ON PARTIAL BARYCENTRIC SUBDIVISION

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Abstract. The l^{th} partial barycentric subdivision is defined for a $(d-1)$-dimensional simplicial complex Δ and studied along with its combinatorial and geometric aspects. We analyze the behavior of the f- and h-vector under the l^{th} partial barycentric subdivision extending previous work of Brenti and Welker on the standard barycentric subdivision – the case $l = d$. We discuss and provide properties of the transformation matrices sending the f- and h-vector of Δ to the f- and h-vector of its l^{th} partial barycentric subdivision. We conclude with open problems.

1. Introduction

For a $(d-1)$-dimensional simplicial complex Δ on the ground set V the barycentric subdivision $\mathrm{sd}(\Delta)$ of Δ is the simplicial complex on the ground set $\Delta \setminus \{\emptyset\}$ with simplices the flags $A_0 \subset A_1 \subset \cdots \subset A_i$ of elements $A_j \in \Delta \setminus \{\emptyset\}$, $0 \le j \le i$. For $0 \le l \le d$, we define the l^{th} partial barycentric subdivision of Δ. This is a geometric subdivision, in the sense of [7], such that $\mathrm{sd}^l(\Delta)$ is a refinement of $\mathrm{sd}^{l-1}(\Delta)$, $\mathrm{sd}^0(\Delta) = \Delta$ and $\mathrm{sd}^d(\Delta) = \mathrm{sd}(\Delta)$. Roughly speaking, the l^{th} partial barycentric subdivision arises when only the simplices of dimension $\ge d - l$ are stellarly subdivided. In the paper, we provide a detailed analysis of the effect of the l^{th} barycentric subdivision operation on the f- and h-vector of a simplicial complex. Most enumerative results will be related to refinements of permutation statistics for the symmetric group. Our results extend the results from [1] for the case $l = d$. We refer the reader also to [2] and [6] for more detailed information in this case.

The paper is organized as follows. We start in Section 2 with geometric and combinatorial descriptions of the l^{th} partial barycentric subdivision. In Section 3 we study the enumerative combinatorics of the l^{th} partial barycentric subdivision. In particular, we relate in Lemma 3.1 and Theorem 3.4 the effect of the l^{th} barycentric subdivision on the f- and h-vector of the simplicial complex Δ to a permutation statistics refining the descent statistics. In Section 4 we analyze the transformation matrices sending the f- and h-vector of the simplicial complex Δ to the corresponding vector for the l^{th} barycentric subdivision. We show that both maps are diagonalizable and provide the eigenvalue structure. Note that by general facts the two matrices are similar. The main result of this section, Theorem 4.10, shows that the eigenvector corresponding to the highest eigenvalue of the h-vector transformation can be chosen such that it is of the form $(0, b_1, \ldots, b_{d-1}, 0)$ for strictly positive

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numbers b_i, $1 \le i \le d-1$. In Section 5 we present some open problems. We ask for explicit descriptions of the eigenvectors and then shift the focus to the local h-vector which has been introduced by Stanley [7]. The local h-vector is a measure for the local effect of a subdivision operation. In particular, general results by Stanley predict that the local h-vector for the l^{th} partial barycentric subdivision is non-negative. For $l = d$ the local h-vector was computed by Stanley in terms of the excedance statistics on derangements. We exhibit some computations and possible approaches to the local h-vector for the l^{th} barycentric subdivision in general.

2. The l^{th} partial barycentric subdivision

2.1. **Geometric definition.** We first give a geometric definition of the l^{th} partial barycentric subdivision. For that we recall some basic facts about the reflection arrangement of the symmetric group S_d permuting the d letters from $[d] := \{1, 2, \ldots, d\}$. The reflection arrangement $\mathcal{B}_d$ in $\mathbb{R}^d$ of the symmetric group S_d consists of the hyperplanes $H_{uv} = \{(x_1, \ldots, x_d) \in \mathbb{R}^d \ : \ x_u - x_v = 0\}$, $1 \le u < v \le d$. To each permutation $w = w_1 \cdots w_d \in S_d$ there corresponds a region (i.e. connected component of the complement) R_w of $\mathcal{B}_d$ given by

$$R_w = \{(\lambda_1, \ldots, \lambda_d) \in \mathbb{R}^d : \lambda_{w_1} > \lambda_{w_2} > \cdots > \lambda_{w_d}\}.$$

Hence the number of regions of $\mathcal{B}_d$ is $d!$. We write $R_{w,+}$ for the intersection of R_w with $\mathbb{R}^d_{\ge 0}$. It is easily seen that geometrically the closure of $R_{w,+}$ is a simplicial cone.

The intersection of the closures of the cones $R_{w,+}$, $w \in S_d$, and the standard $(d-1)$-simplex $\Delta_{d-1} = \{(\lambda_1, \ldots, \lambda_d) \in \mathbb{R}^d \mid \lambda_1 + \cdots + \lambda_d = 1, \lambda_u \ge 0, 1 \le u \le d\}$ induces a simplicial decomposition of Δ_{d-1}. This decomposition is called the *barycentric subdivision* of Δ_{d-1} and is denoted by $\mathrm{sd}(\Delta_{d-1})$.

We generalize this decomposition in the following way. Consider a permutation $w \in S_d$ as an *injective* word of length d on the alphabet $[d]$. Recall that an injective word over an alphabet is a word in which every letter from the alphabet appears at most once. For $0 \le l \le d$ we denote by S_d^l the set of injective words of length l over the alphabet d. For $w = w_1 \ldots w_l \in S_d^l$ we denote by $\mathrm{free}(w) := [d] \setminus \{w_1, \ldots, w_l\}$ the set of letters not occurring in w. Note that for notational convenience we will later index injective words from S_d^l as $w_{d+1-l} \cdots w_d$. We define the cone R_w^l of a word $w = w_1 \cdots w_l \in S_d^l$ to be

$$R_w^l = \{(\lambda_1, \ldots, \lambda_d) \in \mathbb{R}^d \ : \ \lambda_u > \lambda_{w_1} > \cdots > \lambda_{w_l} \text{ for all } u \in \mathrm{free}(w) \ \}.$$

If $l = 0$ then w is the empty word and $R_w^l = \mathbb{R}^d$. We write $R_{w,+}^l$ for the intersection of R_w^l with $\mathbb{R}^d_{\ge 0}$. Again the closure of $R_{w,+}^l$ is a simplicial cone which is the union of all closures of the $R_{v,+}$ for $v \in S_d$ such that w coincides with the last l letters of v. For $0 \le l \le d$ we call the simplicial decomposition of Δ_{d-1} induced by the collection of all $R_{w,+}^l$ for $w \in S_d^l$ the *l^{th} partial barycentric subdivision* of Δ_{d-1} and denote it by $\mathrm{sd}^l(\Delta_{d-1})$. Obviously, we have that $\mathrm{sd}^0(\Delta_{d-1}) = \Delta_{d-1}$, $\mathrm{sd}^d(\Delta_{d-1}) = \mathrm{sd}(\Delta_{d-1})$ and $\mathrm{sd}^l(\Delta_{d-1})$ is a refinement of $\mathrm{sd}^{l-1}(\Delta_{d-1})$ for $1 \le l \le d$. For a $(d-1)$-dimensional simplicial complex Δ on the vertex set $V = [n]$ its l^{th} partial barycentric subdivision is the complex $\mathrm{sd}^l(\Delta)$ which is the subdivision of Δ obtained by replacing

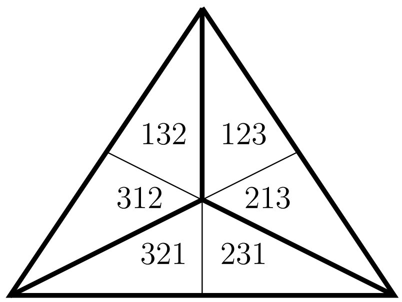

FIGURE 1. First and second partial barycentric subdivision of the 2-simplex

each simplex by its l^{th} partial subdivision. Roughly speaking $\mathrm{sd}^l(\Delta)$ is obtained by stellarly subdividing all k-faces of dimension $d-l \leq k \leq d-1$ in decreasing order of dimension. In Figure 1 we see the first and second barycentric subdivision of the 2-simplex.

By construction the number of cones R_w^l, $w \in S_d^l$, is $\frac{d!}{(d-l)!} = d\cdot(d-1)\cdots(d-l+1)$. Next, we want to get a better understanding of the facial structure of $\mathrm{sd}^l(\Delta_{d-1})$.

We have already seen that the $(d-1)$-dimensional faces of $\mathrm{sd}^l(\Delta_{d-1})$ are in bijection with the injective words in S_d^l. We turn this description of $(d-1)$-dimensional faces into a description by combinatorial objects that are more suitable for studying all faces of $\mathrm{sd}^l(\Delta_{d-1})$.

We start by identifying the faces of Δ_{d-1} with the subsets of $[d]$. In $\mathrm{sd}^l(\Delta_{d-1})$ faces from Δ_{d-1} of dimension $\leq d-l-1$ are not subdivided. The non-subdivided faces can still be represented by subsets of $[d]$ of cardinality $\leq d-l$. Let F be a face that arises when passing from the $(l-1)^{\text{st}}$ subdivision to the l^{th} subdivision. Thus there is a face F_1 of Δ_{d-1} of dimension $d-l$ represented by a set B_1' of cardinality $d-l+1$ such that the barycenter of F_1 is a vertex of F. The other vertices of F either are vertices of a single non-subdivided face $G \subset F_1$ of dimension $< d-l$ represented by a set $B \subset B_1'$ of cardinality $\leq d-l$ or are barycenters of faces $F_2, \ldots, F_r$ of Δ_{d-1} such that $F_1 \subset F_2 \subset \cdots \subset F_r$. In particular, F has $\#B+r$ vertices and hence is of dimension $\#B+r-1$. Let $B_2' \subset \cdots \subset B_r'$ be the sets representing the faces $F_2, \ldots, F_r$ of Δ_{d-1}. In case there is no vertex from a face G we set $B = \emptyset$. Now turn the description of the face by $B|B_1'|\cdots|B_r$ into a description by $B, B_1 = B_1' \setminus B, \cdots, B_r = B_r' \setminus B_{r-1}'$ (see Figure 2 for the corresponding identification of faces in $\mathrm{sd}^1(\Delta_2)$).

This leads to the following definition:

Let $J \subseteq [d]$ be some subset. For numbers $i \geq 1$ and $0 \leq l \leq d$ we call a tuple $|B|B_1|...|B_r|$ of pairwise disjoint subsets, B, $B_1, \ldots, B_r$ of $J \subseteq [d]$ satisfying

- $B \cup B_1 \cup \cdots \cup B_r = J$
- $B_1, \ldots, B_r \neq \emptyset$

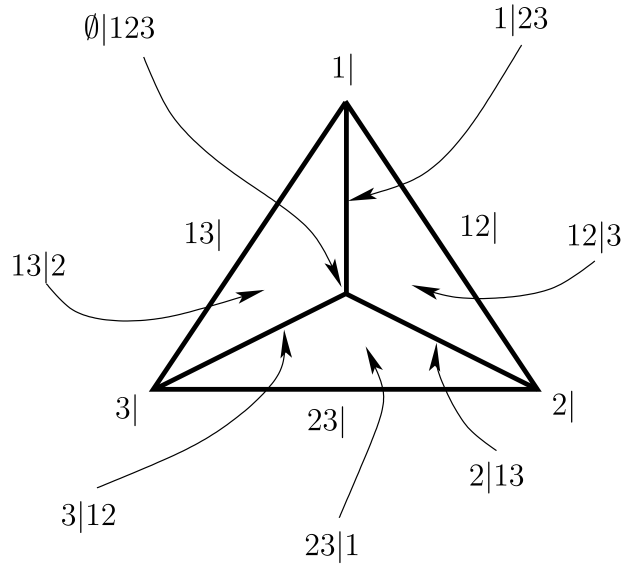

FIGURE 2. Face identification in $\mathrm{sd}^1(\Delta_2)$

a *pointed ordered set partition* of J. We refer to B, $B_1, \ldots, B_r$ as the blocks and to B as the special block of $|B|B_1|\ldots|B_r|$. Note that we allow $r = 0$, i.e. pointed ordered set partition consisting of the special block only. In [3, Sec. 3] our pointed ordered set partitions $|B|B_1|\ldots|B_r|$, written in a slightly different way, are called ordered set partitions. Since in the literature the term ordered set partition often has yet another meaning, we adopt in our terminology the point of view from [3, Sec. 2] where pointed (unordered) set partitions are introduced.

By the discussion above an $(i-1)$-dimensional face of $\mathrm{sd}^l(\Delta_{d-1})$ is represented by a pointed ordered set partition $|B|B_1|\ldots|B_r|$ of $J \subseteq [d]$, if and only if

(P1) $\#B \leq d - l$.
(P2) If $r \geq 1$ then $\#(B \cup B_1) \geq d - l + 1$
(P3) $\#B + r = i$.

Let us discuss the implications of (P1)-(P3):

- For $i = 1$, by (P3) we must have either $\#B = 1$ or $B = \emptyset$ and $r = 1$. This resembles the fact that vertices in $\mathrm{sd}^l(\Delta_{d-1})$ are either vertices of the original simplex, the case $\#B = 1$, or barycenters of faces B_1 of the original simplex, where by (P2) $\#B_1 \geq d - l + 1$.
- For $i = d$, (P3) implies that either $r \geq 1$ and all blocks $B_1, \ldots, B_r$ are singletons or $r = 0$ and $B = [d]$. In particular, we have $J = [d]$. In case $r = 1$ condition (P2) also implies that $\#B \geq d - l$. Thus we can identify an injective word $w = w_1 \cdots w_r$ where $\#\mathrm{free}(w) \geq d - l$ with the pointed ordered set partition $\mathrm{free}(w)|w_1| \cdots |w_r$ for the parameters $i = d$, $J = [d]$ and l.
- For $l = 0$ by (P2) we must have $\#(B \cup B_1) \geq d - l + 1 = d + 1$ for $r \geq 1$ and hence $r = 0$ and $|B|B_1|\ldots|B_r| = |B|$ is some subset B of $[d]$. This

corresponds to the fact that in $\mathrm{sd}(\Delta_{d-1})^0$ no faces is subdivided and hence faces correspond to subsets of $[d]$.

- For $l = d$ by (P1) we must have $\#B \leq 0$ and hence $B = \emptyset$. Thus $r = i$ and $|B_1| \cdots |B_i|$ is a usual ordered set partition of the j-element set $B_1 \cup \cdots \cup B_i = J$ into i (non-empty) blocks. This is the usual description of faces of the full barycentric subdivision of Δ_{d-1}.

Geometrically, the face of $sd^l(\Delta_{d-1})$ corresponding to the pointed ordered set partition $|B|B_1| \cdots |B_r|$ of $J \subseteq [d]$ is given as the set of points $(\lambda_1, \ldots, \lambda_d)$ in the geometric realization of Δ_{d-1} for which

(i) We have $\lambda_u = \lambda_v$ if $u, v \in B_s$ for some $1 \leq s \leq r$.
(ii) We have $\lambda_u > \lambda_v$ if $u \in B_s$ and $v \in B_t$ for some $1 \leq s < t \leq r$.
(iii) We have $\lambda_u > \lambda_v$ for $u \in B$ and $v \in B_1$.
(iv) We have $\lambda_u = 0$ if and only if $u \notin B \cup B_1 \cup \cdots \cup B_r$.

Thus in geometric terms a face corresponding to the pointed ordered set partition $|B|B_1| \cdots |B_r|$ arises as the join of the face corresponding to B in the original simplex with the face corresponding to the chain $B \cup B_1 \subset \cdots \subset B \cup B_1 \cup \cdots \cup B_r$ of the full barycentric subdivision.

Clearly, the number of pointed ordered set partitions of J only depends on $j = \#J$. and hence we write $\mathrm{poS}_d(j, i, l)$ for the number of pointed ordered set partitions of a j-element set J satisfying (P1)-(P3) and call $\mathrm{poS}_d(j, i, l)$ the *pointed ordered Stirling number* for the parameters d,j,i,l. Note, by the discussion above $\mathrm{poS}_d(j, i, d)$ is just the number of usual ordered set partitions of a j-element set into i blocks. Hence $\mathrm{poS}_d(j, i, d) = i!S(j, i)$ where $S(j, i)$ is the usual Stirling number of the second kind counting the number of (unordered) set partitions of a j-element set into i blocks. For $i = j = d$ the discussion above also shows that get $\mathrm{poS}_d(d, d, l) = d!/(d - l)!$ is the number of injective words in S_d^l.

3. f-vector and h-vector transformations

In this section we study the transformation sending the f- and h-vector of a simplicial complex Δ to the f- and h-vector of the l^{th} partial barycentric subdivision of Δ.

Recall that the f-vector $f^\Delta = (f_{-1}^\Delta, \ldots, f_{d-1}^\Delta)$ of a $(d-1)$-dimensional simplicial complex is the vector with its i^{th} entry f_i^Δ counting the i-dimensional faces of Δ. Using this notation, the arguments from the preceding section immediately imply the following lemma generalizing [1, Lem. 1].

Lemma 3.1. *Let Δ be a $(d-1)$-dimensional simplicial complex with f-vector $f^\Delta = (f_{-1}^\Delta, \ldots, f_{d-1}^\Delta)$. Then*

$$f_{i-1}^{\mathrm{sd}^l(\Delta)} = \sum_{j=0}^{d} f_{j-1}^\Delta \cdot \mathrm{poS}_d(j, i, l).$$

Next we study the transformation of the h-vector. Recall that the h-vector of a $(d-1)$-dimensional simplicial complex Δ is the integer vector $h^\Delta = (h_0^\Delta, \ldots, h_d^\Delta)$

defined by

$$h_v^{\Delta} \quad = \quad \sum_{i=0}^{v} \binom{d-i}{v-i} (-1)^{v-i} f_{i-1}^{\Delta}, \tag{1}$$

$0 \le v \le d$. Conversely, the f-vector can be computed from the h-vector by

$$f_{j-1}^{\Delta} \quad = \quad \sum_{u=0}^{d} \binom{d-u}{d-j} h_u^{\Delta}, \tag{2}$$

$0 \le j \le d$. For a permutation $w = w_1 \cdots w_d \in S_d$ we denoted by

$$\mathrm{D}(w) = \{ i \in [d-1] \mid w_i > w_{i+1} \}$$

its decent set and write $\mathrm{des}(w) := \#\,\mathrm{D}(w)$ for its number of descents. Following [1] for $d \ge 1$ and integers i and j we denote by $A_d(j,i)$ the number of permutations $w \in S_d$ such that $w_d = d - i$ and $\mathrm{des}(w) = j$. In particular, $A_d(j,i) = 0$ if $i \le -1$ or $i \ge d$.

In the sequel, we define a refinement of the preceding statistics suitable for the study of our h-vector transformation.

Let $w = w_{d-l+1} \cdots w_d \in S_d^l$ be an injective word of length l and let $\mathrm{free}(w) = \{w_1 < \cdots < w_{d-l}\}$. We define the *descent set* $\mathrm{D}(w)$ of w as follows:

Definition 3.2. *A number $i \in [d-1]$ belongs to the descent set $\mathrm{D}(w)$ of $w = w_{d-l+1} \cdots w_d \in S_d^l$, if i satisfies one of the following two conditions.*

(1) $1 \le i \le d-l$ *and* $w_i > w_{d-l+1}$ *or*
(2) $d-l+1 \le i \le d-1$ *and* $w_i > w_{i+1}$.

We write $\mathrm{des}(w) = \#\,\mathrm{D}(w)$ for the *number of descents* of an injective word $w \in S_d^l$. Note that for $l = d$ condition (1) is never satisfied and and therefore $\mathrm{D}(w)$ is just the usual descent set of the permutation $w \in S_d$.

Example 3.3. *Let $w^1 = 65, w^2 = 34, w^3 = 51 \in S_6^2$, then*

$$\begin{array}{lll} \mathrm{D}(w^1) = \{5\}, & \mathrm{D}(w^2) = \{3,4\}, & \mathrm{D}(w^3) = \{4,5\}, \\ \mathrm{des}(w^1) = 1, & \mathrm{des}(w^2) = 2, & \mathrm{des}(w^3) = 2. \end{array}$$

For all $d \ge 1$, $1 \le l \le d$, and all integers i and j we denote by $A_d(j,i,l)$ the number of all injective words $w_{d-l+1} \cdots w_d \in S_d^l$ such that $\mathrm{des}(w) = j$ and $w_d = d - i$. Then $A_d(j,i,d) = A_d(j,i)$ and $A_d(j,i,l) = 0$ if $i \le -1$ or $i \ge d+1$.

The following is our first main result. The case $l = d$ was treated in [1, Thm. 1].

Theorem 3.4. *Let Δ be a $(d-1)$-dimensional simplicial complex. Then*

$$h_v^{\mathrm{sd}^l(\Delta)} = \sum_{u=0}^{d} A_{d+1}(v,u,l+1) h_u^{\Delta}$$

for $0 \le l \le d$ and $0 \le v \le d$.

Proof. For all $0 \le v \le d$, we have

$$\begin{aligned}
h_v^{\mathrm{sd}^l(\Delta)} &\overset{(1)}{=} \sum_{i=0}^{v} \binom{d-i}{v-i} (-1)^{v-i} f_{i-1}^{\mathrm{sd}^l(\Delta)} \\
&\overset{\text{Lemma 3.1}}{=} \sum_{i=0}^{v} \binom{d-i}{v-i} (-1)^{v-i} \sum_{j=0}^{d} f_{j-1}^{\Delta} \mathrm{poS}_d(j,i,l) \\
&\overset{(2)}{=} \sum_{i=0}^{v} \sum_{j=0}^{d} \binom{d-i}{v-i} (-1)^{v-i} \mathrm{poS}_d(j,i,l) \sum_{u=0}^{j} \binom{d-u}{d-j} h_u^{\Delta} \\
&= \sum_{u=0}^{d} \Big(\sum_{j=0}^{d} \sum_{i=0}^{v} (-1)^{v-i} \binom{d-i}{v-i} \binom{d-u}{d-j} \mathrm{poS}_d(j,i,l) \Big) h_u^{\Delta}.
\end{aligned}$$

Hence it remains to show that

$$(3) \qquad A_{d+1}(v,u,l+1) \;=\; \sum_{j=0}^{d} \sum_{i=0}^{v} (-1)^{v-i} \binom{d-i}{v-i} \binom{d-u}{d-j} \mathrm{poS}_d(j,i,l).$$

We write $S \subseteq_l [d]$ if $S \subseteq \{d-l+1, \ldots, d\}$ or $S = \{s_1 = d-l+1-p < \cdots < s_p = d-l < s_{p+1} < \cdots < s_i\}$ for some $1 \le p \le d-l$. For this we first show:

$$(4) \sum_{S \subseteq_l [d], \#S = i} \# \left\{ w \in S_{d+1}^{l+1} \;\middle|\; \begin{array}{l} \mathrm{D}(w) \subseteq S, \\ w_{d+1} = d+1-u \end{array} \right\} \;=\; \sum_{j=i}^{d} \binom{d-u}{d-j} \mathrm{poS}_d(j,i,l)$$

The left hand side counts pairs of words $w \in S_{d+1}^{l+1}$ with $w_{d+1} = d+1-u$ and i-element sets $S \subseteq_l [d]$ such that $D(w) \subseteq S$. We then consider the right hand side as counting pairs of subsets $J \subseteq [d+1]$ such that $d+1-u$ is the maximal element of $[d+1] \setminus J$, $j = \#J \ge i$ and pointed ordered set partitions of J counted by $\mathrm{poS}_d(j,i,l)$. Note that there are $\binom{d-u}{d-j}$ choices for J. We give an bijection between the objects counted on the left hand side and the ones counted on the right hand side.

Consider an injective word $w = w_{d+1-(l+1)+1} \cdots w_{d+1} \in S_{d+1}^{l+1}$. Let $\mathrm{free}(w) = \{w_1 < \cdots < w_{d-l}\}$ and $\mathrm{D}(w) \subseteq S$ for $S = \{s_1 < \cdots < s_i\} \subseteq_l [d]$ counted in the sum on the left hand side. To this word we associate the pointed ordered set partition $|B|B_1| \cdots |B_{i-p}|$ of $J = \{w_1, \ldots, w_{s_i}\}$, defined as follows. Let $p = \max\{q > 0 \mid s_q < d-l+1\}$ where we treat the maximum over an empty set as 0. Set $B = \{w_{d-l-p+1}, \ldots, w_{s_p = d-l}\}$ if $p > 0$ and $B = \emptyset$ otherwise. Set $B_1 = \{w_1, \ldots, w_{d-l-p}, w_{d-l+1}, \ldots, w_{s_{p+1}}\}$, $B_2 = \{w_{s_{p+1}+1}, \ldots, w_{s_{p+2}}\}$, …, $B_{i-p} = \{w_{s_{i-1}+1}, \ldots, w_{s_i}\}$. Then for $r = i-p$ this partition satisfies the following:

(P1) $\#B = p \le d-l$.
(P2) If $r \ge 1$ then $\#(B \cup B_1) \ge d-l+1$
(P3) $\#B + r = p + i - p = i$.

The number of such pointed ordered set partitions is given by $\mathrm{poS}_d(s_i, i, l)$. Note that w and S can be reconstructed from $|B|B_1| \cdots |B_r|$. For this set $p = \#B$. Define free(w) as the union of B and the $d-l-\#B = d-l-p$ smallest elements of B_1. Then obtain w by first writing down from left to right the remaining elements of B_1 in increasing order, then the elements of B_2 in increasing order, etc. . The set $S = \{s_1 < \cdots < s_i\}$ is obtained by setting $s_1 = d-l+1-p, \ldots, s_p = d-l$ for $p > 0$ and setting $s_{p+q} = d-l+|B_1|+\cdots+|B_q|$ for $1 \leq q \leq i-p$. By construction $S \subseteq_l [d]$ and $D(w) \subseteq S$. Since there are $\binom{d-u}{d-s_i}$ possibilities for choosing an s_i elements subset J of $[d+1]$ such that $\max [d+1] \setminus J = d+1-u$ it follows that we have a bijective mapping from the objects counted by the left hand side of (4) to the objects counted on the right hand side of (4). This proves (4)

Since $\mathrm{poS}_d(j,i,l) = 0$ for $j < i$ we can actually sum from $j = 0$ to d on the right hand side of (4).

Therefore,

$$\sum_{j=0}^{d}\sum_{i=0}^{v}(-1)^{v-i}\binom{d-i}{v-i}\binom{d-u}{d-j}\mathrm{poS}_d(j,i,l)$$

$$\begin{aligned}
&\overset{(4)}{=} \sum_{i=0}^{v}(-1)^{v-i}\binom{d-i}{l-i}\sum_{\{S\subseteq_l[d],\#S=i\}}\#\left\{w\in S_{d+1}^{l+1} \;\middle|\; \begin{array}{l}\mathrm{D}(w)\subseteq S,\\ w_{d+1}=d+1-u\end{array}\right\}\\
&= \sum_{\{S\subseteq_l[d],\#S\leq v\}}(-1)^{v-\#S}\binom{d-\#S}{v-\#S}\#\left\{w\in S_{d+1}^{l+1} \;\middle|\; \begin{array}{l}\mathrm{D}(w)\subseteq S,\\ w_{d+1}=d+1-u\end{array}\right\}\\
&= \sum_{\{S\subseteq_l[d],\#S\leq v\}}(-1)^{v-\#S}\binom{d-\#S}{v-\#S}\sum_{T\subseteq S,T\subset_l[d]}\#\left\{w\in S_{d+1}^{l+1} \;\middle|\; \begin{array}{l}\mathrm{D}(w)=T,\\ w_{d+1}=d+1-u\end{array}\right\}\\
&= \sum_{\{T\subseteq_l[d],\#T\leq v\}}\#\left\{w\in S_{d+1}^{l+1} \;\middle|\; \begin{array}{l}\mathrm{D}(w)=T,\\ w_{d+1}=d+1-u\end{array}\right\}\sum_{\{[d]\supseteq_l S\supseteq T,\#S\leq j\}}(-1)^{v-\#S}\binom{d-\#S}{v-\#S}\\
&= \sum_{\{T\subseteq_l[d],\#T\leq v\}}\#\left\{w\in S_{d+1}^{l+1} \;\middle|\; \begin{array}{l}\mathrm{D}(w)=T,\\ w_{d+1}=d+1-u\end{array}\right\}\sum_{i=\#T}^{v}(-1)^{v-i}\binom{d-i}{v-i}\binom{d-\#T}{i-\#T}.
\end{aligned}$$

But

$$\begin{aligned}
\sum_{i=\#T}^{v}(-1)^{v-i}\binom{d-i}{v-i}\binom{d-\#T}{i-\#T} &= \binom{d-\#T}{i-\#T}\sum_{i=\#T}^{v}(-1)^{j-i}\binom{v-\#T}{i-\#T}\\
&= \delta_{v,\#T}.
\end{aligned}$$

Hence

$$\sum_{j=0}^{d}\sum_{i=0}^{v}(-1)^{v-i}\binom{d-i}{v-i}\binom{d-u}{d-k}\mathrm{poS}_d(j,i,l)$$

$$
\begin{aligned}
&= \sum_{\{T\subseteq[d],\#T=v\}} \#\left\{ w\in S_{d+1}^{l+1} \;\middle|\; \begin{array}{l} \mathrm{D}(w)=T, \\ w_{d+1}=d+1-u \end{array} \right\} \\
&= \#\left\{ w\in S_{d+1}^{l+1} \;\middle|\; \begin{array}{l} \mathrm{des}(w)=v, \\ w_{d+1}=d+1-u \end{array} \right\}. \\
&= A_{d+1}(v,u,l+1)
\end{aligned}
$$

This shows (3) and completes the proof. □

We note that for a $(d-1)$-dimensional simplicial complex Δ and $1 \le l \le d$, the subdivision operation $\mathrm{sd}^l(\bullet)$ non-trivially subdivides each face in top dimension. It follows from Theorem 5.5. in [2] that iterated application of $\mathrm{sd}^l(\bullet)$ will lead to a convergence phenomenon for the f-vector. More precisely, for a $(d-1)$-dimensional simplicial complex Δ, set $\Delta^{(n,l)} := \underbrace{\mathrm{sd}^l(\cdots \mathrm{sd}^l}_{n}(\Delta)\cdots)$ and $f^{(n,l)}(t) = \sum_{i=0}^{d} f_{i-1}^{\Delta^{(n,l)}} t^{d-i}$ then for $n \to \infty$ one root of $f^{(n,l)}(t)$ will go to $-\infty$ and the others converge to complex numbers independent of Δ, only depending on d. This phenomenon was first observed in [1, Thm. 4.2] for the special case of classical barycentric subdivision $\mathrm{sd}^d(\bullet) = \mathrm{sd}(\bullet)$. In addition, in [1, Thm. 3.1] it is shown that for simplicial complexes Δ with non-negative h-vector and $l = d$ the polynomial $f^{(1,d)}(t)$ has only real roots. Simple examples show that this is not the case for general l.

4. The Transformation Matrices

For a $(d-1)$-dimensional simplicial complex Δ we denote by $\mathfrak{H}_{d-1} = (h_{ij}^{(d-1)})_{0\le i,j\le d} \in \mathbb{R}^{(d+1)\times(d+1)}$ the matrix of the linear transformation that sends the h-vector of Δ to the h-vector of $\mathrm{sd}(\Delta)$ and $\mathfrak{H}_{d-1}^l = (h_{ij}^{(d-1,l)})_{0\le i,j\le d} \in \mathbb{R}^{(d+1)\times(d+1)}$ the matrix of the transformation of the h-vector of Δ to the h-vector of $\mathrm{sd}^l(\Delta)$. Thus $\mathfrak{H}_{d-1}^d = \mathfrak{H}_{d-1}$. By [1, Thm. 1] we know $h_{ij}^{(d-1)} = A_{d+1}(j,i)$ and more generally by Theorem 3.4 we know $h_{ij}^{(d-1,l)} = A_{d+1}(j,i,l+1)$.

As an illustration we present the matrices $\mathfrak{H}_d^l$ for $d = 4$ and $l = 1$ and $l = 2$.

$$
\mathfrak{H}_3^1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 & 1 \\ 1 & 1 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}
\qquad
\mathfrak{H}_3^2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 5 & 5 & 3 & 2 & 1 \\ 5 & 5 & 6 & 5 & 5 \\ 1 & 2 & 3 & 5 & 5 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}
$$

The following lemma follows immediately from the definition of $A_{d+1}(j,i,l)$.

Lemma 4.1. *The sum of all entries of $\mathfrak{H}_{d-1}^l$ is given by:*

$$
\sum_{0\le i,j\le d} h_{ij}^{(d-1,l)} = \frac{(d+1)!}{(d-l)!},
$$

and the sum of all entries of each column is given by:

$$\sum_{0\le j\le d} h_{ij}^{(d-1,l)} = \frac{d!}{(d-l)!}, \quad 0\le i\le d.$$

The next simple lemma gives an explicit formula for $\mathfrak{H}_{d-1}^1$ which will serve as the induction base for the proof of monotonicity of the h-vector under partial barycentric subdivision in Corollary 4.5.

Lemma 4.2. *The entries of $\mathfrak{H}_{d-1}^1$ are given by:*

$$h_{ij}^{(d-1,1)} = \begin{cases} 0, & i=0, j\neq 0 \text{ or } i=d, j\neq d; \\ 2, & i=j=1,\dots,d-1; \\ 1, & \text{otherwise.} \end{cases}$$

and hence

$$\mathfrak{H}_{d-1}^1 = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 2 & 1 & \cdots & 1 & 1 \\ 1 & 1 & 2 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & 2 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

Proof. We prove the lemma by describing the entries of an arbitrary row. Let $(A_{d+1}(j,i,2))_{0\le i\le d} \in \mathbb{R}^{(d+1)}$ be the j^{th} row of $\mathfrak{H}_{d-1}^1$. Then by definition the entries $A_{d+1}(j,i,2)$ count the d injective words $w = w_d w_{d+1} \in S_{d+1}^1$ such that $w_{d+1} = d+1-j$ according to their number of descents. Let $w_d = d+1-j'$ and distinguish cases according to the relative size of j and j':

- $j < j' \le d$: Then there are $j'-1$ elements larger then w_d in $\mathrm{free}(w)$ and hence $\mathrm{des}(w_d w_{d+1}) = j'-1$.
- $0 \le j' < j$: Then there are j' elements larger than w_d in $\mathrm{free}(w)$ and there is a descent from w_d to w_{d+1}. Thus $\mathrm{des}(w_d w_{d+1}) = j'+1$.

As a consequence the entries of $(A_{d+1}(j,i,2))_{0\le i\le d}$ are determined:

- $j=0$: For $0\le i\le d-1$ there is a unique word with i descents and there is no word with d descents.
- $j=d$: For $1\le i\le d$ there is a unique word with i descents and there is no word with 0 descents.
- $1\le j\le d-1$: For $0\le i\le j-1$ and for $j+1\le i\le d$ there is a unique word with i descents. In addition, there are two words with j descents.

□

The examples above and the preceding lemma suggest some relations among the entries of $\mathfrak{H}_{d-1}^l$ that we verify in the next lemmas.

Lemma 4.3. *For $0\le i,j,l\le d$,*

$$A_{d+1}(j,i,l+1) = A_{d+1}(d-j,d-i,l+1).$$

Proof. Let us denote by $S^l_{d+1}(j,i)$ the set of injective words $w \in S^l_{d+1}$ such that $\mathrm{des}(w) = j$ and $w_{d+1} = d+1-i$. Thus $A_{d+1}(j,i,l) = \#S^l_{d+1}(j,i)$. To complete the proof it is enough to provide a bijection between $S^{l+1}_{d+1}(j,i)$ and $S^{l+1}_{d+1}(d-j,d-i)$. Let

$$\varphi : S^{l+1}_{d+1}(j,i) \to S^{l+1}_{d+1}(d-j,d-i)$$

be the map that sends $w = w_{d+1-l}\cdots w_{d+1} \in S^{l+1}_{d+1}(i,j)$ to

$$\varphi(w) := d+2-w_{d+1-l}\cdots d+2-w_{d+1}$$

Let $\mathrm{free}(w) = \{w_1 < \cdots < w_{d-l}\}$.

By definition $d+2-w_{d+1} = d+2-(d+1-j) = d+1-(d-j)$. Thus to show $\varphi(w) \in S^l_{d+1}(d-i,d-j)$ it remains to verify that the number of descents of $\varphi(w)$ is $d-i$.

We show that $m \in [d]$ is a descent of w if and only if m is not a descent of $\varphi(w)$. If $m \in [d-l]$ then $w_j > w_{d+1-l}$ if and only if $d+2-w_j < d+2-w_{d+1-l}$. Analogously, if $m \in \{d-l+1,\ldots,d\}$ then $w_m > w_{m+1}$ if and only if $d+2-w_m < d+2-w_{m+1}$.

Therefore, the number of descents of $\varphi(w)$ is $d-i$. This completes the proof since φ is clearly a bijection. □

Proposition 4.4. *For $0 \le i,j \le d$ and $1 \le l \le d$,*

$$A_{d+1}(j,i,l) \quad \le \quad A_{d+1}(j,i,l+1). \tag{5}$$

Proof. As before we denote by $S^l_{d+1}(j,i)$ the set of injective words $w \in S^l_{d+1}$ such that $\mathrm{des}(w) = j$ and $w_{d+1} = d+1-i$. In the sequel, by "$\widehat{\ }$" we mean that the entry below the hat is missing in the permutation. We construct a map

$$\psi : S^l_{d+1}(j,i) \to S^{l+1}_{d+1}(j,i)$$

as follows:

Let $w \in S^l_{d+1}(j,i)$ be an injective word for which p is the number of descents in the first $d+1-l$ positions and $j-p$ descents in the remaining positions for some $0 \le p \le j$. Let $w = w_{d+2-l}\cdots w_{d+1} \in S^l_{d+1}(j,i)$ and $\mathrm{free}(w) = \{w_1 < \cdots < w_{d-l} < w_{d+1-l}\}$ and $w_{d+1} = d+1-i$.

Since w has p descents in the first $d+1-l$ positions it follows that

$$w_1 < \cdots < w_{d+1-l-p} < w_{d+2-l} < w_{d+2-l-p} < \cdots < w_{d+1-l} \quad \text{if } p > 0 \tag{6}$$

$$w_1 < \cdots < w_{d-l} < w_{d+1-l} < w_{d+2-l} \quad \text{if } p = 0 \tag{7}$$

.

We define

$$\psi(w) := \begin{cases} w_{d+2-l-p}w_{d+2-l}w_{d+2-l+1}\cdots w_{d+1} & \text{if } p > 0 \\ w_{d+1-l}w_{d+2-l}w_{d+2-l+1} & \text{if } p = 0 \end{cases}$$

If $p > 0$ then $\psi(w)$ has by (6) $p-1$ descents in positions 1 to $d-l$ and $j-(p-1)$ in positions $d-l+1$ to $d+1$. If $p = 0$ then $\psi(w)$ has by (7) $p = 0$ descents in positions 1 to $d-l$ and j in positions $d-l+1$ to $d+1$. Since the last letter of $\psi(w)$ is $w_{d+1} = d+1-i$ it then follows that $\psi(w) \in S^{l+1}_{d+1}(j,i)$. It is easily checked that ψ is injective. Hence $\psi : S^l_{d+1}(j,i) \hookrightarrow S^{l+1}_{d+1}(j,i)$ which implies $A_{d+1}(j,i,l) \le A_{d+1}(j,i,l+1)$.

$\square$

As a consequence of Theorem 3.4 and Proposition 4.4 we can deduce a result on the growth of the h-vector under l^{th} partial barycentric subdivision.

Corollary 4.5. *Let Δ be a d-dimensional simplicial complex such that $h_i^{\Delta} \geq 0$ for all $0 \leq i \leq d$. Then*

(i) $h_i^{\Delta} \leq h_i^{\mathrm{sd}^l(\Delta)}$ *for $0 \leq l \leq d$ and $0 \leq i \leq d$,*

(ii) $h_i^{\Delta} < h_i^{\mathrm{sd}^l(\Delta)}$ *for $1 \leq l \leq d$ and $1 \leq i \leq d-1$,*

Proof. It is easy to see that $h_0^{\mathrm{sd}^l(\Delta)} = h_0^{\Delta}$ and $h_d^{\mathrm{sd}^l(\Delta)} = h_d^{\Delta}$, thus we are left with the case $1 \leq i \leq d-1$. Since by Theorem 3.4 $h_i^{\mathrm{sd}^l(\Delta)}$ is a non-negative linear combination of the h_j^{Δ} it suffices to show that the entries of the submatrix $(h_{ij}^{(d-1,l)})_{1 \leq i \leq d-1,\, 0 \leq j \leq d}$ are strictly positive. Again, by Equation (5) it is enough to consider the case $l = 1$. Now Lemma 4.2 completes the proof. $\square$

The consequence of the preceding corollary for the smaller class of Cohen-Macaulay simplicial complexes also follows from a very general result by Stanley [7, Theorem 4.10] using the fact that l^{th} partial barycentric subdivision is a quasi-geometric subdivision. Note that $h_i^{\Delta} \geq 0$ for Cohen-Macaulay simplicial complexes.

Let $\mathfrak{F}_{d-1}$ be the matrix of the transformation that sends the f-vector of Δ to the f-vector of $\mathrm{sd}(\Delta)$. We denote by $\mathfrak{F}_{d-1}^l$ the matrix of the transformation from the f-vector of Δ to the f-vector of $\mathrm{sd}^l(\Delta)$. Both matrices $\mathfrak{F}_{d-1}$ and $\mathfrak{F}_{d-1}^l$ are square matrices of order $d+1$, with $\mathfrak{F}_{d-1}^d = \mathfrak{F}_{d-1}$. By Theorem 3.1 the entries of $\mathfrak{F}_{d-1}^l = (f_{ij}^{(d-1,l)})_{0 \leq i,j \leq d}$ are given by $f_{ij}^{(d-1,l)} = \mathrm{poS}_d(j,i,l)$.

The following lemma shows that the matrices $\mathfrak{F}_{d-1}^l$ and $\mathfrak{H}_{d-1}^l$ are diagonalizable.

Proposition 4.6. *For $1 \leq l \leq d-1$:*

(1) *The matrices $\mathfrak{F}_{d-1}^l$ and $\mathfrak{H}_{d-1}^l$ are similar.*

(2) *The matrices $\mathfrak{F}_{d-1}^l$ and $\mathfrak{H}_{d-1}^l$ are diagonalizable with eigenvalues*

$$1, \frac{(d-l+1)!}{(d-l)!}, \ldots, \frac{d!}{(d-l)!}.$$

For $0 \leq l \leq d-1$ the eigenvalues $\frac{(d-l+1)!}{(d-l)!}, \ldots, \frac{d!}{(d-l)!}$ have multiplicity 1 and the eigenvalue 1 has multiplicity $d+1-l$. For $l = d$ the eigenvalues $2!, \ldots, d!$ have multiplicity 1 and the eigenvalue 1 has multiplicity 2.

Proof.

- Since by (1) and (2) the transformation sending the f-vector of a simplicial complex to the h-vector of a simplicial complex is an invertible linear transformation, the first assertion follows.
- Clearly, $\mathfrak{F}_{d-1}^l$ is an upper triangular matrix with diagonal entries

$$\underbrace{1, \ldots, 1}_{(d+1-l)-\text{times}}, \frac{(d-l+1)!}{(d-l)!}, \ldots, \frac{d!}{(d-l)!}.$$

Let $(\mathfrak{F}_{d-1}^l)^{\perp}$ be the transpose of $\mathfrak{F}_{d-1}^l$.

– $0 \le l \le d-1$: The first $(d-l+1)$ unit vectors are eigenvectors of $(\mathfrak{F}_{d-1}^{l})^{\perp}$ for the eigenvalue 1.
– $l = d$: The first 2 unit vectors are eigenvectors of $(\mathfrak{F}_{d-1}^{l})^{\perp}$ for the eigenvalue 1.

The eigenvalues $\frac{(d-l+1)!}{(d-l)!}, \ldots, \frac{d!}{(d-l)!}$ are pairwise different. This implies that $(\mathfrak{F}_{d-1}^{l})^{\perp}$ is diagonalizable. But then $\mathfrak{F}_{d-1}^{l}$ is diagonalizable.

□

Lemma 4.7. *Let $\nu = (\nu_0, \ldots, \nu_d)$ be an eigenvector of the matrix $\mathfrak{H}_{d-1}^{l}$ for the eigenvalue λ such that $\lambda \neq \frac{d!}{(d-l)!}$. Then $\sum_{i=0}^{d} \nu_i = 0$.*

Proof. Since $\mathfrak{H}_{d-1}^{l}\nu = \lambda\nu$ it follows that

$$(1, \ldots, 1)\mathfrak{H}_{d-1}^{l}\nu \;\; = \;\; (1, \ldots, 1)\lambda\nu.$$

But by Lemma 4.1, $(1, \ldots, 1)\mathfrak{H}_{d-1}^{l} = \frac{d!}{(d-l)!}(1, \ldots, 1)$. Therefore, either $\lambda = \frac{d!}{(d-l)!}$ or $\sum_{i=0}^{d} \nu_i = 0$. Since $\lambda \neq \frac{d!}{(d-l)!}$ we are done. □

Next we try to gain a better understanding of the eigenvectors of $\mathfrak{H}_{d-1}^{l}$.

Lemma 4.8. *Let $d \ge 2$ and $\nu_1^{(1)}, \ldots, \nu_1^{(d-l+1)}, \nu_{d-l+2}, \ldots, \nu_{d+1}$ be a basis of eigenvectors of the matrix $\mathfrak{F}_{d-1}^{l}$, where $\nu_1^{(1)}, \ldots, \nu_1^{(d-l+1)}$ are eigenvectors for the eigenvalue 1 and $\nu_{l+1}, \ldots, \nu_{d+1}$ are eigenvectors for the eigenvalues $\{\frac{(d-l+1)!}{(d-l)!}, \ldots, \frac{d!}{(d-l)!}\}$, respectively. Then $(\nu_1^{(1)}, 0), \ldots, (\nu_1^{(d-l+1)}, 0), (\nu_{l+1}, 0), \ldots, (\nu_d, 0)$ are eigenvectors of the matrix $\mathfrak{F}_{d}^{l}$ for the eigenvalues $\{\underbrace{1, \ldots, 1}_{(d-l+1) times}, \frac{(d-l+1)!}{(d-l)!}, \ldots, \frac{d!}{(d-l)!}\}$.*

Proof. Since both $\mathfrak{F}_{d-1}^{l}$ and $\mathfrak{F}_{d}^{l}$ are upper triangular matrices and $\mathfrak{F}_{d-1}^{l}$ is obtained by deleting the $(d+2)^{\text{nd}}$ column and row from $\mathfrak{F}_{d}^{l}$ the assertion follows. □

Let $\hat{\mathfrak{H}}_{d-1}^{l}$ be a matrix obtained be deleting the first and last rows and columns of $\mathfrak{H}_{d-1}^{l}$. Thus $\hat{\mathfrak{H}}_{d-1}^{l}$ is a $d-1$ by $d-1$ square matrix.

Lemma 4.9. *The matrix $\hat{\mathfrak{H}}_{d-1}^{l}$ is diagonalizable.*

Proof. By definition and Theorem 3.4 the first row of $\mathfrak{H}_{d-1}^{l}$ is the first unit vector and the last row of $\mathfrak{H}_{d-1}^{l}$ is the $(d+1)^{\text{st}}$ unit vector. Thus the characteristic polynomial of $\mathfrak{H}_{d-1}^{l}$ splits into $(1-t)^2$ times the characteristic polynomial of $\hat{\mathfrak{H}}_{d-1}^{l}$. Therefore, $\hat{\mathfrak{H}}_{d-1}^{l}$ has for $0 \le l \le d-2$ the eigenvalues

$$\underbrace{1, \ldots, 1}_{(d-l-1)-\text{times}}, \frac{(d-l+1)!}{(d-l)!}, \ldots, \frac{d!}{(d-l)!}.$$

and for $l = d-1, d$ the eigenvalues

$$2!, \ldots, d!.$$

To show that the matrix $\hat{\mathfrak{H}}^l_{d-1}$ is diagonalizable, it is enough to show that for $0 \leq l \leq d-2$ the eigenspace for the eigenvalue 1 is of dimension $d-l-1$.

Let $0 \leq l \leq d-2$. We again consider the full matrix $\mathfrak{H}^l_{d-1}$. Since $\mathfrak{H}^l_{d-1}$ is diagonalizable there is a basis $\omega_1^{(1)}, \ldots, \omega_1^{(d-l+1)}$, $\omega_{d-l+2}, \ldots, \omega_{d+1}$ of $\mathbb{R}^{d+1}$ consisting of eigenvectors of $\mathfrak{H}^l_{d-1}$. We can choose the numbering such that $\omega_1^{(i)}$, $1 \leq i \leq d-l+1$ are eigenvectors for the eigenvalue 1 and ω_j is an eigenvector for the eigenvalues $\frac{(d+1-j)!}{(d-l)!}$, $0 \leq j \leq l$, respectively.

Again, since the first and last row of $\mathfrak{H}^l_{d-1}$ are the first and $(d+1)^{\text{st}}$ unit vector we can choose the eigenvectors of $\mathfrak{H}^l_{d-1}$ for the eigenvalue $\lambda = 1$ as follows: $\omega_1^{(1)}$ and $\omega_1^{(2)}$ can be chosen such that

$$\omega_1^{(1)} = (1, k_{11}, \ldots, k_{1(d-1)}, 0) \text{ and } \omega_1^{(2)} = (0, k_{21}, \ldots, k_{2(d-1)}, 1),$$

and $\omega_1^{(i)}$ can be chosen such that $\omega_1^{(i)} = (0, k_{i1}, \ldots, k_{i(d-1)}, 0)$ for $3 \leq i \leq d+1-l$. Clearly, this implies that deleting the leading and trailing 0 form the $\omega_1^{(i)}$ for $3 \leq i \leq d+1-l$ yields eigenvectors $\hat{\omega_1}^{(i)} = (k_{i1}, \ldots, k_{i(d-1)})$ of $\hat{\mathfrak{H}}^l_{d-1}$ for the eigenvalue $\lambda = 1$. Obviously, the set of vectors $\{\hat{\omega_1}^{(3)}, \ldots, \hat{\omega_1}^{(l+1)}\}$ is linearly independent. Hence we have shown that the dimension of the eigenspace for the eigenvalue 1 of $\hat{\mathfrak{H}}^l_{d-1}$ is $d-l-1$. $\square$

The above lemma is a key ingredient in proving the following theorem.

Theorem 4.10. *Let $d \geq 2$ and let $\omega_1^{(1)}, \ldots, \omega_1^{(d-l+1)}, \omega_{d-l+2}, \ldots, \omega_{d+1}$ be a basis of eigenvectors of the matrix $\mathfrak{H}^l_{d-1}$, where $\omega_1^{(i)}, 1 \leq i \leq d-l+1$ are eigenvectors for the eigenvalue 1 and ω_j is an eigenvector for the eigenvalue $\frac{(j-1)!}{(d-l)!}$, for $d-l+2 \leq j \leq d$.*

1. *Let Δ be a $(d-1)$-dimensional simplicial complex. If we expand h-vector of Δ in terms of eigenvectors of the matrix $\mathfrak{H}^l_{d-1}$, the coefficient of the eigenvector for the eigenvalue $\frac{d!}{(d-l)!}$ is non-zero.*
2. *The first and the last coordinate entry in $\omega_1^{(3)}, \ldots, \omega_1^{(l+1)}, \omega_{l+1}, \ldots, \omega_d$ is zero.*
3. *The vectors $\omega_1^{(1)}$ and $\omega_1^{(2)}$ can be chosen such that*

$$\omega_1^{(1)} = (1, i_1, \ldots, i_{d-1}, 0) \text{ and } \omega_1^{(2)} = (0, j_1, \ldots, j_{d-1}, 1).$$

4. *The vector ω_{d+1} can be chosen such that $\omega_d = (0, b_1, \ldots, b_{d-1}, 0)$ for strictly positive rational numbers b_i, $1 \leq i \leq d-1$.*

Proof. Let us expand the f-vector of Δ in terms of a basis of eigenvectors of the matrix $\mathfrak{F}^l_{d-1}$. Since $f^\Delta_{d-1} \neq 0$ from Lemma 4.8 we deduce that the coefficient of the eigenvector for the highest eigenvalue is non-zero. Since $\mathfrak{F}^l_{d-1}$ and $\mathfrak{H}^l_{d-1}$ are similar so (1) follows.

Assertions (2) and (3) immediately follow from the proof of Lemma 4.9.

For (4) consider the matrix $\hat{\mathfrak{H}}^l_{d-1}$ as defined above. It is easily seen (and also follows Lemma 4.2 and Proposition 4.4) that the entries of $\hat{\mathfrak{H}}^l_{d-1}$ are strictly positive numbers. Therefore, by the Perron-Frobenius Theorem [5] it follows that there is an eigenvector $\hat{\omega}_d^l$ for the eigenvalue $\frac{d!}{(d-l)!}$ with strictly positive entries. Hence $(0, \hat{\omega}_d^l, 0)$ is the required eigenvector. $\square$

5. Open Problems

In this section we discuss a few open problems related to the above work.

Lemma 4.7 describes properties of the eigenvectors of the matrix $\mathfrak{H}_{d-1}^l$ for the eigenvalue λ such that $\lambda \neq \frac{d!}{(d-l)!}$. For the eigenvalue $\lambda = \frac{d!}{(d-l)!}$ we were able to deduce its non-negativity in Theorem 4.10 (4) but were not able to give more structural results or even provide an explicit description. By [2] when applying l^{th} partial barycentric subdivision iteratively the limiting behavior of the h-vector is determined by this eigenvector. Hence some information can be read off from [2]. Nevertheless complete information about that eigenvector would be desirable.

For example, for $d = 4$ we have following eigenvectors, corresponding to the eigenvalues $\frac{4!}{3!}, \frac{4!}{2!}, \frac{4!}{1!}$, for $l = 1, 2, 3$ respectively.

$$\begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ \frac{5}{3} \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ \frac{7}{2} \\ 1 \\ 0 \end{pmatrix}.$$

For $d = 5$ we have the following eigenvectors, corresponding to the eigenvalues $\frac{5!}{4!}, \frac{5!}{3!}, \frac{5!}{2!}, \frac{5!}{1!}$, for $l = 1, 2, 3, 4$ respectively.

$$\begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ \frac{12}{7} \\ \frac{12}{7} \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ \frac{46}{11} \\ \frac{46}{11} \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ \frac{17}{2} \\ \frac{17}{2} \\ 1 \\ 0 \end{pmatrix}.$$

Similarly, for $d = 6$ we have following eigenvectors, corresponding to the eigenvalues $\frac{6!}{5!}, \frac{6!}{4!}, \frac{6!}{3!}, \frac{6!}{2!}, \frac{6!}{1!}$, for $l = 1, 2, 3, 4, 5$ respectively.

$$\begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ \frac{7}{4} \\ \frac{7}{4} \\ \frac{7}{4} \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ \frac{1941}{437} \\ \frac{2146}{437} \\ \frac{1941}{437} \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ \frac{5431}{527} \\ \frac{8906}{527} \\ \frac{5431}{527} \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ \frac{586}{33} \\ \frac{5459}{132} \\ \frac{586}{33} \\ 1 \\ 0 \end{pmatrix}.$$

Thus the following problem appears to be interesting.

Problem 5.1. *Give a description of eigenvectors of the matrices $\mathfrak{F}_{d-1}^l$ and $\mathfrak{H}_{d-1}^l$ for the eigenvalue $\frac{d!}{(d-l)!}$.*

The h-polynomial $h(\mathrm{sd}(\Delta_{d-1}), x) = \sum_{i=0}^{d} h_i^{\mathrm{sd}(\Delta)} x^{d-i}$ of the barycentric subdivision of Δ_{d-1} has the following combinatorial interpretation.

$$h(\mathrm{sd}(\Delta_{d-1}), x) = \sum_{w \in S_d} x^{\mathrm{des}(w)} = \sum_{w \in S_d} x^{\mathrm{ex}(w)}, \tag{8}$$

where $\mathrm{ex}(w)$ denotes the number of *excedances* of $w = w_1 \cdots w_d$, defined by

$$\mathrm{ex}(w) = \#\{i \mid w_i > i\},$$

The first equality follows from [8, Theorem 3.13.1] (it is also a consequence of [1, Thm 1] and Theorem 3.4), and the second is a consequence of [8, Proposition 1.4.3]. In [7], the *local h-polynomial* $\ell_V(\Gamma, x)$ of an arbitrary subdivision (subject to mild conditions) Γ of Δ_{d-1} has been defined. For $\Gamma = \mathrm{sd}(\Delta_{d-1})$ it is given as:

$$\ell_V(\mathrm{sd}(\Delta_{d-1}), x) = \sum_{w \in \mathrm{Der}_d} x^{\mathrm{ex}(w)}, \tag{9}$$

where Der_d denotes the set of all derangements in S_d. We suggest the following:

Problem 5.2. *Give an interpretation of local h-polynomial for the l^{th} partial barycentric subdivision similar to* (9) *in terms of a suitably defined l-excedance statistic on a newly defined set of l-derangements satisfying an analog of* (8).

To approach the problem it seems useful to find a statistic on S_d^l fulfilling a statement analogous to (8). Already this task appears to be hard and challenging.

Problem 5.3. *Define an l-excedance statistic on S_d^l such that the l-excedance and descent statistic on S_d^l are equally distributed; i.e. satisfy an analog of* (9).

For Problem 5.3, we tried different approaches. Despite not yielding a solution to the problem the following idea resulted in some interesting data. We define an injective map say $\chi : S_d^l \to S_d$ in the following way. Let $w \in S_d^l$ such that $w = w_{d-l+1} \cdots w_d$ and $\mathrm{free}(w) = \{w_1 < \cdots < w_{d-l}\}$. Then:

$$\chi(w) = \begin{cases} w_1 \cdots w_{d-1} w_{d-l+1} \cdots w_d, & \text{if } w_{d-l+1} > w_{d-l} \\ \\ w_{d-l-p+1} \cdots w_{d-l} w_1 \cdots w_{d-l-p} w_{d-l+1} \cdots w_d, & \text{if } w_{d-l-p+1} > w_{d-l+1} \\ & \text{and } w_{d-l-p} < w_{d-l+1}. \end{cases}$$

Now define the number of excedances $\mathrm{ex}(w)$ of $w \in S_d^l$ to be number of usual excedances of $\chi(w)$, i.e.

$$\mathrm{ex}(w) := \#\{i \mid \chi(\omega)(i) > i\}.$$

We apply this definition for different values of d and l. For a fixed d, the descent and excedance statistic are equally distributed on S_d^l for $l = 1$ and $l = 2$. But for other values of l the two statistics appear to be different. Nevertheless, the obtained data has some surprising and unexplained symmetry. For example, for S_5^l we have following tables for the number of descents,

# of descents =		$l =$ 4	3	2
	0	1	1	1
	1	1	6	16
	2	1	6	26
	3	1	6	16
	4	1	1	1

and the following table for the number of l-excedances.

# of l-excedances =		$l =$ 4	3	2
	0	1	1	1
	1	1	6	14
	2	1	6	30
	3	1	6	14
	4	1	1	1

Similarly, for S_6^l the number of descents are shown in the following table,

# of descents =		$l =$ 5	4	3	2
	0	1	1	1	1
	1	1	7	22	42
	2	1	7	37	137
	3	1	7	37	137
	4	1	7	22	42
	5	1	1	1	1

and the number of l-excedances are shown in the following table.

# of l-excedances =		$l =$ 5	4	3	2
	0	1	1	1	1
	1	1	7	17	33
	2	1	7	42	146
	3	1	7	42	146
	4	1	7	17	33
	5	1	1	1	1

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References

[1] F. Brenti, V. Welker, *f-vectors of barycentric subdivisions*, Math. Z. **259** (2008) 849-865.

[2] E. Delucchi, L. Sabalka, A. Pixton, f-polynomials of subdivisions of complexes Discrete Math. **312** (2012) 248–257.

[3] R. Ehrenborg, J.Y. Jung, *The topology of restricted partition posets*, Disc. Math. and Th. Comp. Sci., Proceedings, 23rd International Conference on Formal Power Series and Algebraic Combinatorics, (2011) 281-292.
[4] S.R. Gal, *Real root conjecture fails for five- and higher-dimensional spheres*, Discrete Comput. Geom. **34** (2005), 269–284.
[5] R. A. Horn, C. R. Johnson, *Matrix Analysis*, Cambridge University Press, Cambridge (1985).
[6] E. Nevo, T.K. Petersen, B. Tenner, *The γ-vector of a barycentric subdivision*, J. Combin. Theory Ser. A **118** (2011), 1364-1380.
[7] R. P. Stanley, *Subdivision and local h-vectors*, J. Amer. Math. Soc. **5** (1992) 805-851.
[8] R. P. Stanley, *Enumerative Combinatorics*, Vol. 1, Second edition, Cambridge University Press, 2012.

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