WHICH PROPERTIES OF STANLEY-REISNER RINGS AND SIMPLICIAL COMPLEXES ARE TOPOLOGICAL ?

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Abstract. In this article we survey results on topological properties of simplicial complexes Δ, mostly defined via algebraic properties of the Stanley-Reisner ring $\mathbb{K}[\Delta]$. A property of Δ or $\mathbb{K}[\Delta]$ is called topological if it only depends on the homeomorphism type of the geometric realization of Δ (and on $\mathbb{K}$).

1. Introduction

A (finite abstract) simplicial complex Δ is a subset of the power set 2^Ω for some finite non-empty groundset Ω such that $A \subseteq B \in \Delta$ implies $A \in \Delta$. All simplicial complexes will be non-empty. The simplicial complex $\{\emptyset\}$ is allowed. We call an $F \in \Delta$ a face of Δ and an inclusionwise maximal face a facet of Δ. We also write $\bar{F}$ for the simplicial complex 2^F and $\partial\bar{F}$ for the simplicial complex $\bar{F} \setminus \{F\}$.

Let $\mathbb{K}$ be a field and $S_\Omega = \mathbb{K}[x_\omega \,:\, \omega \in \Omega]$ be a polynomial ring over $\mathbb{K}$. For a subset $A \subseteq \Omega$ we write $\mathbf{x}_A$ for $\prod_{\omega\in A} \mathbf{x}_\omega$. The Stanley-Reisner ring or face ring $\mathbb{K}[\Delta]$ of Δ is the quotient S_Ω/I_Δ of S_Ω by the Stanley-Reisner ideal $I_\Delta = (\mathbf{x}_A \,:\, A \notin \Delta, A \subseteq \Omega)$. The set of monomials $\mathbf{x}_N$ for (inclusionwise) minimal non-faces N of Δ is a minimal monomial generating set of Δ.

Relabeling the vertices of Δ preserves the isomorphism type of $\mathbb{K}[\Delta]$. Hence ring theoretic properties and invariants of $\mathbb{K}[\Delta]$ are determined by the combinatorics of Δ and by $\mathbb{K}$. In this survey we will focus on properties and invariants of $\mathbb{K}[\Delta]$ and Δ determined by the topology of the geometric realization of Δ (and the field $\mathbb{K}$).

Basic algebraic topology (see e.g. [9]) teaches us that every simplicial complex comes with a topological space which is called its geometric realization. Recall, that for the definition one chooses points $p_\omega \in \mathbb{R}^d$ for some d, such that for $F \in \Delta$ the p_ω, $\omega \in F$, are affinely independent and $\mathrm{conv}(F) \cap \mathrm{conv}(F') = \mathrm{conv}(F \cap F')$ for $F, F' \in \Delta$. Here for $F \in \Delta$ we denote by $\mathrm{conv}(F)$ the geometric $(\#F-1)$-simplex which is the set of all convex combinations $\sum_{\omega\in F} \lambda_\omega p_\omega$ for $\lambda_\omega \geq 0$, $\omega \in F$, and $\sum_{\omega\in F} \lambda_\omega = 1$. Then $|\Delta| = \bigcup_{F\in\Delta} \mathrm{conv}(F)$ considered as a subspace of $\mathbb{R}^d$ is a geometric realization of Δ. From algebraic topology we know that all geometric realizations are homeomorphic. Given a geometric realization $|\Delta|$ of Δ we write $|\bar{F}|$ for the subspace $\mathrm{conv}(F)$ of $|\Delta|$.

Clearly, the combinatorics of two simplicial complexes with homeomorphic geometric realization can be quite different. Nevertheless, there are surprising results demonstrating that not few properties of Δ or ring theoretic invariants and properties of $\mathbb{K}[\Delta]$ depend only on $\mathbb{K}$ and the homeomorphism type of $|\Delta|$. These are usually called topological invariants or topological properties of Δ or $\mathbb{K}[\Delta]$.

In this article we survey properties and invariants that are topological and give counterexamples for some others. We do not claim completeness but we do our best to at least mention as many related results as possible. We also try to give an overview of the methods from topological combinatorics used in the proofs. For that reason we for example provide two proof of Munkres' result on the toplogical invariance of depth Theorem 3.4. This result is also the first result on topological invariance known to us. We assume the reader to be familiar with basic algebraic topology (see e.g., [9]) and some methods from topological combinatorics (see e.g., [2] or [20]). When proofs use heavy machinery from commutative algebra we will confine ourselves to a brief outline of the proof and references. For definition and facts from commutative algebra used but not defined in the paper we refer the reader to [7].

2. Dimension

In this section we study the Krull dimension of $\mathbb{K}[\Delta]$. For that we need to consider $\mathbb{K}[\Delta]$ as a standard graded $\mathbb{K}$-algebra. As a $\mathbb{K}$-vectorspace we have $\mathbb{K}[\Delta] = \bigoplus_{r=0}^{\infty} A_r$ where A_r is the $\mathbb{K}$-vectorspace of cosets $m + I_\Delta$ of monomials m in S_Ω of degree r. Now by $A_0 = \mathbb{K}$, $A_r A_s \subseteq A_{r+s}$ and the fact that $\mathbb{K}[\Delta]$ is generated by A_1 as a $\mathbb{K}$-algebra it follows, that $\mathbb{K}[\Delta]$ is a standard graded algebra.

Before we can demonstrate that the Krull dimension is a topological invariant we need to introduce some combinatorial invariants of simplicial complexes and relate them to the dimensions of the vectorspaces A_r, $r \geq 0$.

Recall that the dimension of a face F of Δ is given by $\dim(\Delta) = \#F - 1$. We write $\dim(\Delta) = \max_{F\in\Delta} \dim(F)$ for the dimension of Δ and set $f_i = \#\{F \in \Delta \ : \ \dim(F) = i\}$ for all $i \geq -1$. The f-vector of Δ is the vector $\mathfrak{f}^\Delta = (f_{-1}, \ldots, f_{\dim(\Delta)})$ whose entries are the nonzero f_i.

We now show how the f-vector of a simplicial complex determines the Hilbert-series of $\mathbb{K}[\Delta]$. Recall that the Hilbert-series of $\mathbb{K}[\Delta]$ is $\mathrm{Hilb}(\mathbb{K}[\Delta]) = \sum_{r=0}^{\infty} \dim_{\mathbb{K}}(A_r) t^r$, where $\dim_{\mathbb{K}}(A_r)$ denotes the $\mathbb{K}$-Vectorspace dimension of A_r. It is well (see [7, Exercise 10.11]) known that the Hilbert-series of any standard graded $\mathbb{K}$-algebra is a rational function of the form $\frac{h(t)}{(1-t)^d}$ where $d = \dim(\mathbb{K}[\Delta])$ is the Krull-dimension of $\mathbb{K}[\Delta]$ and $h(t)$ a polynomial with $h(1) \neq 0$.

Theorem 2.1. *Let Δ be a simplicial complex with f-vector $\mathfrak{f} = (f_{-1}, \ldots, f_{\dim(\Delta)})$ then*

$$\mathrm{Hilb}(\mathbb{K}[\Delta]) = \frac{\displaystyle\sum_{i=0}^{\dim(\Delta)+1} t^i \, (1-t)^{\dim(\Delta)+1-i} \, f_{i-1}}{(1-t)^{\dim(\Delta)+1}}.$$

In particular, $\dim(\mathbb{K}[\Delta]) = \dim(\Delta) + 1$.

Proof. Since I_Δ is an ideal generated by monomials, it follows that a polynomial from $\mathbb{K}[\Delta]$ lies in I_Δ if and only if each monomial with non-zero coefficient in the polynomial lies in

I_Δ. Thus the cosets $m + I_\Delta$ of the degree r monomials $m \notin I_\Delta$ form a basis of A_r. Now $m+I_\Delta = I_\Delta$ if and only if m is divisible by $\mathbf{x}_N$ for a minimal non-face N. Thus $m+I_\Delta \neq I_\Delta$ if and only if the support $\mathrm{supp}(m) = \{\omega \ : \ x_\omega \ \text{ divides } m\}$ of m lies in Δ. If $i \geq 0$ then for each i-dimensional face $F \in \Delta$ there are $\binom{r-1}{i}$ monomials of degree $r-(i+1)$ in the variables x_ω, $\omega \in F$. If $i = -1$ the unique (-1)-dimensional face $\emptyset$ of Δ corresponds to monomials with empty support and hence contributes only the unique basis element of A_0. It follows that for $r \geq 0$

$$\dim_{\mathbb{K}}(A_r) = \sum_{i=0}^{r-1} \binom{r-1}{i} f_i = \sum_{i=0}^{\infty} \binom{r-1}{i} f_i$$

for arbitrary choices of f_i when $i > \dim(\Delta)$. It follows that

$$\begin{aligned}
\mathrm{Hilb}(\mathbb{K}[\Delta]) &= f_{-1} + \sum_{r=1}^{\infty} \Big(\sum_{i=0}^{\infty} \binom{r-1}{i} f_i \Big) t^i \\
&= f_{-1} + \sum_{i=0}^{\infty} \Big(\sum_{r=1}^{\infty} \binom{r-1}{i} t^r \Big) f_i \\
&= f_{-1} + \sum_{i=0}^{\infty} \frac{t^{i+1}}{(1-t)^{i+1}} f_i \\
&= \frac{\displaystyle\sum_{i=0}^{\dim(\Delta)+1} t^i \, (1-t)^{\dim(\Delta)+1-i} \, f_{i-1}}{(1-t)^{\dim(\Delta)+1}}.
\end{aligned}$$

In the representation of the Hilbert-series as a rational function the numerator polynomial evaluates to $f_{\dim(\Delta)} \neq 0$ at $t = 1$. Thus the Krull dimension of $\mathbb{K}[\Delta]$ is given by the power of $(1-t)$ in the denominator and hence is $\dim(\Delta) + 1$. □

In particular, we see that proving the topological invariance of the Krull dimension of $\mathbb{K}[\Delta]$ and the dimension of Δ is equivalent. Before we deduce the topological invariance of both dimensions, we prove the following lemma. It will serve as the key argument in the proof of the invariance, which could also be deduced by much simpler means. But the lemma will prove to be useful later in more complicated situations. We will use the following notation. We write $\mathrm{link}_\Delta(F) = \{G \in \Delta \ : \ G \cap F = \emptyset, G \cup F \in \Delta\}$ for the link of F in Δ and $\mathrm{star}_\Delta(F) = \{G \in \Delta \ : \ F \cup G \in \Delta\}$ for the (closed) star of F in Δ. For two simplicial complexes Δ and Δ' on disjoint ground sets we denote by $\Delta * \Delta' = \{F \cup F' \ : \ F \in \Delta, F' \in \Delta'\}$ the join of Δ and Δ'. Using the textbook definition (see [9, p.9] of the join operation, we have that the join of the topological spaces $|\Delta| * |\Delta'|$ and $|\Delta * \Delta'|$ are homeomorphic if Δ, $\Delta' \neq \{\emptyset\}$. In case we (for example) have $\Delta = \{\emptyset\}$ the textbook definition implies $|\Delta| * |\Delta'| = \emptyset$ and $|\Delta * \Delta'| = |\Delta'|$. In order to be avoid case distinctions we set $|\Delta| * |\Delta'| = |\Delta'|$ in this case and proceed analogously in case $\Delta' = \{\emptyset\}$. Note that $\mathrm{star}_\Delta(F) = \bar{F} * \mathrm{link}_\Delta(F)$ and hence $|\mathrm{star}_\Delta(F)| = |\bar{F}| * |\mathrm{link}_\Delta(F)|$. For a face F of Δ we write $\Delta \setminus F$ for the simplicial complex $\{G \in \Delta \ : \ F \not\subseteq G\}$ and for a point x

in $|\Delta|$ we write $|\Delta| - x$ for $|\Delta| \setminus \{x\}$. For a simplicial complex Δ we write $\widetilde{H}_i(\Delta, \mathbb{K})$ for the ith reduced simplicial homology groups of Δ with coefficients in $\mathbb{K}$ and for a space X we write $\widetilde{H}_i(X, \mathbb{K})$ for the ith reduced singular homology group of X with coefficients in $\mathbb{K}$. Of course it is well known that $\widetilde{H}_i(\Delta, \mathbb{K}) = \widetilde{H}_i(|\Delta|, \mathbb{K})$. For two simplicial complexes $\Gamma \subseteq \Delta$ we write $H_i(\Delta, \Gamma, \mathbb{K})$ for the simplicial homology of the pair (Δ, Γ) with coefficients in $\mathbb{K}$ and $H_i(X, A, \mathbb{K})$ for the singular homology with coefficients in $\mathbb{K}$ of a pair (X, A) of topological spaces.

Lemma 2.2. *Let Δ be a simplicial complex, F a face of Δ and x a point in the relative interior of $|\bar{F}|$. Then $|\Delta \setminus F|$ is a deformation retract of $|\Delta| - x$ and*

$$H_j(|\Delta|, |\Delta| - x, \mathbb{K}) = \widetilde{H}_{j-\dim(F)-1}(\mathrm{link}_\Delta(F), \mathbb{K}). \tag{1}$$

In particular, we have that

$$\dim(\Delta) = \max\Big\{ j \;:\; \textit{exists } x \in |\Delta| \textit{ such that } H_j(|\Delta|, |\Delta| - x, \mathbb{K}) \neq 0 \Big\}. \tag{2}$$

Proof. Assume our geometric realization is given by points $p_\omega \in \mathbb{R}^d$, $\omega \in \Omega$. Since x is from the relative interior of $|\bar{F}|$ it follows that $x = \sum_{\omega \in F} \lambda_\omega p_\omega$ with all $\lambda_\omega > 0$ and $\sum_{\omega \in F} p_\omega = 1$. Let $y = \sum_{\omega \in \Omega} \mu_\omega p_\omega \in |\Delta|$ given as a convex combination with $\{\omega \;:\; \mu_\omega > 0\} \in \Delta$. The from the fact that each barycentric coordinate defines a continuous map on $|\Delta|$ it follows that $f : y \mapsto \min_{\omega \in F} \frac{\mu_\omega}{\lambda_\omega}$ is a continuous map on $|\Delta|$. Clearly $0 \leq f(y) \leq 1$, $f(y) = 1$ if and only if $y = x$ and $f(y) = 0$ if and only if $y \in |\Delta \setminus F|$. Define the map $g : |\Delta| - x \to |\Delta \setminus F|$ as follows. For $y \in |\Delta| - x$ set $g(y) = \frac{1}{1-f(y)}(y - f(y)x)$. One easily checks that $g(y) \in |\Delta \setminus F|$ and $g(y) = y$ for $y \in |\Delta \setminus F|$. Continuity follows from the continuity of f. Now the standard interpolation between f and the identity of $|\Delta|$ shows the claim (see [12, Lemma 2.2] for detailed calculations).

By excising $|\Delta| - |\mathrm{star}_\Delta(F)|$ we get

$$H_j(|\Delta|, |\Delta| - x, \mathbb{K}) = H_j(|\mathrm{star}_\Delta(F)|, |\mathrm{star}_\Delta(F)| - x, \mathbb{K}).$$

Since $\mathrm{star}_\Delta(F)$ is contractible, it is acyclic. Thus by the long exact sequence in reduced homology we get that $H_j(|\mathrm{star}_\Delta(F)|, |\mathrm{star}_\Delta(F)| - x, \mathbb{K}) = \widetilde{H}_{j-1}(|\mathrm{star}_\Delta(F)| - x, \mathbb{K})$. Since $\mathrm{star}_\Delta(F) \setminus F = \partial\bar{F} * \mathrm{link}_\Delta(F)$ we know from the first part that $|\partial\bar{F}| * |\mathrm{link}_\Delta(F)|$ is a deformation retract of $|\mathrm{star}_\Delta(F)| - x$. Thus

$$\widetilde{H}_{j-1}(|\mathrm{star}_\Delta(F)| - x, \mathbb{K}) = \widetilde{H}_{j-1}(|\partial\bar{F})| * |\mathrm{link}_\Delta(F)|, \mathbb{K}).$$

Now $\partial\bar{F}$ is the boundary of an $\dim(F)$-simplex and hence a triangulation of an $(\dim(F)-1)$-sphere. From

$$\begin{aligned}
\widetilde{H}_{j-1}(|\partial\bar{F})| * |\mathrm{link}_\Delta(F)|, \mathbb{K}) &= \widetilde{H}_{j-1-(\dim(F)-1+1)}(|\mathrm{link}_\Delta(F)|, \mathbb{K}) \\
&= \widetilde{H}_{j-\dim(F)-1}(|\mathrm{link}_\Delta(F)|, \mathbb{K}) \\
&= \widetilde{H}_{j-\dim(F)-1}(\mathrm{link}_\Delta(F), \mathbb{K})
\end{aligned}$$

we now deduce (1).

For (2) consider the following argumentation. Let F be a face of Δ. Pick a point x in the relative interior of $|\bar{F}|$. If F is a facet of dimension $\dim(F) = \dim(\Delta)$. It follows that $\mathrm{link}_\Delta(F) = \{\emptyset\}$. From (2) we deduce $H_j(|\Delta|, |\Delta| - x, \mathbb{K}) \neq 0$ if and only if $j = \dim(F) = \dim(\Delta)$. For an arbitrary face F of Δ the we deduce from $\dim(\mathrm{link}_\Delta(F)) = \dim(\Delta) - \dim(F) - 1$ that $\widetilde{H}_{j-\dim(F)-1}(\mathrm{link}_\Delta(F), \mathbb{K}) = 0$ for $j > \dim(\Delta)$. This implies (2) □

We can now deduce the topological invariance of dimension and Krull dimension.

Theorem 2.3. *Let Δ and Δ' are simplicial complexes such that $|\Delta|$ and $|\Delta'|$ are homeomorphic. Then the Krull dimensions of $\mathbb{K}[\Delta]$ (resp. the dimensions of Δ) and of $\mathbb{K}[\Delta']$ (resp. Δ') coincide.*

Proof. By Theorem 2.1 it suffices to argue that for two simplicial complexes Δ and Δ' with homeomorphic geometric realizations we have $\dim(\Delta) = \dim(\Delta')$.

From the facts that $|\Delta|$ and $|\Delta'|$ are homeomorphic and that homeomorphic spaces have isomorphic homology it follows that:

$$\begin{aligned}
\dim(\Delta) &\overset{(2)}{=} \max\Big\{j \ : \ \text{exists } x \in |\Delta| \text{ such that } H_j(|\Delta|, |\Delta| - x, \mathbb{K}) \neq 0\Big\} \\
&\overset{|\Delta|\cong|\Delta'|}{=} \max\Big\{j \ : \ \text{exists } x \in |\Delta'| \text{ such that } H_j(|\Delta'|, |\Delta'| - x, \mathbb{K}) \neq 0\Big\} \\
&= \dim(\Delta')
\end{aligned}$$

□

The last property which we study in this section is the purity condition. A simplicial complex Δ is called pure if all facets have the same dimension.

Theorem 2.4. *Let Δ and Δ' simplicial complexes such that $|\Delta|$ and $|\Delta'|$ are homeomorphic. Then Δ is pure if and only if Δ' is pure.*

Proof. From Lemma 2.2 we know that for a point x from the relative interior of $|\bar{F}|$ for a face F of Δ we have that

$$H_j(|\Delta|, |\Delta| - x, \mathbb{K}) = \widetilde{H}_{j-\dim(F)-1}(\mathrm{link}_\Delta(F), \mathbb{K}). \tag{3}$$

On the right hand side there can only be a non-zero contribution if $j \geq \dim(F)$. Moreover, there is a non-trivial contribution for $j = \dim(F)$ if and only if F is a facet. Assume Δ is pure and F is a face of Δ. Then there is a facet G of dimension $\dim(\Delta)$ such that $F \subseteq G$. Thus for any x in the relative interior of $|\bar{F}|$ and every open neighborhood $x \in U \subseteq |\Delta|$ there is a $y \in U$ such that y is in the relative interior of $|\bar{G}|$. In particular, for every x in the relative interior of $|\bar{F}|$ and every open neighborhood U of x in $|\Delta|$ there is a $y \in U$ such that $H_{\dim(\Delta)}(|\Delta|, |\Delta| - x, \mathbb{K}) = \mathbb{K}$. Assume Δ' is not pure then there is a face G of dimension $< \dim(\Delta)$. But then for every x form the relative interior of $|\bar{G}|$ there is a small neighbourhood which only contains points y from $|\bar{G}|$. For them $H_{\dim(\Delta)}(|\Delta|, |\Delta| - y, \mathbb{K}) = \widetilde{H}_{\dim(\Delta)-\dim(G)-1}(\mathrm{link}_\Delta(G), \mathbb{K}) = 0$ as $\mathrm{link}_\Delta(G) = \{\emptyset\}$. □

3. Minimal free resolution and depth

In this section we review results from [13] which show the topological invariance of the depth of $\mathbb{K}[\Delta]$ using a formula by Hochster for the Betti-number of its free resolution. Recall that the depth $\mathrm{depth}(\mathbb{K}[\Delta])$ is the maximal number d of elements $f_1, \ldots, f_d \in \mathbb{K}[\Delta]$ such that f_i is a non-zerodivisor on $\mathbb{K}[\Delta]/(f_1, \ldots, f_{i-1})$ for $i = 1, \ldots, d$. We follow Munkres' approach and study this invariant through its relation to minimal free resolutions. In the next paragraphs we review some basic material on minimal free resolutions. In particular, We will easily see that the minimal free resolution as a whole is far from being a topological invariant.

A free resolution of $\mathbb{K}[\Delta]$ over S_Ω is an exact sequence:

$$\mathfrak{F} : \cdots \xrightarrow{\partial_{i+1}} S_\Omega^{b_i} \xrightarrow{\partial_i} \cdots \xrightarrow{\partial_2} S_\Omega^{b_1} \xrightarrow{\partial_1} S_\Omega^{b_0} \xrightarrow{\partial_0} \mathbb{K}[\Delta] \to 0$$

where all maps are S_Ω-module homomorphisms. It is well known that there is a free resolution which minimizes all the b_i simultaneously and has $b_i = 0$ for $i > |\Omega|$. This resolution is unique up to isomorphism and is called the minimal free resolution of $\mathbb{K}[\Delta]$ over S_Ω and the corresponding b_i are called the Betti-numbers of $\mathbb{K}[\Delta]$ as an S_Ω-module. We will write $\beta_i(\mathbb{K}[\Delta])$ or β_i for these b_i.

For our purposes we need a more refined structure of the free resolution. For that we use the multigraded structure of S_Ω which is inherited by $\mathbb{K}[\Delta]$. For a monomial $\prod_{\omega\in\Omega} x_\omega^{\alpha_\omega}$ we call $(\alpha_\omega)_{\omega\in\Omega}$ its multidegree. For $\alpha = (\alpha_\omega)_{\omega\in\Omega} \in \mathbb{N}^\Omega$ we write $\mathbf{x}^\alpha$ for $\prod_{\omega\in\Omega} x_\omega^{\alpha_\omega}$. Then as vectorspaces

$$S_\Omega = \bigoplus_{\alpha\in\mathbb{N}^\Omega} x_\omega^\alpha \mathbb{K}$$

and

$$\mathbb{K}[\Delta] = \bigoplus_{\alpha\in\mathbb{N}^\Omega} A_\alpha$$

where $A_\alpha = 0$ if $\alpha \neq (0)_{\omega\in\Omega}$ and $\mathbf{x}^\alpha \in I_\Delta$ and $\mathbf{x}^\alpha + I_\Delta$ otherwise. We can speak of the scalar multiples of x^α in S_Ω as the α-graded part of S_Ω and of A_α as the α-graded part of $\mathbb{K}[\Delta]$. For $\alpha \in \mathbb{N}^\Omega$ we write $S_\Omega(-\alpha)$ to denote the multigrading on S_Ω where the multiples $\mathbf{x}^{\alpha'}$ form the $\alpha' + \alpha$ graded part. Clearly, $S_\Omega(-\alpha)$ is an $\mathbb{N}^\Omega$-graded S_Ω-module. A muligraded free resolution of $\mathbb{K}[\Delta]$ over S_Ω is an exact sequence:

$$\mathfrak{F} : \cdots \xrightarrow{\partial_{i+1}} \bigoplus_{\alpha\in\mathbb{N}^\Omega} S_\Omega(-\alpha)^{b_{i,\alpha}} \xrightarrow{\partial_i} \cdots \xrightarrow{\partial_2} \bigoplus_{\alpha\in\mathbb{N}^\Omega} S_\Omega(-\alpha)^{b_{1,\alpha}} \xrightarrow{\partial_1} \bigoplus_{\alpha\in\mathbb{N}^\Omega} S_\omega(-\alpha)^{b_{0,\alpha}} \xrightarrow{\partial_0} \mathbb{K}[\Delta] \to 0$$

where all maps are multigraded S_Ω-module homomorphisms. Again it is well known that there is a free resolution which minimizes all the $b_{i,\alpha}$ simultaneously and which satisfies $b_{i,\alpha} = 0$ for $i > |\Omega|$. This resolution is unique up to multigraded isomorphism and is called the multigraded minimal free resolution of $\mathbb{K}[\Delta]$ over S_Ω and the corresponding $b_{i,\alpha}$ are called the multigraded Betti-numbers of $\mathbb{K}[\Delta]$ as an S_Ω-module. We will write $\beta_{i,\alpha}(\mathbb{K}[\Delta])$ or $\beta_{i,\alpha}$ for these $b_{i,\alpha}$.

It is also well known that $\beta_{i,\alpha} = 0$ unless $\alpha \in \{0,1\}^\Omega$. We can identify $\alpha \in \{0,1\}^\Omega$ with the set W of all ω with $\alpha_\omega = 1$. We then write $\beta_{i,W}$ for $\beta_{i,\alpha}$ (resp. $\beta_{i,W}(\mathbb{K}[\Delta])$ for $\beta_{i,\alpha}(\mathbb{K}[\Delta])$.

The connection between the structure of the minimal free resolution of $\mathbb{K}[\Delta]$ and the geometry of Δ is provided through the following formula by Hochster. For its formulation we denote for $W \subseteq \Omega$ by $\Delta_W = \{F \in \Delta \ : \ F \subseteq W\}$ the restriction of Δ to W.

Theorem 3.1 (Hochster formula [10])**.** *Let Δ be a simplicial complex over ground set Ω and let $W \subseteq \Omega$. Then for $i \geq 0$ the multigraded Betti-number $\beta_{i,W}(\mathbb{K}[\Delta])$ is given as*

$$\beta_{i,W}(\mathbb{K}[\Delta]) = \dim_{\mathbb{K}} \left(\widetilde{H}_{\#W-i-1}(\Delta_W, \mathbb{K}) \right).$$

The following is an immediate corollary.

Corollary 3.2. *Let Δ and Δ' be simplicial complexes over Ω and Ω' respectively. If $|\Delta|$ and $|\Delta'|$ are homotopy equivalent then $\beta_{i+\#\Omega,\Omega}(\mathbb{K}[\Delta]) = \beta_{i+\#\Omega',\Omega'}(\mathbb{K}[\Delta'])$ for all $i \geq 0$.*

Proof. By Theorem 3.1 we have

$$\begin{aligned}
\beta_{i+\#\Omega,\Omega}(\mathbb{K}[\Delta]) &= \dim_{\mathbb{K}} \left(\widetilde{H}_{\#\Omega-i-\#\Omega-1}(\Delta_\Omega, \mathbb{K}) \right) \\
&= \dim_{\mathbb{K}} \left(\widetilde{H}_{i-1}(\Delta_\Omega, \mathbb{K}) \right) \\
&= \dim_{\mathbb{K}} \left(\widetilde{H}_{i-1}(|\Delta|, \mathbb{K}) \right) \\
&= \dim_{\mathbb{K}} \left(\widetilde{H}_{i-1}(|\Delta'|, \mathbb{K}) \right) \\
&= \dim_{\mathbb{K}} \left(\widetilde{H}_{i-1}(\Delta'_{\Omega'}, \mathbb{K}) \right) \\
&= \dim_{\mathbb{K}} (\widetilde{H}_{\#\Omega'-i-\#\Omega'-1}(\Delta'_{\Omega'}, \mathbb{K}) \Big) \\
&= \beta_{i+\#\Omega',\Omega'}(\mathbb{K}[\Delta'])
\end{aligned}$$

□

On the other hand the set of topologies that arise among the restrictions Δ_W for subsets W of the ground set can be very different for simplicial complexes with homeomorphic geometric realization.

For example consider for a simplicial complex Δ over ground set Ω its barycentric subdivision $\mathrm{sd}(\Delta)$; that is the simplicial complex on group set $\Delta \setminus \{\emptyset\}$ with simplices $\{F_0, \ldots, F_i\}$ being sets of non-empty faces of Δ which if suitable numbered satisfy $F_0 \subset F_1 \subset \cdots \subset F_i$. It is well known that $|\Delta|$ and $|\mathrm{sd}(\Delta)|$ are homeomorphic. Indeed the geometric realizations can be choosen such that $|\Delta| = |\mathrm{sd}(\Delta)|$ by the following construction. Assume the geometric realization $|\Delta| \subseteq \mathbb{R}^d$ has simplicies that are convex hulls of points $p_\omega \in \mathbb{R}^d$, $\omega \in \Omega$. For $F \in \Delta \setminus \{\emptyset\}$ set $p_F = \frac{1}{\#F} \sum_{\omega \in F} p_\omega$. Then one can show that for a face $\{F_0, \ldots, F_i\}$ of $\mathrm{sd}(\Delta)$ the p_{F_i}, $i = 0, \ldots, i$ are affinely independent and define a geometric realization $|\mathrm{sd}(\Delta)|$ of $\mathrm{sd}(\Delta)$. When speaking of a simplicial complex and its barycentric subdivision we will assume assume that the geometric realizations are chosen in that way. In particular, $|\Delta| = |\mathrm{sd}(\Delta)|$.

Let $\Delta = \partial 2^{\{1,\ldots,n\}}$ be the boundary of the $(n-1)$-simplex. For any $W \subseteq \{1,\ldots,n\}$, $W \neq \emptyset, \{1,\ldots,n\}$, we have that Δ_W is a simplex and hence contractible and acyclic. For $\mathrm{sd}(\Delta)$ any restriction to $W = \{F, F'\}$ for $F, F \in \Delta$ such that $F \not\subseteq F'$ and $F' \not\subseteq F$ is a 0-sphere and hence has homology of rank 1 in dimension 0. Similarly, for any face $F \in \Delta \setminus \{\emptyset\}$ and $W = \partial \bar{F}$ we have that Δ_W is a triangulation of a $(\dim(F)-1)$-sphere and hence has homology of rank 1 concentrated in dimension $\dim(F)-1$.

Finally we can relate the depth of $\mathbb{K}[\Delta]$ to its minimal free resolution. The following is the Auslander-Buchsbaum formula (see [7, Theorem 19.9]) in our context. Recall that the projective dimension of $\mathbb{K}[\Delta]$ is the maximal i for which $\beta_i(\mathbb{K}[\Delta]) \neq 0$.

Theorem 3.3 (Auslander-Buchsbaum formula))**.** *Let Δ be a simplicial complex over ground set Ω. Then*

$$\mathrm{depth}(\mathbb{K}[\Delta]) = \#\Omega - \mathrm{pd}(\mathbb{K}[\Delta]).$$

Theorem 3.3 allowed Munkres to use Theorem 3.1 in order to deduce the topological invariance of the depth from the invariance of the difference of the cardinality of the ground set and the projective dimension. For that let us introduce a homological version of depth. The following homological version of depth which is obviously a topologcial invariant of a simplicial complex Δ over ground set Ω

$$\mathrm{hdepth}(\Delta) = \min_i \left\{ \begin{array}{c} \widetilde{H}_i(|\Delta|, \mathbb{K}) \neq 0 \text{ or} \\ H_i(|\Delta|, |\Delta| - x, \mathbb{K}) \neq 0 \text{ for some } x \in |\Delta| \end{array} \right\} + 1.$$

Theorem 3.4. *Let Δ be a simplicial complex over ground set Ω. Then*

$$\mathrm{pd}(\mathbb{K}[\Delta]) = \#\Omega - \mathrm{hdepth}(\Delta).$$

In particular, if Δ' is a simplicial complex over ground set Ω' such that $|\Delta|$ and $|\Delta'|$ are homeomorphic then $\#\Omega - \mathrm{pd}(\mathbb{K}[\Delta]) = \#\Omega' - \mathrm{pd}(\mathbb{K}[\Delta'])$.

Clearly, the second part of the theorem is an immediate consequence of the first. We will prove the first part in the next section. For that we need to recall several lemmas from [13] that are of independent interest in topological combinatorics.

Finally, by Theorem 3.3 the following theorem is equivalent to Theorem 3.4

Theorem 3.5. *Let Δ be a simplicial complex over ground set Ω. Then*

$$\mathrm{depth}(\mathbb{K}[\Delta]) = \mathrm{hdepth}(\Delta).$$

In particular, if Δ' is a simplicial complex such that $|\Delta|$ and $|\Delta'|$ are homeomorphic then $\mathrm{depth}(\mathbb{K}[\Delta]) = \mathrm{depth}(\mathbb{K}[\Delta'])$.

We will present independent proofs of the two equivalent theorems Theorem 3.4 and Theorem 3.5. The first in Section 4 proves Theorem 3.4 and follows the lines of Munkres' proof. For this proof one has to develop tools from topological combinatorics which are of independent interest. In Section 5 we prove Theorem 3.5 in a rather straightforward manner but use deep facts about local cohomology.

4. Munkres' proof of Theorem 3.4 and Theorem 3.5

First we define a covering of the barycentric subdivision of a simplicial complex which carries a lot of structural information but which is not covered by most texts on methods in topological combinatorics (e.g. [2]). For a simplicial complex Δ and a face $F \in \Delta$ we denote by $\mathrm{dblock}_\Delta(F)$ the subcomplex of $\mathrm{sd}(\Delta)$ which consists of all subsets of faces of the form $\{F = F_0 \subset\subseteq \cdots \subset F_i\}$. The simplicial complex $\mathrm{dblock}_\Delta(F)$ is called the dual block to F. By definition, $\mathrm{dblock}_\Delta(F)$ is a subcomplex of $\mathrm{star}_{\mathrm{sd}(\Delta)}(\{F\})$. As we have observed before as a star of a simplicial complex $\mathrm{star}_{\mathrm{sd}(\Delta)}(\{F\}) = \mathit{overline}\{F\} * \mathrm{link}_{\mathrm{sd}(\Delta)}(\{F\})$. The dual block has a similar decomposition as $\mathrm{dblock}_\Delta(F) = \overline{\{F\}} * \mathrm{lblock}_\Delta(F)$, where $\mathrm{lblock}_\Delta(F)$ consists of all $\{F_1 \subset \cdots \subset F_i\} \in \mathrm{sd}(\Delta)$ for which F is a proper subset of F_1. In particular, as a cone $|\mathrm{dblock}_\Delta(F)|$ is contractible and hence acyclic. We can also decompose $\mathrm{link}_{\mathrm{sd}(\Delta)}(\{F\}) = \mathrm{sd}(\partial \bar{F}) * \mathrm{lblock}_\Delta(F)$. Thus

$$\mathrm{star}_\Delta(F) = \overline{\{F\}} * \mathrm{sd}(\partial \bar{F}) * \mathrm{lblock}_\Delta(F). \tag{4}$$

Thus the pairs $(\mathrm{star}_{\mathrm{sd}(\Delta)}(F), \mathrm{link}_{\mathrm{sd}(\Delta)}(F))$ and $(\mathrm{dblock}_\Delta(F), \mathrm{lblock}_\Delta(F))$ exhibit analogous structural properties. The following lemma, which is an analog of Lemma 2.2, shows that these structural similarities lead to analogous homological behavior.

Lemma 4.1. *Let Δ be a simplicial complex and $F \in \Delta \setminus \{\emptyset\}$ a face of Δ. For an point x in the relative interior of $|\bar{F}|$ we have*

$$H_j(|\Delta|, |\Delta| - x, \mathbb{K}) = \widetilde{H}_{j-\dim(F)-1}(\mathrm{lblock}_\Delta(F), \mathbb{K}).$$

Proof. By excising $|\Delta| \setminus |\mathrm{dblock}_\Delta(F)|$ we obtain

$$H_j(|\Delta|, |\Delta| - p, \mathbb{K}) = H_j(|\mathrm{dblock}_\Delta(F)|, |\mathrm{dblock}_\Delta(F)| - x, \mathbb{K}).$$

Since $\mathrm{dblock}_\Delta(F)$ is contractible the long exact sequence in homology shows

$$H_j(|\mathrm{dblock}_\Delta(F)|, |\mathrm{dblock}_\Delta(F)| - x, \mathbb{K}) = \widetilde{H}_{j-1}(|\mathrm{dblock}_\Delta(F)| - x, \mathbb{K}).$$

Using (4) we obtain

$$|\mathrm{dblock}_\Delta(F)| = |\overline{\{F\}}| * |\mathrm{sd}(\partial \bar{F})| * |\mathrm{lblock}_\Delta(F)|.$$

Since x is taken from the relative interior of $|\bar{F}|$ and $|\bar{F}| = |\mathrm{sd}(\bar{F})| = |\overline{\{F\}}| * |\mathrm{sd}(\partial \bar{F})|$ we can see analogous to the proof of Lemma 2.2 that $|\mathrm{sd}(\partial \bar{F})| * |\mathrm{lblock}_\Delta(F)|$ is a deformation retract of $|\mathrm{dblock}_\Delta(F)| - x = |\{\bar{F}\}| * |\mathrm{sd}(\partial \bar{F})| * |\mathrm{lblock}_\Delta(F)| - x$. Thus

$$H_j(|\Delta|, |\Delta| - x, \mathbb{K}) = \widetilde{H}_{j-1}(|\mathrm{sd}(\partial \bar{F})| * |\mathrm{lblock}_\Delta(F)|, \mathbb{K}).$$

From the fact that $|\mathrm{sd}(\partial \bar{F})|$ is a $(\dim(F) - 1)$-sphere we infer

$$\widetilde{H}_{j-1}(|\mathrm{sd}(\partial \bar{F})| * |\mathrm{lblock}_\Delta(F)|, \mathbb{K}) = \widetilde{H}_{j-\dim(F)-1}(|\mathrm{lblock}_\Delta(F)|, \mathbb{K}) = \widetilde{H}_{j-\dim(F)-1}(\mathrm{lblock}_\Delta(F), \mathbb{K})$$

. ∎

Next we study collections of dual blocks. Let Δ be a simplicial complex. For a face $F \in \Delta$ set $m_F = \max_{F \subseteq G \in \Delta} \dim(G)$. It follows from $\mathrm{dblock}_\Delta(F) = \overline{\{F\}} * \mathrm{lblock}_\Delta(F)$ and (4) that $\dim(\mathrm{dblock}_\Delta(F)) = m_F - \dim(F) \leq \dim(\Delta) - \dim(F)$. We call $m_F - \dim(F)$ also the codimension of F in Δ and set $\mathrm{fcodim}(F) = \mathrm{fdim}(\mathrm{dblock}_\Delta(F)) = \dim(\Delta) - \dim(F)$ which we call the formal codimension of F and the formal dimension of $\mathrm{dblock}_\Delta(F)$.

We collect in $\mathfrak{Db}_\Delta$ all $\mathrm{dblock}_\Delta(F)$ for $F \in \Delta \setminus \{\emptyset\}$. We say that a collection $\mathfrak{C} \subseteq \mathrm{dblock}_\Delta(F)$ is a block-subcomplex if $\mathrm{dblock}_\Delta(F) \in \mathfrak{C}$ and $F \subseteq G \in \Delta$ implies that $\mathrm{dblock}_\Delta(G) \in \mathfrak{C}$. For a block-subcomplex $\mathfrak{C} \subseteq \mathfrak{Db}_\Delta$ we write $\mathfrak{C}^{\langle k \rangle}$ for the collection of all $\mathrm{dblock}_\Delta(F) \in \mathfrak{C}$ for $F \in \Delta \setminus \{\emptyset\}$ such that $\mathrm{fdim}(\mathrm{dblock}_\Delta(F)) \leq k$ or equivalently $\mathrm{fcodim}(F) \leq k$. Note that $\mathfrak{C}^{\langle k \rangle}$ is also a block-subcomplex. If $\mathfrak{C}$ is a block-subcomplex then we call the set $\mathrm{Face}_{\mathfrak{C}} = \{F \in \Delta \ : \ \mathrm{dblock}_\Delta(F) \in \mathfrak{C}\}$ the face set of $\mathfrak{C}$. Clearly, $\mathfrak{C} = \{\mathrm{dblock}_\Delta(F) \ : \ F \in \mathrm{Face}_{\mathfrak{C}}\}$.

For a block-subcomplex $\mathfrak{C} \subseteq \mathfrak{Db}_\Delta$ we write $|\mathfrak{C}|$ for

$$| \bigcup_{\mathrm{dblock}_\Delta(F) \in \mathfrak{C}} \mathrm{dblock}_\Delta(F)| \subseteq |\mathrm{sd}(\Delta)| = |\Delta|.$$

Lemma 4.2. *Let Δ be a simplicial complex and $\mathfrak{C} \subseteq \mathfrak{Db}_\Delta$ a block-subcomplex. Then for a number $k \geq 0$ we have*

$$H_i\Big(\mathfrak{Db}_\Delta^{\langle k \rangle} \cup \mathfrak{C}, \mathfrak{Db}_\Delta^{\langle k-1 \rangle} \cup \mathfrak{C}, \mathbb{K}\Big) = \bigoplus_{\substack{F \in \Delta \setminus \mathrm{Face}_{\mathfrak{C}} \\ \mathrm{fcodim}(F) = k}} H_i\Big(\mathrm{dblock}_\Delta(F), \partial \mathrm{sd}(\bar{F}) * \mathrm{lblock}_\Delta(F), \mathbb{K}\Big)$$

Proof. For $F, F' \in \Delta$ we have that $\mathrm{dblock}_\Delta(F) \cap \mathrm{dblock}_\Delta(F') =$ if $F \cup F' \notin \Delta$ and $\mathrm{dblock}_\Delta(F \cup F')$ otherwise. Since $\mathrm{dblock}_\Delta(F) = \overline{\{F\}} * \mathrm{lblock}_\Delta(F)$ it then follows that

$$\Big|\mathfrak{Db}_\Delta^{\langle k \rangle} \cup \mathfrak{C}\Big| / \Big|\mathfrak{Db}_\Delta^{\langle k-1 \rangle} \cup \mathfrak{C}\Big|$$

is a wedge of the suspensions of $|\mathrm{lblock}_\Delta(F)|$ for F of formal codimension k and such that $F \notin \mathrm{Face}_{\mathfrak{C}}$. Hence

$$H_i\Big(\mathfrak{Db}_\Delta^{\langle k \rangle} \cup \mathfrak{C}, \mathfrak{Db}_\Delta^{\langle k-1 \rangle} \cup \mathfrak{C}, \mathbb{K}\Big) = \bigoplus_{\substack{F \in \Delta \setminus \mathrm{Face}_{\mathfrak{C}} \\ \mathrm{fcodim}(F) = k}} \widetilde{H}_{i-1}(\mathrm{lblock}_\Delta(F), \mathbb{K}).$$

Since $\mathrm{dblock}_\Delta(F)$ is contractible and hence acyclic it follows that

$$\widetilde{H}_{i-1}(\mathrm{lblock}_\Delta(F), \mathbb{K}) = H_i(\mathrm{dblock}_\Delta(F), \mathrm{lblock}_\Delta(F), \mathbb{K}).$$

This completes the proof. $\square$

Consider a subcomplex $\Gamma \subseteq \Delta$ of a simplicial complex Δ such that $\Gamma \neq \{\emptyset\}$. Note that in this situation $\Gamma \setminus \{\emptyset\}$ is a subset of the ground set of $\mathrm{sd}(\Delta)$. Moreover, $\mathrm{sd}(\Gamma)$ is a subcomplex of $\mathrm{sd}(\Delta)$. Now if Γ is a proper subcomplex then $\mathrm{sd}(\Delta)_{\Delta \setminus \Gamma}$ is the subcomplex of $\mathrm{sd}(\Delta)$ with simplices $\{F_0 \subset \cdots \subset F_i\}$ such that $F_0, \ldots, F_i \in \Delta \setminus \Gamma$. We write $\mathfrak{Db}_{\Delta \setminus \Gamma}$ for the set of simplicial complexes $\mathrm{dblock}_\Delta(F)_{\Delta \setminus \Gamma}$ for $F \in \Delta \setminus \Gamma$. Clearly, $\mathfrak{Db}_{\Delta \setminus \Gamma}$ is a block-subcomplex of $\mathfrak{Db}_\Delta$.

Lemma 4.3. *Let Δ be a simplicial complex and $\Gamma \subset \Delta$ a proper subcomplex $\neq \{\emptyset\}$. Assume that for some $0 \leq M \leq \dim(\Delta)$ we have that $H_i(|\Delta|, |\Delta| - x, \mathbb{K}) = 0$ for all $x \in |\Gamma|$ and $0 \leq i < M$. Then*

(i) $H_j(|\mathrm{sd}(\Delta)|, |\mathrm{sd}(\Delta)_{\Delta\setminus\Gamma}|, \mathbb{K}) = 0$ *for* $0 \leq j < M - \dim(\Gamma)$.
(ii) $H_{M-\dim(\Gamma)}(|\mathrm{sd}(\Delta)|, |\mathrm{sd}(\Delta)_{\Delta\setminus\Gamma}|, \mathbb{K})$ *is isomorphic to the cokernel of*

$$\begin{tikzcd} H_{M-\dim(\Gamma)+1}\Big(|\mathfrak{Db}_\Delta^{\langle \dim(\Delta)-\dim(\Gamma)+1\rangle} \cup \mathfrak{Db}_{\Delta-\Gamma}|, |\mathfrak{Db}_\Delta^{\langle \dim(\Delta)-\dim(\Gamma)\rangle} \cup \mathfrak{Db}_{\Delta\setminus\Gamma}|\Big) \arrow[d, "\partial^*"] \\ H_{M-\dim(\Gamma)}\Big(|\mathfrak{Db}_\Delta^{\langle \dim(\Delta)-\dim(\Gamma)\rangle} \cup \mathfrak{Db}_{\Delta\setminus\Gamma}|, |\mathfrak{Db}_\Delta^{\langle \dim(\Delta)-\dim(\Gamma)-1\rangle} \cup \mathfrak{Db}_{\Delta-\Gamma}|\Big) \end{tikzcd} \tag{5}$$

Proof. Claim 1: For $i \leq j$ we have

$$H_i(|\mathfrak{Db}_\Delta^{\langle j+\dim(\Delta)-M+1\rangle} \cup \mathfrak{Db}_{\Delta\setminus\Gamma}|, |\mathfrak{Db}_\Delta^{\langle j+\dim(\Delta)-M\rangle} \cup \mathfrak{Db}_{\Delta\setminus\Gamma}|, \mathbb{K}) = 0 \tag{6}$$

$\lhd$ Proof of Claim: By Lemma 4.2 we know that the homology group on the left hand side of (6) decomposes as a direct sum of groups $H_i(\mathrm{dblock}(F), \mathrm{lblock}(F), \mathbb{K}) = \widetilde{H}_{i-1}(\mathrm{lblock}_\Delta(F), \mathbb{K})$ for faces $F \in \Delta$ of formal codimension

$$\mathrm{fcodim}(F) = \dim(\Delta) - \dim(F) = j + \dim(\Delta) - M + 1$$

that are not in Γ. By Lemma 4.1 we have that $H_{i-1}(\mathrm{lblock}_\Delta(F), \mathbb{K}) = H_{i+\dim(F)}(|\Delta|, |\Delta| - x, \mathbb{K})$ for any x in the interior of $|\bar{F}|$. By assumption this group vanishes for $i + \dim(F) < M$. Now

$$i + \dim(F) = i + \dim(\Delta) - (j + \dim(\Delta) - M + 1) = M + (i - j) - 1.$$

Since $M + (i - j) - 1 < M$ for $i \leq j$ the assertion follows. $\rhd$

Claim 2: For $i \leq j$ and $\ell \geq 1$ we have

$$H_i\Big(|\mathfrak{Db}_\Delta^{\langle j+\dim(\Delta)-M+\ell\rangle}|, |\mathfrak{Db}_\Delta^{\langle j+\dim(\Delta)-M\rangle}|, \mathbb{K}\Big) = 0. \tag{7}$$

In particular,

$$H_i\Big(|\Delta|, |\mathfrak{Db}_\Delta^{\langle j+\dim(\Delta)-M\rangle}|, \mathbb{K}\Big) = 0. \tag{8}$$

$\lhd$ Proof of Claim: Since $|\mathfrak{Db}_\Delta^{\langle j+\dim(\Delta)-M+\ell\rangle}| = |\Delta|$ for $\ell \geq M - j$ we get (8) as a direct consequence of (7).

We prove (7) by induction on ℓ. For $\ell = 1$ the assertion coincides with Claim 1.

Let $\ell \geq 2$. Set $K = j + \dim(\Delta) - M$ and consider the long exact sequence

$$\begin{tikzcd} \cdots \arrow[r] & H_i\Big(|\mathfrak{Db}_\Delta^{\langle K+\ell-1\rangle}|, |\mathfrak{Db}_\Delta^{\langle K\rangle}, \mathbb{K}\Big)|, \mathbb{K}) \arrow[r] & H_i\Big(|\mathfrak{Db}_\Delta^{\langle K+\ell\rangle}|, |\mathfrak{Db}_\Delta^{\langle K\rangle}|, \mathbb{K}\Big) \arrow[d] \\ \cdots & H_i(|\mathfrak{Db}_\Delta^{\langle K+\ell\rangle}|, |\mathfrak{Db}_\Delta^{\langle K\rangle}|, \mathbb{K}) \arrow[l] & H_i\Big(|\mathfrak{Db}_\Delta^{\langle K+\ell-1\rangle}|, |\mathfrak{Db}_\Delta^{\langle K\rangle}|, \mathbb{K}\Big) \arrow[l] \end{tikzcd}$$

of the triple

$$(|\mathfrak{Db}_\Delta^{\langle K+\ell\rangle}|, |\mathfrak{Db}_\Delta^{\langle K+\ell-1\rangle}|, |\mathfrak{Db}_\Delta^{\langle K\rangle}|).$$

By induction we can deduce the vanishing all homology groups except for

$$H_i(|\mathfrak{Db}_\Delta^{\langle K+\ell\rangle}|, |\mathfrak{Db}_\Delta^{\langle K\rangle}|, \mathbb{K}).$$

The fact that the sequence is exact then implies also the vanishing of this group. ▷

Claim 3: For $i < \dim(\Delta) - \dim(\Gamma)$ we have

$$\mathfrak{Db}_\Delta^{\langle i\rangle} \cup \mathfrak{Db}_{\Delta\setminus\Gamma} = \mathfrak{Db}_{\Delta\setminus\Gamma}.$$

◁ Proof of Claim: For a face F of Γ the formal dimension of $\mathrm{dblock}_\Delta(F)$ is at least $\dim(\Delta) - \dim(\Gamma)$. This shows that $|\mathfrak{Db}_\Delta^{\langle i\rangle}| \subseteq |\mathfrak{Db}_{\Delta\setminus\Gamma}|$ for $i < \dim(\Delta) - \dim(\Gamma)$ and implies the assertion. ▷

Now we are in position to prove part (i) and (ii) of the lemma.

◁ Proof of (i): For $i < \dim(\Delta) - \dim(\Gamma)$ we have

$$\begin{aligned} H_i(|\mathrm{sd}(\Delta)|, |\mathrm{sd}(\Delta)_{\Delta\setminus\Gamma}|, \mathbb{K}) &= H_i(|\mathfrak{Db}_\Delta|, |\mathfrak{Db}_{\Delta\setminus\Gamma}|, \mathbb{K}) \\ &\overset{\text{Claim 3}}{=} H_i(|\mathfrak{Db}_\Delta|, |\mathfrak{Db}_\Delta^{\langle j\rangle} \cup \mathfrak{Db}_{\Delta\setminus\Gamma}|, \mathbb{K}) \\ &\overset{\text{Claim 2}}{=} 0. \end{aligned}$$

▷

◁ Proof of (ii): Let ℓ be such that $|\mathfrak{Db}_\Delta^{\langle j+\dim(\Delta)-M+\ell\rangle}| = |\mathfrak{Db}_\Delta|$. Setting $i = j = M - \dim(\Gamma)$ in Claim 2 we obtain:

$$H_{M-\dim(\Gamma)}(\Big(|\mathfrak{Db}_\Delta|, |\mathfrak{Db}_\Delta^{\langle \dim(\Delta)-\dim(\Gamma)\rangle}|, \mathbb{K}\Big) = 0. \tag{9}$$

Setting $i = j = M - \dim(\Gamma) + 1$ in Claim 2 we obtain

$$H_{M-\dim(\Gamma)+1)}(\Big(|\mathfrak{Db}_\Delta|, |\mathfrak{Db}_\Delta^{\langle \dim(\Delta)-\dim(\Gamma)+1\rangle}|, \mathbb{K}\Big) = 0. \tag{10}$$

Using long exact sequences of triples in rows and columns and (10) to obtain the 0 on the top of the first column and (9) to obtain the 0 at the end of the second row we derive the following commutative diagram with exact rows and columns. In the diagram we write D for $\dim(\Delta)$ and G for $\dim(\Gamma)$.

$$\begin{tikzcd}
0 & & & \\
H_{M-G+1}\Big({|\Delta|, \atop |\mathfrak{Db}_\Delta^{\langle D-G\rangle} \cup \mathfrak{Db}_{\Delta\setminus\Gamma}|}, \mathbb{K}\Big) \arrow[u] \arrow[r] & H_{M-G+1}\Big({|\mathfrak{Db}_\Delta^{\langle D-G\rangle} \cup \mathfrak{Db}_{\Delta\setminus\Gamma}|, \atop |\mathfrak{Db}_{\Delta\setminus\Gamma}|}, \mathbb{K}\Big) \arrow[r] \arrow[d, equal] & H_{M-G}\Big({|\mathfrak{Db}_\Delta|, \atop |\mathfrak{Db}_{\Delta\setminus\Gamma}|}, \mathbb{K}\Big) \arrow[r] & 0 \\
H_{M-G+1}\Big({|\mathfrak{Db}_\Delta^{\langle D-G+1\rangle} \cup \mathfrak{Db}_{\Delta\setminus\Gamma}|, \atop |\mathfrak{Db}_\Delta^{\langle D-G\rangle} \cup \mathfrak{Db}_{\Delta\setminus\Gamma}|}, \mathbb{K}\Big) \arrow[u] \arrow[r] & H_{M-G}\Big({|\mathfrak{Db}_\Delta^{\langle D-G\rangle} \cup \mathfrak{Db}_{\Delta\setminus\Gamma}|, \atop |\mathfrak{Db}_\Delta^{\langle D-G-1\rangle} \cup \mathfrak{Db}_{\Delta\setminus\Gamma}|}, \mathbb{K}\Big) & &
\end{tikzcd}$$

Note that the equality in the second column is a consequence of Claim 3. Since $|\mathrm{sd}(\Delta)| = |\mathfrak{Db}_\Delta|$ and $|\mathrm{sd}(\Delta)_{\Gamma-\Delta}| = |\mathfrak{Db}_{\Delta-\Gamma}|$ it suffices to show that by the exactness of the second row it follows that $H_{M-G}\Big(|\mathfrak{Db}_\Delta|, |\mathfrak{Db}_{\Delta-\Gamma}|, \mathbb{K}\Big)$ is isomorphic to the image of (5). By the exactness of the diagram above it follows that $H_{M-G}\Big(|\mathfrak{Db}_\Delta|, |\mathfrak{Db}_{\Delta-\Gamma}|, \mathbb{K}\Big)$ is isomorphic to the cokernel of the left map in the second row of the above diagram. From the fact that the diagram is commutative and the exactness of the first column the assertion then follows. $\triangleright$ $\square$

Lemma 4.4. *Let Δ be a simplicial complex over ground set Ω such that $\Delta \neq 2^\Omega$. Assume further that M is a number such that for all $x \in |\Delta|$ and all $i < M$ we have $H_i(|\Delta|, |\Delta| - x, \mathbb{K}) = 0$. Then the following are equiavlent:*

(i) *There is an $x \in |\Delta|$ for which $H_M(|\Delta|, |\Delta| - x, \mathbb{K}) \neq 0$.*
(ii) *There is a subcomplex $\Gamma \subseteq \Delta$, $\Gamma \neq \{\emptyset\}$ such that for every x in the relative interior of $|\Gamma|$ we have $H_M(|\Delta|, |\Delta| - x, \mathbb{K}) \neq 0$.*
(iii) *There is a face $F \neq \emptyset$ of Δ such that for every x in the relative interior of $|\bar{F}|$ we have $H_M(|\Delta|, |\Delta| - x, \mathbb{K}) \neq 0$.*
(iv) *There is face $F \neq \emptyset$ of Δ such that such that for $\Gamma = \bar{F}$ we have*

$$H_{M-\dim(\Gamma)}(|\mathrm{sd}(\Delta)|, |\mathrm{sd}(\Delta)_{\Delta-\Gamma}|, \mathbb{K}) \neq 0.$$

Proof. The implications (iii) $\Leftrightarrow$ (ii) $\Rightarrow$ (i) and (v) $\Rightarrow$ (iv) are valid for trivial reasons.

First we show (i) $\Rightarrow$ (iii). By Lemma 4.1 we know that the homology groups $H_M(|\Delta|, |\Delta| - x, \mathbb{K})$ are isomorphic whenever x is choosen from the relative interior of $|\bar{F}|$ for a fixed face F of Δ. This implies the assertion.

Before we show (iv) $\Leftrightarrow$ (iii) we analyze

$$H_{M-\dim(\Gamma)}(|\mathrm{sd}(\Delta)|, |\mathrm{sd}(\Delta)_{\Delta\setminus\Gamma}|, \mathbb{K})$$

more closely in case $\Gamma = \bar{F}$ for a non-empty face F of Δ. By Lemma 4.3(ii) the hmology group is isomorphic to the cokernel of the map from (5). By Lemma 4.2 we know that

(A)
$$H_{M-\dim(\Gamma)+1}(|\mathfrak{Db}_\Delta^{\langle \dim(\Delta)-\dim(\Gamma)\rangle} \cup \mathfrak{Db}_{\Delta-\Gamma}|, |\mathfrak{Db}_\Delta^{\langle \dim(\Delta)-\dim(\Gamma)-1\rangle} \cup \mathfrak{Db}_{\Delta-\Gamma}|, \mathbb{K})$$
is isomorphic to a direct sum of homology groups $H_{M-\dim(\Gamma)}(\mathrm{lblock}_\Delta(G), \mathbb{K})$ for $G \in \Gamma$ of formal codimension $\mathrm{fcodim}(G) = \dim(\Delta) - \dim(\Gamma)$ or equivalently of dimension $\dim(\Gamma)$. By $\Gamma = \bar{F}$ only $F \in \Gamma$ satisfies this condition and it follows that the homology group is isomorphic to $H_{M-\dim(\Gamma)}(\mathrm{lblock}_\Delta(F), \mathbb{K})$.

(B)
$$H_{M-\dim(\Gamma)}(|\mathfrak{Db}_\Delta^{\langle \dim(\Delta)-\dim(\Gamma)+1\rangle} \cup \mathfrak{Db}_{\Delta\setminus\Gamma}|, |\mathfrak{Db}_\Delta^{\langle \dim(\Delta)-\dim(\Gamma)\rangle} \cup \mathfrak{Db}_{\Delta\setminus\Gamma}|, \mathbb{K})$$
is isomorphic to a direct sum of homology groups $H_{M-\dim(\Gamma)-1}(\mathrm{lblock}_\Delta(G), \mathbb{K})$ for $G \in \Gamma$ of formal codimension $\mathrm{fcodim}(G) = \dim(\Delta) - \dim(\Gamma) + 1$ or equivalently of dimension $\dim(\Gamma) - 1$. By $\Gamma = \bar{F}$ it follows that the homology group is isomorphic to the direct sum $H_{M-\dim(\Gamma)-1}(\mathrm{lblock}_\Delta(F \setminus \{\omega\}), \mathbb{K})$ for $\omega \in F$.

Now we can prove (iv) $\Rightarrow$ (iii). By assumption the cokernel of (5) is nontrivial. Thus it follows from (A) that $H_{M-\dim(\Gamma)}(\mathrm{lblock}_\Delta(F),\mathbb{K})$ is non-trivial. By Lemma 4.1 the latter is isomorphic to $H_M(|\Delta|,|\Delta|-x,\mathbb{K})$ for any x in the interior of $|\bar{F}|$. This implies (iii)

To prove (iii) $\Rightarrow$ (iv) By Lemma 4.1 each group $H_{M-\dim(\Gamma)-1}(\mathrm{lblock}_\Delta(F\setminus\{\omega\}),\mathbb{K})$ for $\omega\in F$ is isomorphic to $H_{M-1}(|\Delta|,|\Delta|-x,\mathbb{K})$ for x in the relative interior of $|\overline{F\setminus\{\omega\}}|$. By assumption the latter group is trivial. Thus by (A) and (B) the cokernel of (5) is isomorphic to $H_{M-\dim(\Gamma)}(\mathrm{lblock}_\Delta(F),\mathbb{K})$. By Lemma 4.1 the latter is isomorphic to $H_M(|\Delta|,|\Delta|-x,\mathbb{K})$ for x in the relative interior of $|\bar{F}|$. Thus it is non-trivial by the hypothesis of (iii). Now the assertion follows. $\square$

We now show that hdepth is a homological version of depth.

Lemma 4.5. *Let Δ be a simplicial complex over ground set Ω and $T\subseteq\Omega$. Then*

(i) $\widetilde{H}_{j-\#T}(|\Delta|\setminus|\Delta_T|,\mathbb{K})=0$ *for* $j<\mathrm{hdepth}(\Delta)-1$
(ii) $\widetilde{H}_{j-\#T}(|\Delta|\setminus|\Delta_T|,\mathbb{K})=0$ *for* $j\leq\mathrm{hdepth}(\Delta)-1$ *if* $\Delta_T\neq\bar{T}$.

Proof. If $T=\emptyset$ then $\Delta_T=\{\emptyset\}=\bar{\emptyset}$ and $\widetilde{H}_{j-\#T}(|\Delta|\setminus|\Delta_T|,\mathbb{K})=\widetilde{H}_j(|\Delta|,\mathbb{K})$ which vanishes for $j<\mathrm{hdepth}(\Delta)-1$ by definition.

If $T=\Omega$ then $\Delta_T=\Delta$ and Then $\widetilde{H}_{j-\#T}(|\Delta|\setminus|\Delta_T|,\mathbb{K})=\widetilde{H}_{j-\#\Omega}(\emptyset,\mathbb{K})=0$ for all $j-\#\Omega\neq-1$. Now $\mathrm{hdepth}(\Delta)\leq\#\Omega$ and therefore for $j<\mathrm{hdepth}(\Delta)-1$ we have $j-\#\Omega<-1$. If $\Delta\neq 2^\Omega=\bar{\Omega}$ then $\mathrm{hdepth}(\Delta)<\#\Omega$ and for $j\leq\mathrm{hdepth}(\Delta)-1$ we have $j-\#\Omega<-1$.

Now let $T\neq\emptyset,\Omega$. Consider the long exact sequence

$$\cdots\to H_{i+1}(|\Delta|,|\Delta|\setminus|\Delta_T|,\mathbb{K})\to\widetilde{H}_i(|\Delta|\setminus|\Delta_T|,\mathbb{K})\to\widetilde{H}_i(|\Delta|,\mathbb{K})\to\cdots.$$

The group on the right hand side vanishes for $i<\mathrm{hdepth}(\Delta)-1$ by definition. By Lemma 4.3(i) and the definition of $\mathrm{hdepth}(\Delta)$ the group on the left hand side vanishes for $i+1<\mathrm{hdepth}(\Delta)-1-\dim(\Delta_T)$. Therefore, $\widetilde{H}_i(|\Delta|-|\Delta_T|,\mathbb{K})=0$ for $i+1<\mathrm{hdepth}(\Delta)-1-\dim(\Delta_T)$. Since $\dim(\Delta_T)\leq\#T-1$ with equality if and only if $\Delta_T=\bar{T}$ the assertions (i) and (ii) now follow. $\square$

We are now in position to prove the following proposition which will immediately implies Theorem 3.4.

Proposition 4.6. *Let Δ be a simplicial complex on ground set Ω and set Then*

(i) $\mathrm{hdepth}(\Delta)=\#\Omega-\max_i\{\beta_i(\mathbb{K}[\Delta])\neq 0\}=\#\Omega-\mathrm{pd}(\mathbb{K}[\Delta])$.
(ii) *Let $\emptyset\neq W\in\Delta$ and assume that $H_{\mathrm{hdepth}(\Delta)-|T|}(|\Delta|,|\Delta|-x,\mathbb{K})\neq 0$ for some x in the relative interior of $|\bar{W}|$ then $H_i(|\Delta_W|,\mathbb{K})=0$ for $i\neq 0$.*

Proof. (i)
Case 1: $\Delta=\bar{F}$ for some $F\subseteq\Omega$ is a full simplex.

Then $\mathbb{K}[\Delta]=\mathbb{K}[x_\omega\ :\ \omega\in\Omega\setminus F]$ and by simple homological algebra $\beta_i(\mathbb{K}[\Delta])=0$ for $i>\#\Omega-\#F$ and $\beta_{\#\Omega-\#F}(\mathbb{K}[\Delta])=1$ for $i=\#\Omega-\#F$. Thus $\mathrm{pd}(\mathbb{K}[\Delta]=\#\Omega-\#F$. Thus we need to show that $\mathrm{hdepth}(\Delta)=\#F$.

If $F = \emptyset$ then $\widetilde{H}_i(|\Delta|, \mathbb{K}) = 0$ for $i > -1$ and $\mathbb{K}$ for $i = -1$. Obviously, there are no x in the relative interior of $|\Delta|$. Thus $\operatorname{hdepth}(\Delta) = (-1) + 1 = 0 = \#F$.

Now assume that $F \neq \emptyset$. Since $\Delta = \bar{F}$ is a full simplex we have $\widetilde{H}_i(|\Delta|, \mathbb{K}) = H_i(|\Delta|, |\Delta| - x, \mathbb{K}) = 0$ for all x in the boundary of the simplex and all $i \geq -1$. If x is in the relative interior of $|\Delta|$ then $H_i(|\Delta|, |\Delta| - x, \mathbb{K})$ is 0 for $i < \dim(F)$ and $\mathbb{K}$ for $i = \dim(F)$. Thus $\operatorname{hdepth}(\Delta) = \dim(F) + 1 = \#F$.

Since in both cases $\operatorname{hdepth}(\Delta) = \#F$ the assertion (i) follows.

Case 2: Δ is not a full simplex.

By Hochster's formula Theorem 3.1 we know that

$$\beta_{i,W} = \dim_{\mathbb{K}} \left(\widetilde{H}_{\#W-i-1}(\Delta_W, \mathbb{K}) \right) \text{ and therefore}$$

$$\operatorname{pd}(\mathbb{K}[\Delta]) = \max_i \{\beta_i(\mathbb{K}[\Delta]) \neq 0\} = \max_i \{\widetilde{H}_{\#W-i-1}(\Delta_W, \mathbb{K}) \neq 0 \text{ for some } W \subseteq \Omega\}.$$

Recall that $\widetilde{H}_{\#W-i-1}(\Delta_W, \mathbb{K}) = \widetilde{H}_{\#W-i-1}(|\Delta| - |\Delta_{\Omega \setminus W}|, \mathbb{K})$. We apply Lemma 4.5 to $T = \Omega \setminus W$ and deduce that $\widetilde{H}_{j-\#\Omega+\#W}(\Delta_W, \mathbb{K}) = 0$ for $j < \operatorname{hdepth}(\Delta) - 1$. It follows that $\widetilde{H}_{\#W-i-1}(\Delta_W, \mathbb{K}) = 0$ for $i > \#\Omega - \operatorname{hdepth}(\Delta)$. Hence we infer $\operatorname{pd}(\mathbb{K}[\Delta]) \leq \#\Omega - \operatorname{hdepth}(\Delta)$.

It remains to show that there is a $W \subseteq \Omega$ such that $\widetilde{H}_{\#W-i-1}(\Delta_W, \mathbb{K}) \neq 0$ for $i = \#\Omega - \operatorname{hdepth}(\Delta)$.

If $\widetilde{H}_{\operatorname{hdepth}(\Delta)-1}(\Delta, \mathbb{K}) \neq 0$ then for $W = \Omega$ one has $\widetilde{H}_{\operatorname{hdepth}(\Delta)-1}(\Delta_W, \mathbb{K}) \neq 0$. Thus for $i = \#W - \operatorname{hdepth}(\Delta)$ one has $\widetilde{H}_{\#W-i-1}(\Delta_W, \mathbb{K}) \neq 0$ and the assertion follows.

If $\widetilde{H}_{\operatorname{hdepth}(\Delta)-1}(\Delta, \mathbb{K}) = 0$ then there is some $x \in \Delta$ such that $H_i(|\Delta|, |\Delta| - x, \mathbb{K}) \neq 0$ for $i = \operatorname{hdepth}(\Delta) - 1$ and $H_i(|\Delta|, |\Delta| - y, \mathbb{K}) = 0$ for $i < \operatorname{hdepth}(\Delta) - 1$ and any $y \in |\Delta|$. Thus we can apply Lemma 4.4 for $M = \operatorname{hdepth}(\Delta) - 1$. It follows that There is face $T \neq \emptyset$ of Δ such that such that for $\Gamma = \bar{T}$ we have

$$H_{\operatorname{hdepth}(\Delta)-1-\dim(\Gamma)}(|\operatorname{sd}(\Delta)|, |\operatorname{sd}(\Delta)_{\Delta-\Gamma}|, \mathbb{K}) = H_{\operatorname{hdepth}(\Delta)-\#T}(|\Delta|, |\Delta| \setminus |\Delta_T|, \mathbb{K}) \neq 0.$$

For $W = \Omega \setminus T$ we obtain that $H_{\operatorname{hdepth}(\Delta)-\#\Omega+\#W}(|\Delta|, |\Delta_W|, \mathbb{K}) \neq 0$. Since $-\#\Omega + \#W \leq 0$ we know from $\widetilde{H}_{\operatorname{hdepth}(\Delta)-1}(\Delta, \mathbb{K}) = 0$ that $H_{\operatorname{hdepth}(\Delta)-\#\Omega+\#W}(|\Delta|, \mathbb{K}) = 0 = H_{\operatorname{hdepth}(\Delta)-\#\Omega+\#W}(|\Delta|, \mathbb{K}) = 0$. The long exact sequence

$$\begin{array}{ccccc} \cdots & \longrightarrow & \widetilde{H}_{\operatorname{hdepth}(\Delta)-\#\Omega+\#W}(|\Delta|, \mathbb{K}) & \longrightarrow & H_{\operatorname{hdepth}(\Delta)-\#\Omega+\#W}(|\Delta|, |\Delta_W|, \mathbb{K}) \\ & & & & \downarrow \\ \cdots & \longleftarrow & \widetilde{H}_{\operatorname{hdepth}(\Delta)-\#\Omega+\#W}(|\Delta|, \mathbb{K}) & \longleftarrow & \widetilde{H}_{\operatorname{hdepth}(\Delta)-\#\Omega+\#W-1}(|\Delta_W|, \mathbb{K}) \end{array}$$

now shows that

$$0 \neq H_{\operatorname{hdepth}(\Delta)-\#\Omega+\#W}(|\Delta|, |\Delta_W|, \mathbb{K}) \cong \widetilde{H}_{\operatorname{hdepth}(\Delta)-\#\Omega+\#W-1}(|\Delta_W|, \mathbb{K}).$$

Thus for $i = \#\Omega - \operatorname{hdepth}(\Delta)$ we obtain $\widetilde{H}_{\#W-i-1}(|\Delta_W|, \mathbb{K}) \neq 0$. $\square$

5. Local cohomology proof of Theorem 3.4 and Theorem 3.5

In this section we use local cohomology (see [4] for definitions and basic properties) to prove Theorem 3.5 and hence Theorem 3.4. This verification is much shorter then the one from Section 4 but builds on substantially more deep theory from commutative algebra. Topologically, the simplification comes from the fact that here we can work with links, which are easier to control than the induced subcomplexes used in the previous section.

Let $\mathbb{K}[\Delta] = \bigoplus_{r=0}^{\infty} A_r$ be the vectorspace decomposition of $\mathbb{K}[\Delta]$ as a standard graded algebra as in Section 2. We write $\mathfrak{m} = \bigoplus_{r=1}^{\infty} A_r$ for the unique graded maximal ideal of $\mathbb{K}[\Delta]$ and $H^i_{\mathfrak{m}}(\mathbb{K}[\Delta])$ for the ith local cohomology module of $\mathbb{K}[\Delta]$. The local cohomology $H^i_{\mathfrak{m}}(\mathbb{K}[\Delta])$ is itself a graded module and the following formula by Hochster expresses its Hilbert series in homological terms (see e.g. [18, Theorem 4.1]).

Theorem 5.1 (Hochster Formula for Local Cohomology)**.** *Let Δ be a simplicial complex over ground set Ω. Then*

$$\mathrm{Hilb}(H^i_{\mathfrak{m}}(\mathbb{K}[\Delta])) = \sum_{F\in\Delta} \dim_{\mathbb{K}} \left(\widetilde{H}_{i-\dim(F)-2}(\mathrm{link}_\Delta(F),\mathbb{K})\right)\frac{1}{(t-1)^{\#F}}.$$

Local cohomology is a powerful tool which encodes many invariants of a module. Here the following fact will be important. We formulate this very general fact for Stanley-Reisner rings only.

Theorem 5.2. *Let Δ be a simplicial complex over ground set Ω. Then*

$$\dim(\mathbb{K}[\Delta]) = \max_i H^i_{\mathfrak{m}}(\mathbb{K}[\Delta]) \neq 0$$

and

$$\mathrm{depth}(\mathbb{K}(\Delta)) = \min_i H^i_{\mathfrak{m}}(\mathbb{K}[\Delta]) \neq 0.$$

Using Theorem 5.1 we immediately obtain the following corollary.

Corollary 5.3. *Let Δ be a simplicial complex over ground set Ω. Then*

$$\dim(\mathbb{K}[\Delta]) = \max_i \{\widetilde{H}_{i-\dim(F)-2}(\mathrm{link}_\Delta(F),\mathbb{K}) \neq 0 \textit{ für ein } F\in\Delta\}$$

and

$$\mathrm{depth}(\mathbb{K}[\Delta]) = \min_i \{\widetilde{H}_{i-\dim(F)-2}(\mathrm{link}_\Delta(F),\mathbb{K}) \neq 0 \textit{ für ein } F\in\Delta\}.$$

Now we already in position to prove Theorem 3.5.

Proof or Theorem 3.5. If $F=\emptyset$ then $\mathrm{link}_\Delta(F) = \Delta$ and

$$\widetilde{H}_{i-\dim(F)-2}(\mathrm{link}_\Delta(F),\mathbb{K}) = \widetilde{H}_{i-1}(\Delta,\mathbb{K}) = \widetilde{H}_{i-1}(|\Delta|,\mathbb{K}). \tag{11}$$

If $F\neq\emptyset$ and x is a point from the relative interior of $|\bar{F}|$ then by Lemma 2.2 we have:

$$\widetilde{H}_{i-\dim(F)-2}(\mathrm{link}_\Delta(F),\mathbb{K}) = H_{i-1}(|\Delta|,|\Delta|-x,\mathbb{K}). \tag{12}$$

The minimal i for which at least one of homology groups on the right hand side of (11) or (12) is non-zero is exactly $\mathrm{hdepth}(\Delta)-1$. Thus Theorem 3.5 follows. ☐

6. Cohen-Macaulay, Gorenstein, Buchsbaum

A ring R is called Cohen-Macaulay if $\dim(R) = \operatorname{depth}(R)$, i.e. its depth equals it Krull dimension. As an immediate consequence of Theorem 2.3 and Theorem 3.5 we obtain the following result by Munkres (see [13, Corollary 3.4]).

Theorem 6.1 (Munkres)**.** *Let Δ and Δ' be simplicial complexes such that $|\Delta|$ and $|\Delta'|$ are homeomorphic. Then $\mathbb{K}[\Delta]$ is Cohen-Macaulay if and only if $\mathbb{K}[\Delta']$ is Cohen-Macaulay.*

As a further consequence of Theorem 3.5 and (11) and (12) we obtain the following criterion for Cohen-Macaulayness by Reisner [14].

Theorem 6.2 (Reisner's Criterion)**.** *Let Δ be a simplicial complex over ground set Ω. Then $\mathbb{K}[\Delta]$ is Cohen-Macaulay if and only if for all $F \in \Delta$*

$$\widetilde{H}_i(\operatorname{link}_\Delta(F), \mathbb{K}) = 0 \text{ for all } i < \dim(\operatorname{link}_\Delta(F)).$$

In particular, if $\mathbb{K}[\Delta]$ is Cohen-Macaulay then so if $\mathbb{K}[\operatorname{link}_\Delta(F)]$ for all $F \in \Delta$.

For a Cohen-Macaulay $\mathbb{K}[\Delta]$ the Betti-number $\beta_{\operatorname{pd}(\mathbb{K}[\Delta]}(\mathbb{K}[\Delta])$ is called the Cohen-Macaulay type of $\mathbb{K}[\Delta]$. The Cohen-Macaulay $\mathbb{K}[\Delta]$ of type 1 are called Gorenstein.

Example 6.3. Let $\Omega = \{1,2,3,4\}$ and Δ the simplicial complex over Ω with facets $\{1,2,3\}$ and $\{2,3,4\}$. Then $\mathbb{K}[\Delta] = S/I_\Delta = S_\Omega/(x_1x_4)$ and

$$0 \to S_\Omega \xrightarrow{\left(x_1x_4\right)} S_\Omega \xrightarrow{m \mapsto m+I_\Delta} \mathbb{K}[\Delta] \to 0$$

is the minimal free resolution. In particular, $\operatorname{pd}(\mathbb{K}[\Delta]) = 1$ and $\beta_1(\mathbb{K}[\Delta]) = 1$. Thus by Theorem 3.3 we have $\operatorname{depth}(\mathbb{K}[\Delta]) = \#\Omega - 1 = 3$. Since $\dim(\Delta) = 2$ it follows that $\dim(\mathbb{K}[\Delta]) = 3$. Thus $\mathbb{K}[\Delta]$ is Cohen-Macaulay and of type 1 and therefore $\mathbb{K}[\Delta]$ is Gorenstein.

Now consider Δ' over ground set $\Omega' = \{1,2,3,4,5\}$ with facets $\{1,2,3\}$, $\{2,3,4\}$ and $\{1,2,5\}$. Then $\mathbb{K}[\Delta] = S_{\Omega'}/I_{\Delta'} = S_{\Omega'}/(x_1x_4, x_3x_5, x_4x_5)$. It can be checked the the minimal free resolution is given by

$$0 \to S_{\Omega'}^2 \xrightarrow{\begin{pmatrix} x_3x_4 & -x_1x_4 & 0 \\ 0 & x_4 & -x_3 \end{pmatrix}} S_{\Omega'}^3 \xrightarrow{\begin{pmatrix} x_1x_4 \\ x_3x_5 \\ x_4x_5 \end{pmatrix}} S_{\Omega'} \xrightarrow{m \mapsto m+I_{\Delta'}} \mathbb{K}[\Delta'] \to 0.$$

In particular, $\operatorname{pd}(\mathbb{K}[\Delta']) = 2$ and $\beta_2(\mathbb{K}[\Delta]) = 2$. Thus by Theorem 3.3 we have $\operatorname{depth}(\mathbb{K}[\Delta']) = \#\Omega' - 2 = 3$. Since $\dim(\Delta') = 2$ it follows that $\dim(\mathbb{K}[\Delta']) = 3$. Thus $\mathbb{K}[\Delta]$ is again Cohen-Macaulay but of type 2 and hence not Gorenstein.

Both $|\Delta|$ and $|\Delta'|$ are homeomorphic to a 2-ball. It follows that the Gorentein property is not topological.

The following will allow us to deduce the topological invariance of a property which is slightly stronger than Gorenstein. simplicial complex Δ is called Gorenstein* (over $\mathbb{K}$) if $\mathbb{K}[\Delta]$ is Gorenstein and $\widetilde{H}_{\dim(\Delta)}(\Delta, \mathbb{K}) \neq 0$. To study the topological invariance of the

Gorenstein* property, we need a few more definitions. For a simplicial complex Δ we define its core $\mathrm{core}(\Delta)$ as the induced subcomplex $\Delta_{\mathrm{core}(\Omega)}$ where $\mathrm{core}(\Omega)$ is the set of all $\omega \in \Omega$ such that $\mathrm{star}_\Delta(\omega) \neq \Delta$. It follows that $\Delta = 2^{\Omega \setminus \mathrm{core}(\Omega)} * \mathrm{core}(\Delta)$ and $\dim(\Delta) = \dim(\Delta_{\mathrm{core}(\Omega)}) + \#\Omega - \#\mathrm{core}(\Omega)$.

Theorem 6.4. *Let Δ be a simplicial complex over ground set Ω. Then the following are equivalent.*

(i) $\mathbb{K}[\Delta]$ *is Gorenstein.*

(ii) *For all $F \in \mathrm{core}(\Delta)$ we have*

$$\widetilde{H}_i(\mathrm{link}_{\mathrm{core}(\Delta)}(F), \mathbb{K}) = \begin{cases} \mathbb{K} & \text{if } i = \dim(\mathrm{link}_{\mathrm{core}(\Delta)}(F)) \\ 0 & \text{if } i < \dim(\mathrm{link}_{\mathrm{core}(\Delta)}(F)) \end{cases}$$

(iii) *For all $x \in |\mathrm{core}(\Delta)|$ we have*

$$\widetilde{H}_i(|\mathrm{core}(\Delta)|, \mathbb{K}) = H_i(|\mathrm{core}(\Delta)|, |\mathrm{core}(\Delta)| - x, \mathbb{K}) = \begin{cases} \mathbb{K} & \text{if } i = \dim(\mathrm{link}_{\mathrm{core}(\Delta)}(F)) \\ 0 & \text{if } i < \dim(\mathrm{link}_{\mathrm{core}(\Delta)}(F)) \end{cases}$$

Proof. The equivalence of (ii) and (iii) again follows from Lemma 2.2.

The equivalence of (i) and (ii) is much harder and was orginally proved in [16]. A detailed proof of this fact can be found in [5, Section 5.5.]. □

It follows that if Δ is a simplicial complex for which $\mathbb{K}[\Delta]$ is Gorenstein then $\mathbb{K}[\mathrm{core}(\Delta)]$ is Gorenstein as well. Condition (ii) from Theorem 6.4 then implies for $F = \emptyset$ that $\widetilde{H}_{\dim(\mathrm{core}(\Delta))}(\mathrm{core}(\Delta), \mathbb{K}) \neq 0$ and hence $\mathrm{core}(\Delta)$ is Gorenstein*. Thus any simplicial complex Δ for which $\mathbb{K}[\Delta]$ is Gorenstein has a decomposition $\Delta = 2^{\Omega \setminus \mathrm{core}(\Omega)} * \mathrm{core}(\Delta)$ and $\mathrm{core}(\Delta)$ is Gorenstein*.

Corollary 6.5. *Let Δ be a simplicial complex over ground set Ω and Δ' a simplicial complex over ground set Ω' such that*

- $\mathrm{core}(\Delta) = \Delta$ *and* $\mathrm{core}(\Delta') = \Delta'$,
- $|\Delta|$ *is homeomorphic to* $|\Delta'|$.

Then $\mathbb{K}[\Delta]$ is Gorenstein if and only if $\mathbb{K}[\Delta']$ is Gorenstein*.*

Proof. The result follows from Theorem 6.4(iii) and the fact that $\mathrm{core}(\Delta) = \Delta$ and $\mathrm{core}(\Delta') = \Delta'$. □

Next we consider the Buchsbaum property of $\mathbb{K}[\Delta]$. We refer the reader to [19] for the general theory. For its definition we need the concept of a weak $\mathbb{K}[\Delta]$-sequence. A sequence $f_1, \ldots, f_r$ of elements from the maximal graded ideal of $\mathbb{K}[\Delta]$ is called a weak $\mathbb{K}[\Delta]$ sequence if $\mathfrak{m}((f_1, \ldots, f_{i-1} : f_i) \subseteq (f_1, \ldots, f_{i-1})$ for $i = 1, \ldots, r$. Now $\mathbb{K}[\Delta]$ is called Buchsbaum if every system of parameters is a weak $\mathbb{K}[\Delta]$-sequence. The following is an analog of Reisner's criterion for Buchsbaum rings proved by Schenzel in [15].

Theorem 6.6. *Let Δ be a simplicial complex over ground set Ω. Then the following are equivalent.*

(i) $\mathbb{K}[\Delta]$ *is Buchsbaum.*

(ii) *For all* $F \in \Delta$, $F \neq \emptyset$ *we have* $\widetilde{H}_i(\mathrm{link}_\Delta(F), \mathbb{K}) = 0$ *for* $i < \dim(\mathrm{link}_\Delta(F))$.
(iii) *For all* $x \in |\Delta|$ *we have* $H_i(|\Delta|, |\Delta| - x, \mathbb{K}) = 0$ *for* $i < \dim(\Delta)$.

The equivalence of (ii) and (iii) is again in immediate consequence of Lemma 2.2. The equivalence of (i) and (ii) is Theorem 3.2 in [15]. Its proof first shows a characterization of Buchsbaum $\mathbb{K}[\Delta]$ as those $\mathbb{K}[\Delta]$ for which the localization at all prime ideals different from the graded maximal ideal is Cohen-Macaulay. Using this characterization the equivalence can be reduced to Reisner's criterion Theorem 6.2.

Since condition (iii) from Theorem 6.6 is obviously a topological property, we obtain the following immediate corollary.

Corollary 6.7. *Let* Δ *and* Δ' *be simplicial complexes such that* $|\Delta|$ *and* $|\Delta'|$ *are homeomorphic. Then* $\mathbb{K}[\Delta]$ *is Buchsbaum if and only if* $\mathbb{K}[\Delta']$ *is Buchsbaum.*

7. n-purity, n-Cohen-Macaulay and n-Buchsbaum

A simplicial complex Δ over ground set Ω is called n-pure if for any subset $W \subseteq \Omega$ of cardinality $\#W < n$ we have that $\Delta_{\Omega \setminus W}$ is pure and $\dim(\Delta) = \dim(\Delta_{\Omega\setminus W})$. In particular, 1-pure is the usual pure property.

For $n \geq 3$ the n-pure property is not topological.

Example 7.1. Let Δ be the simplicial complex over ground set $\Omega = \{1, \ldots, n+2\}$ for some $n \geq 1$ with facets $\{i,j\}$ for $1 \leq i < j \leq n$. Then Δ is $(n+1)$-pure. The deletion of an vertex set of size $< n+1$ leaves a connected 1-dimensional simplicial complex. Consider $\Delta' = \mathrm{sd}(\Delta)$ on ground set $\Omega' = 2^\Omega \setminus \{\emptyset\}$. Clearly, $|\Delta|$ and $|\Delta'|$ are homeomorphic. We set $W = \{\{1\},\{2\}\}$ and get that $\Delta'_{\Omega'\setminus W}$ is a simplicial complex with two connected components. One component is a connected 1-dimensional simplicial complex and the other the 0-dimensional complex $\overline{\{\{1,2\}\}}$. In particular, $\Delta'_{\Omega'\setminus W}$ is not pure. Thus Δ' is not $(n+1)$-pure for $n+1 \geq 3 > 2 = \#W$.

Theorem 7.2. *Let* Δ *be a pure simplicial complex over ground set* Ω*. Then the following are equivalent:*
(i) Δ *is 2-pure*
(ii) *If* F *is a face of* Δ *such that* $H_{\dim(\Delta)}(|\Delta|, |\Delta| - x, \mathbb{K}) = 0$ *for all* x *in the relative interior of* $|\bar{F}|$ *then* $\dim(F) \leq \dim(\Delta) - 2$.
(iii) *For any simplicial complex* Δ' *such that* $|\Delta'|$ *and* $|\Delta|$ *are homeomorphic and for all faces* F *of* Δ' *and all* x *from the relative interior of* $|\bar{F}|$ *we have* $H_{\dim(\Delta')}(|\Delta'|, |\Delta'| - x, \mathbb{K}) = 0$.

Proof.

(i) $\Rightarrow$ (ii)

Let F be a face of Δ of dimension $\dim(\Delta) - 1$. Since Δ is pure there must be a facet G of dimension $\dim(\Delta)$ containing F. Let ω be the unique vertex in $G \setminus F$. Since $\Delta_{\Omega\setminus\{\omega\}}$ is of the same dimension as Δ it follows that there must be at least a second facet containing

F. In particular, writing 0 as $\dim(\Delta) - \dim(F) - 1$ we get

$$H_{\dim(\Delta)}(|\Delta|, |\Delta| - x, \mathbb{K}) \overset{Lemma\ 2.2}{=} \widetilde{H}_0(\mathrm{link}_\Delta(F), \mathbb{K}) \neq 0$$

for every x in the relative interior of $|\bar{F}|$.

Let F be a face of Δ of dimensions $\dim(\Delta)$. It follows that

$$H_{\dim(\Delta)}(|\Delta|, |\Delta| - x, \mathbb{K}) \overset{Lemma\ 2.2}{=} \widetilde{H}_{-1}(\mathrm{link}_\Delta(F), \mathbb{K}) = \mathbb{K} \neq 0$$

for any x in the relative interior of $|\bar{F}|$.

These two facts imply (ii).

(ii) $\Rightarrow$ (i)

By assumption, for a face F of dimension $\dim(\Delta) - 1$ we have that

$$H_{\dim(\Delta)}(|\Delta|, |\Delta| - x, \mathbb{K}) = \widetilde{H}_0(\mathrm{link}_\Delta(F), \mathbb{K}) \neq 0.$$

As a consequence there are at least two facets containing F. This implies that for any $\omega \notin F$ there is a facet of dimension $\dim(\Delta)$ containing F in $\Delta_{\Omega \setminus \{\omega\}}$. In particular, $\Delta_{\Omega \setminus \{\omega\}}$ is pure.

(iii) $\Rightarrow$ (ii)

This is obvious.

(ii) $\Rightarrow$ (iii)

Since $|\Delta'|$ is homeomorphic to $|\Delta|$ it follows from Theorem 2.4 that Δ' is pure of the same dimension as Δ. Assume there is a face F of Δ' such that $H_{\dim(\Delta')}(|\Delta'|, |\Delta'| - x, \mathbb{K}) = 0$ for some x from the relative interior of $|\bar{F}|$ and $\dim(F) \geq \dim(\Delta') - 1$. If $\dim(F) = \dim(\Delta')$ then

$$H_{\dim(\Delta)}(|\Delta|, |\Delta| - x, \mathbb{K}) \overset{Lemma\ 2.2}{=} \widetilde{H}_{-1}(\mathrm{link}_\Delta(F), \mathbb{K}) = \mathbb{K} \neq 0.$$

Thus we have $\dim(F) = \dim(\Delta') - 1$. Note that our assumptions imply that for any x' from the relative interior of $|F|$ we have $H_{\dim(\Delta')}(|\Delta'|, |\Delta'| - x', \mathbb{K}) = 0$. But then (ii) shows that the homemomorphic image of the relative interior of $|\bar{F}|$, which is an open $\dim(F)$-ball, must be covered by the relative interiors of $|\bar{G}|$ for faces G of Δ of dimension $\leq \dim(\Delta) - 2 < \dim(F)$. The latter is impossible in the geometric realization of a simplicial complex. Thus (iii) follows. $\square$

The next corollary immediately follows from the fact that condition (iii) in Theorem 7.2 only depends on the homeomorphism type of the geometric realization.

Corollary 7.3. *Let Δ and Δ' be two pure simplicial complexes such that $|\Delta|$ and $|\Delta'|$ are homeomorphic. Then Δ is 2-pure if and only if Δ' is 2-pure.*

A simplicial complex Δ over ground set Ω is called n-Cohen-Macaulay (over $\mathbb{K}$) if for any subset $W \subseteq \Omega$ of cardinality $\#W < n$ we have that $\mathbb{K}[\Delta_{\Omega \setminus W}]$ is Cohen-Macaulay and $\dim(\Delta) = \dim(\Delta_{\Omega \setminus W})$. In particular, 1-Cohen-Macaulay is the usual Cohen-Macaulay property of $\mathbb{K}[\Delta]$.

If Δ and Δ' are the simplicial complexes from Example 7.1 then the arguments in the example show that for $n \geq 2$ we have that $\mathbb{K}[\Delta]$ is $(n+1)$-Cohen-Macaulay but Δ' is not. Thus for $n \geq 3$ the property of being n-Cohen-Macaulay is not topological.

In the thesis of J. Walker [21, Theorem 9.8] it is proved that 2-Cohen-Macaulayness is indeed a topological property. Following idea from [12] we will provide a proof of this result below. As a preparation we need to study properties of links.

Lemma 7.4. *If F is a face of Δ such that for any face $F \subseteq G \in \Delta$ we have $\widetilde{H}_i(\mathrm{link}_\Delta(G), \mathbb{K}) = 0$ for $i < \dim(\mathrm{link}_\Delta(G))$ then for any face $G' \in \mathrm{link}_\Delta(F)$ we have that $\widetilde{H}_i(\mathrm{link}_{\mathrm{link}_\Delta(F)}(G'), \mathbb{K}) = 0$ for $i < \dim(\mathrm{link}_{\mathrm{link}_\Delta(F)}(G'))$.*

In particular, it follows that

(i) *if $\mathbb{K}[\Delta]$ is Cohen-Macaulay then so is $\mathbb{K}[\mathrm{link}_\Delta(F)]$ for every $F \in \Delta$.*
(ii) *if $\mathbb{K}[\Delta]$ is Buchsbaum, then $\mathbb{K}[\mathrm{link}_\Delta(F)]$ is Cohen-Macaulay for every $\emptyset \neq F \in \Delta$.*
(iii) *if $\mathbb{K}[\Delta]$ is 2-Cohen-Macaulay then so is $\mathbb{K}[\mathrm{link}_\Delta(F)]$.*

Proof. If $G' \in \mathrm{link}_\Delta(F)$ then $G = F \cup G' \in \Delta$. Then

$$\begin{aligned} \mathrm{link}_\Delta(G) &= \{H \subseteq \Omega \ : \ H \cap G = \emptyset \text{ and } H \cup G \in \Delta\} \\ &= \{H \subseteq \Omega \ : \ H \cap G' = \emptyset \text{ and } H \cup G' \in \mathrm{link}_\Delta(F)\} \\ &= \mathrm{link}_{\mathrm{link}_\Delta(F)}(G'). \end{aligned}$$

This implies the first assertion of the lemma. The claims (i) about the Cohen-Macaulay and (ii) aboout the Buchsbaum property follow from Theorem 6.2 and Theorem 6.6. For (iii) we argue as follows. By (i) we already know that $\mathbb{K}[\mathrm{link}_\Delta(F)]$ is Cohen-Macaulay for all $F \in \Delta$. Let $\omega \in \Omega$ and set $W = \Omega \setminus \{\omega\}$. If $F \cap W \neq \emptyset$ then there is nothing to show. If $F \cap W = \emptyset$ then

$$\begin{aligned} (\mathrm{link}_\Delta(F))_W &= \{G \subseteq W \ : \ G \cap F = \emptyset \text{ and } G \cup F \in \mathrm{link}_\Delta(F)\} \\ &= \{G \subseteq W \ : \ G \cap F = \emptyset \text{ and } G \cup F \in \mathrm{link}_{\Delta_W}(F)\} \qquad = \mathrm{link}_{\Delta_W}(F). \end{aligned}$$

Now the facts that Δ_W is Cohen-Macaulay and $\dim(\Delta_W) = \dim(\Delta)$ imply the claim. $\square$

As a last prerequisite for a topological characterization of 2-Cohen-Macaulayness we need the following simple fact about chain complexes.

Lemma 7.5. *Let Δ be a simplicial complex and $H \subseteq K$ faces of Δ. Then there is a commutative diagram*

$$\begin{tikzcd} \widetilde{H}_i(\mathrm{link}_\Delta(H), \mathbb{K}) \arrow[r] \arrow[dd, equal] & H_i(\mathrm{link}_\Delta(H), \mathrm{link}_\Delta(H) \setminus (K \setminus H), \mathbb{K}) \arrow[d, equal] \\ & \widetilde{H}_{i+\dim(H)-\dim(K)}(\mathrm{link}_\Delta(K), \mathbb{K}) \arrow[d, equal] \\ H_{i+\dim(H)+1}(\Delta, \Delta \setminus H, \mathbb{K}) \arrow[r] & H_{i+\dim(H)+1}(\Delta, \Delta \setminus K, \mathbb{K}) \end{tikzcd}$$

Where the maps in the rows are given by the long exact sequences of the

$$\textit{pair } (\mathrm{link}_\Delta(H), \mathrm{link}_\Delta(H) \setminus (K \setminus H)) \textit{ and the triple } (\Delta, \Delta \setminus K, \Delta \setminus H)$$

and the maps in the columns are isomorphisms.

Proof. Consider for a simplicial complex Δ', a face E of Δ'. For $i \geq -1$ let $C_{i+\dim(E)+1}(\Delta', \Delta' \setminus E, \mathbb{K})$ be the simplicial chain group in dimension $i + \dim(E) + 1$ and the reduced simplicial chain group $\widetilde{C}_i(\mathrm{link}_\Delta(E), \mathbb{K})$ in dimension i. The first chain group has as a basis the faces E' of Δ' such that $E \subseteq E'$ and $\dim(E') = i + \dim(E) + 1$, the second a has as a basis faces $E'' \in \mathrm{link}_\Delta(E)$ with $\dim(E'') = i$. Now mapping E'' to $E'' \cup E$ establishes a bijection with the two bases which after choosing appropriate orientations extends to an isomorphism of chain complexes.

This fact explains all isomorphism in the columns of the asserted diagram. It is then easily checked that these isomorphisms commutes with the exact sequences of the pair and the triple. The assertion then follows (see [12, Theorem 2.1] for more details). □

Now we are in position to state and prove a result which will immediately imply the result by Walker [21, Theorem 9.8]. For the formulation and the proof of the next theorem we again mostly follow [12].

Theorem 7.6. *Let Δ be a simplicial complex on ground set Ω such that $\mathbb{K}[\Delta]$ is Cohen-Macaualay. Then the following are equivalent*

(i) *Δ is 2-Cohen-Macaulay.*
(ii) *For all $\emptyset \neq F \in \Delta$ the map*

$$\widetilde{H}_{\dim(\Delta)}(\Delta, \mathbb{K}) \to H_{\dim(\Delta)}(\Delta, \Delta \setminus F, \mathbb{K}) \tag{13}$$

from the long exact sequence of the pair $(\Delta, \Delta \setminus F)$ is surjective.

(iii) *For all $\emptyset \neq F \in \Delta$ we have $\widetilde{H}_{\dim(\Delta)-1}(\Delta \setminus F, \mathbb{K}) = 0$.*
(iv) *For all $x \in |\Delta|$ we have $\widetilde{H}_{\dim(\Delta)-1}(|\Delta| - x, \mathbb{K}) = 0$.*

Proof.

(i) ⇒ (ii)

We prove the assertion by induction on $\dim(F)$ for arbitrary Δ for which $\mathbb{K}[\Delta]$ is 2-Cohen-Macaulay. If $\dim(F) = 0$ then $\Delta \setminus F = \Delta_{\Omega \setminus F}$ which is Cohen-Macaulay of dimension $\dim(\Delta)$ by assumption. It follows that $\widetilde{H}_{\dim(\Delta)-1}(\Delta \setminus F, \mathbb{K}) = 0$. Hence by the exactness of the long exact sequence of the pair $(\Delta, \Delta \setminus F)$ the map in (13) must be surjective.

Now let F be a face of dimension $\dim(F) > 0$ and let $\omega \in F$ be some fixed element. We set $G = F \setminus \{\omega\}$. From Lemma 7.4 we know that $\mathrm{link}_\Delta(G)$ is 2-Cohen-Macaulay of dimension $\dim(\Delta) - \dim(G) - 1$. Hence by induction we know that the map

$$\widetilde{H}_{i-\dim(G)-1}(\mathrm{link}_\Delta(G), \mathbb{K}) \to H_{i-\dim(G)-1}(\mathrm{link}_\Delta(G), \mathrm{link}_\Delta(F) \setminus \{\omega\}, \mathbb{K})$$

is surjective. Thus by Lemma 7.5 for $H = G$ and $K = F$ we obtain that the map

$$H_i(\Delta, \Delta \setminus G, \mathbb{K}) \to H_{\dim(\Delta)}(\Delta, \Delta \setminus F, \mathbb{K})$$

is surjective. Again by induction we know that the map

$$\widetilde{H}_{\dim(\Delta)}(\Delta, \mathbb{K}) \to H_{\dim(\Delta)}(\Delta, \Delta, \backslash G, \mathbb{K})$$

is surjective.

By the naturality of the maps it follows that the composition map $\widetilde{H}_{\dim(\Delta)}(\Delta, \mathbb{K}) \to H_{\dim(\Delta)}(\Delta, \Delta \setminus F, \mathbb{K})$ is surjective.

(ii) $\Rightarrow$ (i)

Let $\omega \in \Omega$ and $F \in \Delta$.

If $F \cup \{\omega\} \notin \Delta$ then $\mathrm{link}_{\Delta_{\Omega\setminus\{\omega\}}}(F) = \mathrm{link}_\Delta(F)$. Since Δ is Cohen-Macaulay it follows from Theorem 6.2 that $\widetilde{H}_i(\mathrm{link}_{\Delta_{\Omega\setminus\{\omega\}}}(F), \mathbb{K}) = 0$ for $i < \dim(\mathrm{link}_{\Delta_{\Omega\setminus\{\omega\}}}) = \dim(\mathrm{link}_\Delta(F))$.

We are left with the case when $G = F \cup \{\omega\} \in \Delta$. For that consider the commutative diagram

$$\begin{tikzcd}
\widetilde{H}_{\dim(\Delta)}(\Delta) \arrow[r] \arrow[d] & H_{\dim(\Delta)}(\Delta, \Delta \setminus G, \mathbb{K}) \\
H_{\dim(\Delta)}(\Delta, \Delta \setminus F, \mathbb{K}) \arrow[ur] &
\end{tikzcd}$$

with maps induced by the long exact sequence of the pairs $(\Delta, \Delta \setminus G)$, $(\Delta, \Delta \setminus F)$ and the triple $(\Delta, \Delta \setminus F, \Delta \setminus G)$. The map in the first row is surjective by (ii). Thus the diagonal map is surjective too. By Lemma 7.5 for $H = F$ and $K = G$ we deduce that the map

$$\widetilde{H}_{\dim(\Delta)-\dim(F)-1}(\mathrm{link}_\Delta(F), \mathbb{K}) \to H_{\dim(\Delta)-\dim(F)-1}(\mathrm{link}_\Delta(F), \mathrm{link}_\Delta(F) \setminus \{\omega\}, \mathbb{K})$$

is surjective as well. Since $\mathbb{K}[\mathrm{link}_\Delta(F)]$ is Cohen-Macaulay it follows by Theorem 6.2 that $\widetilde{H}_i(\mathrm{link}_\Delta(F), \mathbb{K}) = 0$ for $i < \dim(\Delta) - \dim(F) - 1$. By Lemma 7.5 $H_i(\mathrm{link}_\Delta(F), \mathrm{link}_\Delta(F) \setminus \{\omega\}, \mathbb{K}) = \widetilde{H}_{i-1}(\mathrm{link}_\Delta(G), \mathbb{K})$. Since $\mathbb{K}[\mathrm{link}_\Delta(G)]$ is also Cohen-Macaulay again by Theorem 6.2 we obtain is $\widetilde{H}_{i-1}(\mathrm{link}_\Delta(G), \mathbb{K}) = 0$ for $i - 1 < \dim(\Delta) - \dim(G) - 1 = \dim(\Delta) - \dim(F) - 2$.

Hence in the long exact sequence of the pair $(\mathrm{link}_\Delta(F), \mathrm{link}_\Delta(F) \setminus \{\omega\})$. We have that

$$\widetilde{H}_i(\mathrm{link}_{\Delta\setminus\{\omega\}}(F), \mathbb{K}) = \widetilde{H}_i(\mathrm{link}_\Delta(F) \setminus \{\omega\}, \mathbb{K}) = 0$$

for $i < \dim(\Delta) - \dim(F) - 1$.

Now it follows from Theorem 6.2 that $\mathbb{K}[\Delta \setminus \{\omega\}]$ is Cohen-Macaualay and hence Δ is 2-Cohen-Macaulay.

(ii) $\Leftrightarrow$ (iii)

Consider the exact sequence

$$\cdots \to \widetilde{H}_{\dim(\Delta)}(\Delta, \mathbb{K}) \to H_{\dim(\Delta)}(\Delta, \Delta \setminus F, \mathbb{K}) \to \widetilde{H}_{\dim(\Delta)}(\Delta \setminus F, \mathbb{K}) \to \widetilde{H}_{\dim(\Delta)-1}(\Delta, \mathbb{K}) \to$$

Since $\mathbb{K}[\Delta]$ is Cohen-Macaulay we know by Theorem 6.2 that $\widetilde{H}_{\dim(\Delta)-1}(\Delta, \mathbb{K}) = 0$. It follows that $\widetilde{H}_{\dim(\Delta)}(\Delta \setminus F, \mathbb{K}) = 0$ if and only if the map $\widetilde{H}_{\dim(\Delta)}(\Delta, \mathbb{K}) \to H_{\dim(\Delta)}(\Delta, \Delta \setminus F, \mathbb{K})$ is surjective.

(iii) $\Leftrightarrow$ (iv)

We know by Lemma 2.2 that $|\Delta_{\Omega\setminus F}|$ is a deformation retract of $|\Delta| - x$ for x in the relative interior of $|\bar{F}|$. In particular, the homology groups of the two spaces coincide. $\square$

The next corollary is an immediate consequence of the fact that condition (iv) of Theorem 7.6 depends only on the homeomorphism type of $|\Delta|$.

Corollary 7.7 (Walker). *Let Δ and Δ' be two simplicial complexes for which $\mathbb{K}[\Delta]$ and $\mathbb{K}[\Delta']$ are Cohen-Macaulay and such that $|\Delta|$ and $|\Delta'|$ are homeomorphic. Then Δ is 2-Cohen-Macaulay if and only if Δ' is 2-Cohen-Macaulay.*

A simplicial complex Δ over ground set Ω is called n-Buchsbaum (over $\mathbb{K}$) if for any subset $W \subseteq \Omega$ of cardinality $\#W < n$ we have that $\mathbb{K}[\Delta_{\Omega\setminus W}]$ is Buchsbaum and $\dim(\Delta) = \dim(\Delta_{\Omega\setminus W})$. In particular, 1-Buchsbaum is the usual Buchsbaum property for $\mathbb{K}[\Delta]$.

Analogous to the case of the Cohen-Macaulay property n-Buchsbaum is not a topological property for $n \geq 3$.

For the $n = 2$ there is the following result [12, Theorem 4.3].

Theorem 7.8. *Let Δ be a simplicial complex such that $\mathbb{K}[\Delta]$ is Buchsbaum. Then the following are equivalent.*

(i) *Δ is 2-Buchsbaum*

(ii) *For any $x \in |\Delta|$ and any neighbourhood U of x in Δ there exists an open set V such that*

  (a) *$x \in V \subseteq U$*

  (b) *The inclusion $|\Delta| \setminus V \hookrightarrow |\Delta| - x$ induces an isomorphisms*

$$\widetilde{H}_i(|\Delta| - V, \mathbb{K}) \to \widetilde{H}_i(|\Delta| - x, \mathbb{K})$$

  for all $i \geq 0$

  (c) *For any $y \in V$ we have $\widetilde{H}_{\dim(\Delta)-1}(|\Delta| - y, \mathbb{K}) = 0$.*

The proof of Theorem 7.8 in [12] is based on arguments similar to those used in the proof of Theorem 7.6. But the deduction becomes more technical and more involved. We refer the reader to the paper [12] for details. Condition (ii) of the preceding result obvious only depends on the homeomorphism type of $|\Delta|$. Therefore, Theorem 7.8 immediately implies the following corollary (see [12, Corollary 4.4]).

Corollary 7.9 (Miyazaki). *Let Δ and Δ' be two simplicial complexes for which $\mathbb{K}[\Delta]$ and $\mathbb{K}[\Delta']$ are Buchsbaum and such that $|\Delta|$ and $|\Delta'|$ are homeomorphic. Then Δ is 2-Buchsbaum if and only if Δ' is 2-Buchsbaum.*

Building on condition (iv) of Theorem 7.6 one can define the class of Buchsbaum* simplicial complexes. A simplicial complex Δ such that $\mathbb{K}[\Delta]$ is Buchsbaum is called Buchsbaum* (over $\mathbb{K}$) if $\widetilde{H}_{\dim(\Delta)-1}(|\Delta|, \mathbb{K}) = \widetilde{H}_{\dim(\Delta)-1}(|\Delta|, \mathbb{K})$ for all $x \in \Delta$. By definition the Buchsbaum* property depends only on the homeomorphism type of $|\Delta|$. In the following results (see [1, Proposition 2.5, 2.8]) the relation of this property to the properties Gorenstein*, 2-Cohen-Macaulay and 2-Buchsbaum is clarified.

Lemma 7.10. *Let Δ be a simplicial complex.*

(i) *If* $\mathbb{K}[\Delta]$ *is Cohen-Macaulay then*

$$\Delta \text{ is } 2 - \text{Cohen-Macaulay} \Leftrightarrow \Delta \text{ Buchbaum}^*.$$

(ii) *If* $\mathbb{K}[\Delta]$ *is Gorenstein then*

$$\Delta \text{ Gorenstein}^* \Leftrightarrow \Delta \text{ Buchsbaum}^*.$$

(iii) *If* Δ *is Buchsbaum** *then* Δ *is* 2*-Buchsbaum.*

The statement in (i) is immediate from the that fact that by Theorem 6.2 we have that $\widetilde{H}_{\dim(\Delta)-1}(\Delta, \mathbb{K}) = 0$ for a Cohen-Macaulay Δ. Statements (ii) and (iii) follows by arguments similar to those used in the proof of Theorem 7.6

8. Other properties

In this section we go over other properties of $\mathbb{K}[\Delta]$ studied in the literature for which the question of whether the property is topological or not was considered. We do not think that the list exhaustive but we have included all results known to us.

An interesting strengthening of the Cohen-Macaulay property was studied in [11]. Here a simplicial complex Δ for which $\mathbb{K}[\Delta]$ is Cohen-Macaulay is called uniformly Cohen-Macaulay (over $\mathbb{K}$) if $\mathbb{K}[\Delta \setminus F]$ is Cohen-Macaulay and $\dim(\Delta \setminus F) = \dim(\Delta)$ for every facet F of Δ. The authors show the following topological characterization in [11, Theorem 1.1].

Theorem 8.1. *Let* Δ *be a Cohen-Macaulay simplicial complex. Then the following are equivalent.*

(i) Δ *is uniformly Cohen-Macaulay.*

(ii) *For every* $x \in |\Delta|$ *the map* $\widetilde{H}_{\dim(\Delta)}(|\Delta|, \mathbb{K}) \to \widetilde{H}_{\dim(\Delta)}(|\Delta|, |\Delta| - x, \mathbb{K})$ *from the long exact sequence of the pair* $(|\Delta|, |\Delta| - x)$ *is an inclusion.*

Clearly, condition (ii) from the theorem depends only the homeomorphism type of $|\Delta|$ and hence the property is topological.

In commutative algebra the Cohen-Macaulay property of a ring is equivalent to the ring having Serre's property (S_d) for the Krull dimension d of the ring. We refer the reader to [5, p. 62] for the defintion of property (S_r) in general. It can be shown, again using Hochster's formula Theorem 5.1 on the local cohomology of $\mathbb{K}[\Delta]$, that $\mathbb{K}[\Delta]$ has property (S_r) if and only if $\widetilde{H}_i(\mathrm{link}_\Delta(F), \mathbb{K}) = 0$ for all $i < \min\{r-1, \dim(\Delta) - \dim(F) - 1\}$. Clearly for $r = d$ we recover Reisner's criterion Theorem 6.2 for the Cohen-Macaulay property of $\mathbb{K}[\Delta]$. In [22, Theorem 4.4] Yanagawa showed that the property (S_r) is topological for any r, which is a vast generalization of Munkres' result Theorem 6.1 on Cohen-Macaulayness.

Theorem 8.2 (Yanagawa)**.** *Let* Δ *and* Δ' *be two simplicial complexes such that* $|\Delta|$ *and* $|\Delta'|$ *are homeomorphic and* $r \geq 0$ *a number. Then* $\mathbb{K}[\Delta]$ *has property* (S_r) *if and only of* $\mathbb{K}[\Delta']$ *has property* (S_r)*.*

The original proof from [22, Theorem 4.4] uses quite heavy machinery from commutative algebra. Recently a short proof was given in [8, Corollary 3].

Equally natural as these weakenings and strengthenings of the Cohen-Macaulay condition are generalizations of the Cohen-Macaulay condition towards pairs of simplicial complexes. A pair (Δ,Γ) of simplicial complexes consists of two simplicial complexes over the same ground set Ω such that Γ is a subcomplex of Δ. For a relative simplicial complex (Δ,Γ) its Stanley-Reisner ideal is the ideal $I_{\Delta,\Gamma}$ in $\mathbb{K}[\Delta]$ generated by the monomials $\mathbf{x}_F$ for $F \in \Delta \setminus \Gamma$. The relative simplicial complex (Δ,Γ) is called Cohen-Macaulay (over $\mathbb{K}$), if the S_Ω module $I_{\Delta,\Gamma}$ is. As for rings the equality of depth and dimension defines Cohen-Macaulayness for modules.

In [18] Stanley deduces the topological invariance of the Cohen-Macaulay property from results in [17, Corollary 5.4]. Topological invariance here means that the property only depends on the homemorphism type of the pair $(|\Delta|,|\Gamma|)$. The proof heavily relies on a relative version of Reisner's criterion Theorem 6.2.

Theorem 8.3 (Stanley)**.** *Let (Δ,Γ) and (Δ',Γ') be two pairs of simplicial complexes such that $(|\Delta|,|\Gamma|)$ and $(|\Delta|',|\Gamma|')$ are homeomorphic pairs of spaces. Then $I_{\Delta,\Gamma}$ is Cohen-Macaulay if and only if $I_{\Delta',\Gamma'}$ is Cohen-Macaulay.*

In the 90s motivated by a series of interesting non-pure simplicial complexes arising in combinatorics, Stanley [18, p. 87] defined the notion of a sequentially Cohen-Macaulay module. We do not want to work with the general definition here. Using [18, Proposition 2.11] we rather define sequential Cohen-Macaulayness for Stanley-Reisner rings $\mathbb{K}[\Delta]$ only. Let Δ be a simplicial complex. For a number $0 \le i \le \dim(\Delta)$ let Δ_i be the simplicial complex of all $F \in \Delta$ such that there is a facet $G \in \Delta$ such that $\dim(G) = i$ and $F \subseteq G$. Then one calls $\mathbb{K}[\Delta]$ sequentially Cohen-Macaulay (over $\mathbb{K}$) if for all $0 \le i \le \dim(\Delta)$ the relative simplicial complex

$$(\Delta_i, \Delta_i \cap (\Delta_{i+1} \cup \cdots \cup \Delta_{\dim(\Delta)})$$

is Cohen-Macaulay over $\mathbb{K}$ (see [6, 3] for equivalent formulations).

Stanley's result on the sequential Cohen-Macaulay property follows from Theorem 8.3.

In [21, Theorem 4.1.6] Wachs provides an obviously topological property which is equivalent to sequential Cohen-Macaulayness.

Theorem 8.4. *Let Δ be a simplicial complex. Then $\mathbb{K}[\Delta]$ is sequentially Cohen-Macaulay if and only if for all $0 \le j < i \le \dim(\Delta)$ and $x \in |\Delta_i|$ we have*

$$\widetilde{H}_j(|\Delta_i|,\mathbb{K}) = H_j(|\Delta_i|,|\Delta_i| - x,\mathbb{K}) = 0.$$

So either using Theorem 8.3 or using Theorem 8.4 we get the Stanley's result as a corollary.

Corollary 8.5 (Stanley)**.** *Let Δ and Δ' be simplicial complexes such that $|\Delta|$ and $|\Delta'|$ are homeomorphic. Then $\mathbb{K}[\Delta]$ is sequentially Cohen-Macaulay if and only if $\mathbb{K}[\Delta']$ is.*

In a similar fashion sequential versions have been attached to other properties of Δ or $\mathbb{K}[\Delta]$. In [8, Corollary 7] the topological invariance of the sequential (S_r) properties is proved. In [3] sequential connectivity and sequential acyclicity are shown to be topological properties.

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