,|\Gamma\cup\Delta|-p;k) \\ & \uparrow & & \uparrow & & \uparrow \epsilon_* \\ & H_{d-1}(|\Delta|-p,\partial|\Delta|;k) & \to & H_{d-1}(|\Delta|,\partial|\Delta|;k) & \xrightarrow{\delta_*} & H_{d-1}(|\Delta|,|\Delta|-p;k). \end{array}$$

Our assumptions on Δ and p imply that $\widetilde{H}_{d-1}(|\Delta|-p,\partial|\Delta|;k)=0$ and $H_{d-1}(|\Delta|,\partial|\Delta|;k)\cong H_{d-1}(|\Delta|,|\Delta|-p;k)\cong k$. It follows that the map δ_* is an isomorphism. Since ϵ_* is an isomorphism by excision, the commutativity of the square on the right implies that the map $\widetilde{\delta}_*$ is surjective. These facts and the exactness of the top row of the diagram imply that

$$\dim_k H_{d-1}(|\Gamma\cup\Delta|,\partial|\Delta|;k) \;=\; \dim_k H_{d-1}(|\Gamma\cup\Delta|-p,\partial|\Delta|;k) \,+\, 1. \tag{3.3}$$

Our assumption (c) and the long exact homology sequences for the pairs $(|\Gamma\cup\Delta|,\partial|\Delta|)$ and $(|\Gamma\cup\Delta|-p,\partial|\Delta|)$ yield exact sequences of the form

$$0 \longrightarrow \widetilde{H}_{d-1}(|\Gamma\cup\Delta|;k) \to H_{d-1}(|\Gamma\cup\Delta|,\partial|\Delta|;k) \to \widetilde{H}_{d-2}(\partial|\Delta|;k) \longrightarrow 0$$

and

$$0 \longrightarrow \widetilde{H}_{d-1}(|\Gamma\cup\Delta|-p;k) \to H_{d-1}(|\Gamma\cup\Delta|-p,\partial|\Delta|;k) \to \widetilde{H}_{d-2}(\partial|\Delta|;k) \to 0.$$

These sequences and (3.3) imply that

$$\dim_k \widetilde{H}_{d-1}(|\Gamma\cup\Delta|;k) \;=\; \dim_k \widetilde{H}_{d-1}(|\Gamma\cup\Delta|-p;k) \,+\, 1. \tag{3.4}$$

Finally, we consider the exact sequence

$$0 \to \widetilde{H}_{d-1}(|\Gamma\cup\Delta|-p;k) \to \widetilde{H}_{d-1}(|\Gamma\cup\Delta|;k) \xrightarrow{\widetilde{\rho}_*} H_{d-1}(|\Gamma\cup\Delta|,|\Gamma\cup\Delta|-p;k)=k,$$

coming from the long exact homology sequence for the pair $(|\Gamma\cup\Delta|,|\Gamma\cup|\Delta|-p)$, where $H_{d-1}(|\Gamma\cup\Delta|,|\Gamma\cup\Delta|-p;k)\cong H_{d-1}(|\Delta|,|\Delta|-p;k)\cong k$. Equation (3.4) shows that $\widetilde{\rho}_*$ cannot be the zero map. This forces $\widetilde{\rho}_*$ to be surjective.

Case 3: $p\in|\Gamma\cap\Delta|=\partial|\Delta|$. From the Mayer-Vietoris sequence for the pairs $(|\Delta|,|\Gamma|)$ and $(|\Delta|-p,|\Gamma|-p)$ and the naturality of such sequences, we get a commutative diagram as follows:

$$
\begin{array}{ccccccc}
\widetilde{H}_{d-2}(\partial|\Delta|;k) & \xrightarrow{\alpha_*} & \begin{array}{c}\widetilde{H}_{d-2}(|\Gamma|;k)\\ \oplus \\ \widetilde{H}_{d-2}(|\Delta|;k)\end{array} & \longrightarrow & \widetilde{H}_{d-2}(|\Gamma\cup\Delta|;k) & \longrightarrow & \widetilde{H}_{d-3}(\partial|\Delta|;k) \\
 & & \beta_* \big\uparrow & & \widetilde{\iota}_* \big\uparrow & & \gamma_* \big\uparrow \\
 & & \begin{array}{c}\widetilde{H}_{d-2}(|\Gamma|-p;k)\\ \oplus \\ \widetilde{H}_{d-2}(|\Delta|-p;k)\end{array} & \longrightarrow & \widetilde{H}_{d-2}(|\Gamma\cup\Delta|-p;k) & \longrightarrow & \widetilde{H}_{d-3}(\partial|\Delta|-p;k).
\end{array}
$$

We claim that the map α_*, induced by the inclusions of $\partial|\Delta|$ into $|\Gamma|$ and $|\Delta|$, is the zero map. Indeed, for the map $\widetilde{H}_{d-2}(\partial|\Delta|;k) \to \widetilde{H}_{d-2}(|\Gamma|;k)$ induced by inclusion, this holds by condition (c). For the other map, consider the exact sequence

$$\widetilde{H}_{d-1}(|\Delta|;k) \longrightarrow H_{d-1}(|\Delta|,\partial|\Delta|;k) \xrightarrow{\partial_*} \widetilde{H}_{d-2}(\partial|\Delta|;k) \xrightarrow{j_*} \widetilde{H}_{d-2}(|\Delta|;k),$$

obtained from the long exact homology sequence of the pair $(|\Delta|,\partial|\Delta|)$. Since Δ is a $(d-1)$-dimensional connected orientable homology manifold with nonempty boundary $\partial\Delta$, we have $\widetilde{H}_{d-1}(|\Delta|;k) = 0$ and $H_{d-1}(|\Delta|,\partial|\Delta|;k) \cong k$. Since $\partial\Delta$ is a $(d-2)$-dimensional connected orientable homology manifold without boundary, we have $\widetilde{H}_{d-2}(\partial|\Delta|;k) \cong k$. It follows that ∂_* is an isomorphism and hence that $j_* : \widetilde{H}_{d-2}(\partial|\Delta|;k) \to \widetilde{H}_{d-2}(|\Delta|;k)$ is the zero map.

Finally, we note that the maps $\widetilde{H}_{d-2}(|\Gamma|-p;k) \to \widetilde{H}_{d-2}(|\Gamma|;k)$, $\widetilde{H}_{d-2}(|\Delta|-p;k) \to \widetilde{H}_{d-2}(|\Delta|;k)$ and $\widetilde{H}_{d-3}(\partial|\Delta|-p;k) \to \widetilde{H}_{d-3}(\partial|\Delta|;k)$, induced by inclusions, are isomorphisms. This follows by our assumption (i) in the first case, by condition (a) of our assumption (ii) and Proposition 2.7 in the third case and by the long exact homology sequence of the pair $(|\Delta|,|\Delta|-p)$ in the second case, if one observes that $p \in \partial|\Delta|$ implies $H_i(|\Delta|,|\Delta|-p;k) = 0$ for all i. Thus the vertical maps β_* and γ_* in the diagram are isomorphisms.

Given the above, it follows by diagram chasing that the map $\widetilde{\iota}_* : \widetilde{H}_{d-2}(|\Gamma\cup\Delta|-p;k) \to \widetilde{H}_{d-2}(|\Gamma\cup\Delta|;k)$ is injective and hence an isomorphism. Thus (1.1) holds in this case as well. □

Remark 3.12. Example 2.11 (iii) shows that condition (c) in Theorem 3.10 and Corollary 3.11 cannot be dropped.

4. The graph of a Buchsbaum* complex

The graph of a simplicial complex Δ is defined as the abstract graph $\mathcal{G}(\Delta)$ whose nodes are the vertices of Δ and whose edges are the one-dimensional simplices. This section shows that a result of Nevo [16] on the rigidity of graphs of doubly Cohen-Macaulay complexes extends easily to those of connected Buchsbaum* complexes. Since the proof follows from that of [16] by minor modifications, we will only indicate those points in the proof where some modification is actually needed.

Let $\mathcal{G}$ be an abstract graph (without loops or multiple edges) on the set of nodes V and let $\|x\|$ denote the Euclidean length of $x \in \mathbb{R}^d$. A map $f : V \to \mathbb{R}^d$ is called *$\mathcal{G}$-rigid* if there exists $\varepsilon > 0$ with the following property: if $g : V \to \mathbb{R}^d$ is a map satisfying $\|f(v)-g(v)\| < \varepsilon$ for every $v \in V$ and $\|g(u)-g(v)\| = \|f(u)-f(v)\|$ for every edge $\{u, v\}$ of $\mathcal{G}$, then we have $\|g(u) - g(v)\| = \|f(u) - f(v)\|$ for all $u, v \in V$. The graph $\mathcal{G}$ is called *generically d-rigid* if the set of all $\mathcal{G}$-rigid maps $f : V \to \mathbb{R}^d$ is open and dense in the topological vector space of all maps $f : V \to \mathbb{R}^d$.

Theorem 4.1. *Let Δ be a $(d-1)$-dimensional connected simplicial complex which is Buchsbaum* over some field k. If $d \geq 3$, then the graph $\mathcal{G}(\Delta)$ is generically d-rigid.*

Proof. As in the proof of [16, Theorem 1.3], it suffices to show that for $d \geq 2$ every such complex Δ admits a decomposition into minimal $(d-1)$-cycle complexes, as in [16, Theorem 3.4]. The case $d = 2$ is covered by [16, Theorem 3.4], since every one-dimensional connected Buchsbaum* complex is doubly Cohen-Macaulay (see Example 2.6). Thus we may assume that $d \geq 3$ and proceed by induction on d. Let v be any vertex of Δ and note that the induction hypothesis applies to the complex $\mathrm{lk}_\Delta(v)$, which is doubly Cohen-Macaulay over k by Corollary 2.12. Note also that if s is a minimal $(d-2)$-cycle for $\mathrm{lk}_\Delta(v)$, then there exists a $(d-1)$-chain c for $\Delta \smallsetminus v$ such that $\partial_{d-1}(c) = s$. Indeed, this follows from condition (ii) of Proposition 2.3, since s is trivial as an element of $\widetilde{H}_{d-2}(\Delta; k)$. The remainder of the proof follows that of [16, Theorem 3.4] without change. □

Given a positive integer m, an abstract graph $\mathcal{G}$ is said to be *m-connected* if $\mathcal{G}$ has at least $m+1$ nodes and any graph obtained from $\mathcal{G}$ by deleting $m-1$ or fewer nodes and their incident edges is connected (necessarily with at least one edge). Part (i) of the following corollary is a special case of a result independently found by Björner [6]. Example 2.11 (i) shows that part (ii) is not valid if Buchsbaum* is replaced by doubly Buchsbaum.

Corollary 4.2. *Let Δ be a connected simplicial complex of dimension $d - 1 \geq 1$.*

(i) *If Δ is Buchsbaum over some field, then the graph $\mathcal{G}(\Delta)$ is $(d-1)$-connected.*
(ii) *If Δ is Buchsbaum* over some field, then the graph $\mathcal{G}(\Delta)$ is d-connected.*

Proof. Proposition 3.4 implies that a one-dimensional connected simplicial complex Γ is m-Buchsbaum* if and only if Γ is $(m + 1)$-connected as a graph. Therefore, both parts of the corollary follow from the special case $i = 1$ of Theorem 3.7. Part (ii) also follows from Theorem 4.1 for $d \geq 3$, since every generically d-rigid graph is d-connected, and from the discussion in Example 2.6 for $d = 2$. □

5. Face enumeration

This section is concerned with enumerative properties of Buchsbaum* complexes. Let Δ be a $(d-1)$-dimensional simplicial complex. For $0 \leq i \leq d$, we denote by $\mathfrak{f}_{i-1}(\Delta)$ the number of faces of Δ of dimension $i - 1$ (in particular, we have $\mathfrak{f}_{-1}(\Delta) = 1$ unless $\Delta = \varnothing$).

The $\mathfrak{h}$-vector of Δ is the sequence $\mathfrak{h}(\Delta) = (\mathfrak{h}_0(\Delta), \mathfrak{h}_1(\Delta), \dots, \mathfrak{h}_d(\Delta))$ defined by

$$\mathfrak{h}_j(\Delta) \;=\; \sum_{i=0}^{j} (-1)^{j-i} \binom{d-i}{d-j} \mathfrak{f}_{i-1}(\Delta). \tag{5.1}$$

We refer the reader to [22, Chapter II] for the importance of this concept and recall that the numbers $\mathfrak{h}_j(\Delta)$ are nonnegative integers, if Δ is Cohen-Macaulay over k.

The $\mathfrak{h}'$-vector of Δ is the sequence $\mathfrak{h}'(\Delta) = (\mathfrak{h}'_0(\Delta), \mathfrak{h}'_1(\Delta), \dots, \mathfrak{h}'_d(\Delta))$ defined by

$$\mathfrak{h}'_j(\Delta) \;=\; \mathfrak{h}_j(\Delta) + \binom{d}{j} \sum_{i=0}^{j-1} (-1)^{j-i-1} \widetilde{\beta}_{i-1}(\Delta), \tag{5.2}$$

where $\widetilde{\beta}_{i-1}(\Delta) = \dim_k \widetilde{H}_{i-1}(\Delta; k)$. We note that $\mathfrak{h}'_d(\Delta) = \widetilde{\beta}_{d-1}(\Delta)$ and that if Δ is Cohen-Macaulay over k, then $\mathfrak{h}'(\Delta) = \mathfrak{h}(\Delta)$. It was proved by Schenzel [20] that if Δ is Buchsbaum over an infinite field k, then

$$\mathfrak{h}'_i(\Delta) \;=\; \dim_k \left(k[\Delta]/(\Theta_1, \dots, \Theta_d)\right)_i \tag{5.3}$$

for $0 \le i \le d$, where $(\Theta_1, \dots, \Theta_d)$ is a linear system of parameters for $k[\Delta]$ (here we denote by A_i the ith graded component of a graded algebra A). Thus if Δ is Buchsbaum over k, then the $\mathfrak{h}'(\Delta)$ are nonnegative integers which may depend on the characteristic of k.

The following proposition gives an analogue for the $\mathfrak{h}'$-vector of a Buchsbaum* complex to a well-known recursive formula for the $\mathfrak{h}$-vector of a pure simplicial complex; see (5.6) below. We will write $\Delta/v = \{\sigma \smallsetminus \{v\} : \sigma \in \Delta,\, v \in \sigma\}$ for the link of a vertex v of Δ.

Proposition 5.1. *Let Δ be a $(d-1)$-dimensional Buchsbaum* simplicial complex over k. For each vertex v of Δ and $0 \le j \le d$ we have*

$$\mathfrak{h}'_j(\Delta) \;=\; \mathfrak{h}'_j(\Delta \smallsetminus v) + \mathfrak{h}_{j-1}(\Delta/v), \tag{5.4}$$

where $\mathfrak{h}_{-1}(\Delta/v) = 0$ by convention.

Proof. Applied to $\Delta \smallsetminus v$, equation (5.2) yields

$$\mathfrak{h}'_j(\Delta \smallsetminus v) \;=\; \mathfrak{h}_j(\Delta \smallsetminus v) + \binom{d}{j} \sum_{i=0}^{j-1} (-1)^{j-i-1} \widetilde{\beta}_{i-1}(\Delta \smallsetminus v). \tag{5.5}$$

It is well known (see, for instance, [1, Lemma 4.1]) that

$$\mathfrak{h}_j(\Delta) \;=\; \mathfrak{h}_j(\Delta \smallsetminus v) + \mathfrak{h}_{j-1}(\Delta/v) \tag{5.6}$$

holds for $0 \le j \le d$. Combining equations (5.5) and (5.6), we get

$$\mathfrak{h}'_j(\Delta \smallsetminus v) + \mathfrak{h}_{j-1}(\Delta/v) \;=\; \mathfrak{h}_j(\Delta) + \binom{d}{j} \sum_{i=0}^{j-1} (-1)^{j-i-1} \widetilde{\beta}_{i-1}(\Delta \smallsetminus v). \tag{5.7}$$

Since Δ is Buchsbaum* over k, we have $\widetilde{\beta}_{i-1}(\Delta \smallsetminus v) = \widetilde{\beta}_{i-1}(\Delta)$ for $i \le d-1$. Hence the right-hand side of (5.7) is equal to that of (5.2) and the result follows. $\square$

The $\mathfrak{h}''$-vector of Δ is the sequence $\mathfrak{h}''(\Delta) = (\mathfrak{h}''_0(\Delta), \mathfrak{h}''_1(\Delta), \dots, \mathfrak{h}''_d(\Delta))$ defined by

$$\mathfrak{h}''_j(\Delta) \;=\; \mathfrak{h}'_j(\Delta) - \binom{d}{j}\widetilde{\beta}_{j-1}(\Delta) \;=\; \mathfrak{h}_j(\Delta) + \binom{d}{j}\sum_{i=0}^{j}(-1)^{j-i-1}\widetilde{\beta}_{j-1}(\Delta) \tag{5.8}$$

for $0 \le j \le d-1$ and $\mathfrak{h}''_d(\Delta) = \widetilde{\beta}_{d-1}(\Delta) = \mathfrak{h}'_d(\Delta)$. The $\mathfrak{h}''$-vector was introduced by Kalai (see [17, Section 7]) as the "correct" $\mathfrak{h}$-vector for orientable homology manifolds and shown to have nonnegative entries for every Buchsbaum simplicial complex Δ over k in [18, Theorem 3.4].

Proposition 5.2. *Let Δ be a $(d-1)$-dimensional Buchsbaum* simplicial complex over k. For each vertex v of Δ and $0 \le j \le d$ we have*

$$\mathfrak{h}''_j(\Delta) \;=\; \mathfrak{h}''_j(\Delta \smallsetminus v) + \mathfrak{h}_{j-1}(\Delta/v), \tag{5.9}$$

where $\mathfrak{h}_{-1}(\Delta/v) = 0$ by convention.

Proof. The proposed equation is equivalent to (5.4) for $j = d$ and follows as in the proof of Proposition 5.1 for $0 \le j \le d-1$. □

Recall that a simplicial complex Δ is called *flag* if every minimal non-face of Δ has at most two elements. As an application of Proposition 5.1, we will show (Corollary 5.4) that among all Buchsbaum* flag simplicial complexes of dimension $d-1$, the simplicial join of d copies of the zero-dimensional sphere has the minimum $\mathfrak{h}'$-vector. This result generalizes one of [1] on Cohen-Macaulay complexes to the setting of Buchsbaum complexes. The question of formulating such a generalization provided the initial motivation for introducing the class of Buchsbaum* complexes.

The following proposition extends [21, Theorem 2.1] (see also [22, Theorem 9.1]) in the setting of Buchsbaum complexes.

Proposition 5.3. *Let Δ be a simplicial complex of dimension $d-1$ and Γ be a subcomplex of dimension $e-1$. Assume that no set of $e+1$ vertices of Γ is a face of Δ (this condition holds automatically if $d = e$). If both Γ and Δ are Buchsbaum over k, then $\mathfrak{h}'_i(\Gamma) \le \mathfrak{h}'_i(\Delta)$ holds for all $0 \le i \le d$.*

Proof. By equation (5.3), a proof of the proposition can be given by simply replacing the term $\mathfrak{h}$-vector with $\mathfrak{h}'$-vector in the proof of [21, Theorem 2.1]. □

Corollary 5.4. *If Δ is a $(d-1)$-dimensional flag simplicial complex which is Buchsbaum* over k, then the inequalities*

$$\mathfrak{h}'_i(\Delta) \ge \binom{d}{i} \tag{5.10}$$

hold for $0 \le i \le d$.

Proof. In view of Propositions 5.1 and 5.3, this follows by replacing $\mathfrak{h}$-vectors by $\mathfrak{h}'$-vectors in the argument of [1, Section 4] and using the fact (Corollary 2.9) that Δ is doubly Buchsbaum over k. □

We note that by the result of [18] on the nonnegativity of $\mathfrak{h}''(\Delta)$, mentioned earlier, we have $\mathfrak{h}'_i(\Delta) \geq \binom{d}{i}\widetilde{\beta}_{i-1}(\Delta)$ for every $(d-1)$-dimensional Buchsbaum complex Δ.

We conclude this section with two results on the face enumeration of Buchsbaum* complexes. They both extend results of Nevo [16] on doubly Cohen-Macaulay complexes and rely heavily on Theorem 4.1.

Proposition 5.5. *Let Δ be a $(d-1)$-dimensional Buchsbaum* simplicial complex over some field and let n be the number of vertices of Δ. If $d \geq 3$, then $\mathfrak{f}_i(\Delta) \geq \mathfrak{f}_i(n,d)$ holds for $0 \leq i \leq d-1$, where*

$$\mathfrak{f}_i(n,d) = \begin{cases} \binom{d}{i}n - \binom{d+1}{i+1}i, & \text{if } 0 \leq i \leq d-2 \\ (d-1)n - (d+1)(d-2), & \text{if } i = d-1 \end{cases}$$

is the number of i-dimensional faces of a stacked $(d-1)$-dimensional sphere with n vertices.

Proof. The assertion follows from Theorem 4.1 and the discussion in [16, Section 1]. □

The $\mathfrak{g}$-vector of Δ is the sequence $(\mathfrak{g}_0(\Delta), \mathfrak{g}_1(\Delta), \dots, \mathfrak{g}_{\lfloor d/2 \rfloor}(\Delta))$, defined by $\mathfrak{g}_i(\Delta) = \mathfrak{h}_i(\Delta) - \mathfrak{h}_{i-1}(\Delta)$ for $i \geq 1$ and $\mathfrak{g}_0(\Delta) = \mathfrak{h}_0(\Delta) = 1$. Recall that a sequence $(a_0, \dots, a_r)$ of nonnegative integers is called an *M-vector* if there exists a standard graded k-algebra $A = A_0 \oplus \cdots \oplus A_r$ such that $\dim_k A_i = a_i$ for every i; see [4, Section 4.2] for details and for further information.

Proposition 5.6. *Let Δ be a connected $(d-1)$-dimensional simplicial complex which is Buchsbaum* over some field. If $d \geq 4$, then $(\mathfrak{g}_0(\Delta), \mathfrak{g}_1(\Delta), \mathfrak{g}_2(\Delta))$ is an M-vector.*

Proof. Since Δ is connected, we have $\mathfrak{h}'_i(\Delta) = \mathfrak{h}_i(\Delta)$ for $i \leq 2$. Therefore (5.3) continues to hold for $i \leq 2$, if $\mathfrak{h}'_i(\Delta)$ is replaced by $\mathfrak{h}_i(\Delta)$. Thus the assertion follows from Theorem 4.1 as in the discussion in [16, Section 2]. □

The following question is an extension to Buchsbaum* complexes of a question posed by A. Björner and E. Swartz (see [24, Problem 4.2]) for doubly Cohen-Macaulay complexes. An example pointed out by E. Swartz (personal communication with the authors) shows that for $d = 7$, the analogous question with the numbers $\mathfrak{h}''_i(\Delta)$ replaced by the $\mathfrak{h}''_i(\Delta)$ has a negative answer.

Question 5.7. Are the following true for every $(d-1)$-dimensional Buchsbaum* simplicial complex Δ over k?

(i) $\mathfrak{h}''_i(\Delta) \leq \mathfrak{h}''_{d-i}(\Delta)$ for $0 \leq i \leq \lfloor d/2 \rfloor$.
(ii) $(\mathfrak{g}''_0(\Delta), \dots, \mathfrak{g}''_{\lfloor d/2 \rfloor}(\Delta))$ is an M-vector, where $\mathfrak{g}''_i(\Delta) = \mathfrak{h}''_i(\Delta) - \mathfrak{h}''_{i-1}(\Delta)$ for $i \geq 1$ and $\mathfrak{g}''_0(\Delta) = 1$.

6. Further results

This section summarizes some interesting results which appeared in the literature after this paper was publicized, giving further important properties of Buchsbaum* complexes.

1. Let Δ be a $(d-1)$dimensional Buchsbaum simplicial complex over an infinite field k and let $\Theta = (\Theta_1, \Theta_2, \ldots, \Theta_d)$ be a linear system of parameters for $k[\Delta]$. Consider the ring $\overline{k[\Delta]} = (k[\Delta]/\Theta)/I$, where $I = \bigoplus_{i=1}^{d-1} \mathrm{Soc}(k[\Delta]/\Theta)_i$. It was proved by Novik and Swartz in [18, Theorem 3.5] that we have

$$\dim_k \left(\mathrm{Soc}(k[\Delta]/\Theta)\right)_i \ \geq \ \binom{d}{i} \widetilde{\beta}_{i-1}(\Delta) \tag{6.1}$$

or, equivalently,

$$\mathfrak{h}''_i(\Delta) \ \geq \ \dim_k \ \overline{k[\Delta]}_i, \tag{6.2}$$

for $0 \leq i \leq d$ and in [19, Theorem 1.3] that equality holds in (6.1) if Δ is a connected orientable homology manifold over k. Nagel [15, Theorem 1.1] showed that equality holds in (6.1) for every Buchsbaum* complex Δ over k and all $0 \leq i \leq d$. He also showed [15, Theorem 1.2] that if Δ is Buchsbaum* over k, then $\overline{k[\Delta]}$ is a level ring of Cohen-Macaulay type $\widetilde{\beta}_{i-1}(\Delta)$ and socle degree d. This result provides an analogue to [19, Theorem 1.4], which states that $\overline{k[\Delta]}$ is a Gorenstein ring for every connected orientable homology manifold Δ over k.

2. One of the standard constructions on simplicial complexes is rank selection on balanced complexes; see [22, Section III.4] for an exposition of these concepts. Responding to a question raised in an earlier version of this paper, Browder and Klee [3] have shown that if a balanced simplicial complex Δ is Buchsbaum* over k, then so is every rank selected subcomplex of Δ. Among other results, they have also generalized the inequalities (5.10) to m-Buchsbaum* complexes and treated the case of equality.

3. It was shown by I. Novik (personal communication with the authors) that, under the assumptions of Corollary 5.4, the inequalities (5.10) can be strengthened to

$$\mathfrak{h}''_i(\Delta) \geq \binom{d}{i} \tag{6.3}$$

for $0 \leq i \leq d-2$. The proof uses results from [18] and then follows the general outline of the proof of Corollary 5.4.

Acknowledgments

The authors thank Anders Björner for suggesting the terminology Buchsbaum* complex and for pointing out useful references. They also thank Isabella Novik and Ed Swartz for their comments on an earlier version of this paper and for their help with the formulation of Question 5.7. The first author was supported by the 70/4/8755 ELKE research fund of the University of Athens.

References

[1] C.A. Athanasiadis, *Some combinatorial properties of flag simplicial pseudomanifolds and spheres*, Ark. Mat. **49** (2011), 17–29.

[2] K. Baclawski, *Cohen-Macaulay connectivity and geometric lattices*, European J. Combin. **3** (1982), 293–305.

[3] J. Browder and S. Klee, *Lower bounds for Buchsbaum* complexes*, European J. Combin. **32** (2011), 146–153.

[4] W. Bruns and J. Herzog, Cohen-Macaulay rings, Cambridge Studies in Advanced Mathematics **39**, Cambridge University Press, Cambridge, 1998.

[5] A. Björner, *Topological methods*, in *Handbook of Combinatorics* (R.L. Graham, M. Grötschel and L. Lovász, eds.), North Holland, Amsterdam, 1995, pp. 1819–1872.

[6] A. Björner, *Mixed connectivity properties of polytopes, cell complexes and posets*, manuscript in preparation.

[7] A. Björner and T. Hibi, *Betti numbers of Buchsbaum complexes*, Math. Scand. **67** (1990), 193–196.

[8] M.K. Chari, *Two decompositions in topological combinatorics with applications to matroid complexes*, Trans. Amer. Math. Soc. **349** (1997), 3925–3943.

[9] G. Fløystad, *Cohen-Macaulay cell complexes*, in *Algebraic and Geometric Combinatorics*, Contemporary Mathematics **423**, American Mathematical Society, Providence, RI, 2007, pp. 205–220.

[10] H.-G. Gräbe, *The canonical module of a Stanley-Reisner ring*, J. Algebra **86** (1984), 272–281.

[11] G. Kalai, *Rigidity and the lower bound theorem*, Invent. Math. **88** (1987), 125–151.

[12] M. Miyazaki, *On 2-Buchsbaum complexes*, J. Math. Kyoto Univ. **30** (1990), 367–392.

[13] J.R. Munkres, Elements of Algebraic Topology, Addison-Wesley, Reading, MA, 1984.

[14] J.R. Munkres, *Topological results in combinatorics*, Michigan Math. J. **31** (1984), 113–128.

[15] U. Nagel, *Level algebras through Buchsbaum* manifolds*, Collect. Math. (to appear).

[16] E. Nevo, *Rigidity and the lower bound theorem for doubly Cohen-Macaulay complexes*, Discrete Comput. Geom. **39** (2008), 411–418.

[17] I. Novik, *Upper bound theorems for homology manifolds*, Israel J. Math. **108** (1998), 45–82.

[18] I. Novik and E. Swartz, *Socles of Buchsbaum modules, complexes and posets*, Adv. Math. **222** (2009), 2059–2084.

[19] I. Novik and E. Swartz, *Gorenstein rings through face rings of manifolds*, Compos. Math. **144** (2009), 993–1000.

[20] P. Schenzel, *On the number of faces of a simplicial complex and purity of Frobenius*, Math. Z. **178** (1981), 125–142.

[21] R.P. Stanley, *A monotonicity property of h-vectors and h^*-vectors*, European J. Combin. **14** (1993), 251–258.

[22] R.P. Stanley, Combinatorics and Commutative Algebra, second edition, Birkhäuser, Basel, 1996.

[23] J. Stückrad and W. Vogel, Buchsbaum rings and applications, Springer-Verlag, Berlin, 1986.

[24] E. Swartz, *g-elements, finite buildings and higher Cohen-Macaulay connectivity*, J. Combin. Theory Series A **113** (2006), 1305–1320.

[25] J.W. Walker, Topology and Combinatorics of Ordered Sets, Ph.D Thesis, MIT, 1981.

Department of Mathematics, Division of Algebra-Geometry, University of Athens, Panepistimioupolis, Athens 15784, Greece

E-mail address: caath@math.uoa.gr

Fachbereich Mathematik und Informatik, Philipps-Universität Marburg, 35032 Marburg, Germany

E-mail address: welker@mathematik.uni-marburg.de