TYPE-B GENERALIZED TRIANGULATIONS AND DETERMINANTAL IDEALS

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Abstract. For $n \geq 3$, let Ω_n be the set of line segments between the vertices of a convex n-gon. For $j \geq 2$, a j-crossing is a set of j line segments pairwise intersecting in the relative interior of the n-gon. For $k \geq 1$, let $\Delta_{n,k}$ be the simplicial complex of (type-A) generalized triangulations, i.e. the simplicial complex of subsets of Ω_n not containing any $(k+1)$-crossing.

The complex $\Delta_{n,k}$ has been the central object of numerous papers. Here we continue this work by considering the complex of type-B generalized triangulations. For this we identify line-segments in Ω_{2n} which can be transformed into each other by a $180°$-rotation of the $2n$-gon. Let $\mathcal{F}_n$ be the set Ω_{2n} after identification, then the complex $\mathcal{D}_{n,k}$ of type-B generalized triangulations is the simplicial complex of subsets of $\mathcal{F}_n$ not containing any $(k+1)$-crossing in the above sense. For $k = 1$, we have that $\mathcal{D}_{n,1}$ is the simplicial complex of type-B triangulations of the $2n$-gon as defined in [33] and decomposes into a join of an $(n-1)$-simplex and the boundary of the n-dimensional cyclohedron. We demonstrate that $\mathcal{D}_{n,k}$ is a pure, $k(n-k)-1+kn$ dimensional complex that decomposes into a $kn-1$-simplex and a $k(n-k)-1$ dimensional homology sphere. For $k = n-2$ we show that this homology-sphere is in fact the boundary of a cyclic polytope. We provide a lower and an upper bound for the number of maximal faces of $\mathcal{D}_{n,k}$.

On the algebraical side we give a term-order on the monomials in the variables $X_{ij}, 1 \leq i,j \leq n$, such that the corresponding initial ideal of the determinantal ideal generated by the $(k+1)$ times $(k+1)$ minors of the generic $n \times n$ matrix contains the Stanley-Reisner ideal of $\mathcal{D}_{n,k}$. We show that the minors form a Gröbner-Basis whenever $k \in \{1, n-2, n-1\}$ thereby proving the equality of both ideals and the unimodality of the h-vector of the determinantal ideal in these cases. We conjecture this result to be true for all values of $k < n$.

1. Introduction and Basic Definitions

Generalized associahedra have been subject to fruitful and intensive study recently (see for example [13]). In this research associahedra are defined uniformly for all root systems – the classical associahedron being the type-A case. In a second stream originating in the work of Dress, Koolen & Moulton [8] and Nakamigawa [27] (see also [9], [20], [21]) a generalization of the type-A associahedron into a different direction has been shown to exhibit very nice combinatorial, geometric and algebraic properties.

In this paper we carry out an analogous generalization for the type-B associahedron – the cyclohedron (see for example [33], [30]). We verify similar nice combinatorial, geometric and algebraic properties. In particular, we show that it is a

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homology sphere, give bounds on the number of facets and relate it to determinantal ideals.

Note, that by polytope duality there are two polytopes – one simple, one simplicial – that can be called associahedron (resp. cyclohedron). In this paper, by associahedron (resp. cyclohedron) we will always mean the simplicial polytope.

The paper is organized as follows. This section gives a brief introduction to the topic and provides basic definitions on simplicial complexes and Gröbner bases.

In Section 2 we recall the results on the generalized type-A associahedron. In Section 3 we then state the main results of this paper. In Sections 4, 5 and 7 we then provide the proofs of the main results.

In order to recall known facts and formulate our own results we need to introduce basic notions about simplicial complexes and Gröbner bases.

A simplicial complex on vertex-set V is a set-system $\Delta \subset 2^V$ such that $\sigma \in \Delta$ implies $\tau \in \Delta$ for all $\tau \subset \sigma$. Elements of Δ are called faces and inclusionwise maximal faces are called facets. The dimension $\dim \sigma$ of a face $\sigma \in \Delta$ is the number of elements in σ reduced by one. The dimension $\dim \Delta$ of the complex Δ is the maximal dimension of a facet in Δ. If all facets have the same dimension the complex is said to be pure. The f-vector of a $(d-1)$-dimensional complex $f(\Delta) = (f_{-1}, f_0, \ldots, f_{d-1})$ counts the faces by their dimension, i.e. f_i is the number of faces of dimension i. The h-vector is a transformation of the f-vector defined as follows. We set $h(t) := f(t-1) = \sum_{i=0}^{d} h_i t^{d-i}$ – the h-polynomial – for $f(t) := \sum_{i=0}^{d-1} f_{i-1} t^{d-i}$ – the f-polynomial – and obtain the h-vector by $h(\Delta) = (h_0, \ldots, h_d)$. A simplicial complex may be defined by its inclusion-maximal faces, the facets, or by its inclusion-minimal nonfaces. The ideal I_Δ in $T = k[x_v \mid v \in V]$ generated by $\prod_{v\in\sigma} x_v$ for the inclusion minimal non-faces σ is called Stanley-Reisner ideal of Δ and $k[\Delta] := T/I_\Delta$ the Stanley-Reisner ring of Δ. The Krull-dimension of $k[\Delta]$ is well known to equal $\dim \Delta + 1$ and the entry f_{d-1} for $d-1 = \dim \Delta$ is known to be the multiplicity of $k[\Delta]$ (see [4, Chapter 5] for more details) . More generally, the Hilbert-series of $k[\Delta]$ for a d-dimensional simplicial complex Δ with h-vector $(h_0, \ldots, h_d)$ is given by

$$\mathrm{Hilb}(k[\Delta], t) = \frac{h_0 + \cdots + h_d t^d}{(1-t)^{d+1}}.$$

For simplicial complexes Δ, Σ over disjoint vertex-sets V_1, V_2 one defines the join $\Delta \star \Sigma$ as the simplicial complex on vertex-set $V_1 \cup V_2$ and simplices $\tau = \delta \cup \sigma$ for $\delta \in \Delta, \sigma \in \Sigma$. The minimal non-faces of $\Delta * \Sigma$ are the minimal non-faces of Δ and of Σ. In particular, $k[\Delta * \Sigma] = k[\Delta] \otimes k[\Sigma]$. Note, that in the special case when Σ is simplex all minimal non-faces of $\Delta * \Sigma$ are minimal non-faces of Δ. This case will become crucial later in this work.

Since the Stanley-Reisner ideals are monomial ideals (i.e., ideals generated by monomials), we will need Gröbner basis theory and initial ideals for establishing the link between the Stanley-Reisner ideals of our complexes and determinantal ideals in Section 3. In the following paragraph we introduce the basic terminology of Gröbner basis theory in our setting (for more details see [1]).

Let $\preceq$ be a term order on the monomials in the polynomial ring $S = k[x_1, \ldots, x_n]$. For a polynomial $f \in S$ we let $\mathrm{lm}_\preceq(f)$ be the leading monomial of f, i.e. the largest monomial appearing with non-zero coefficient in f with respect to $\preceq$. For a subset $M \subset S$ let $\mathrm{in}_\preceq(M)$ be the ideal generated by the set of monomials $\{\mathrm{lm}_\preceq(f) \big| f \in$

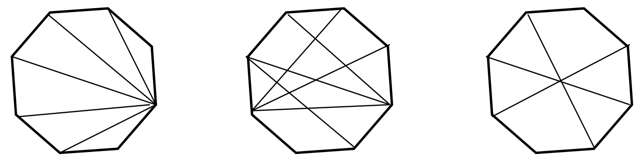

FIGURE 1. Generalized type-A k-triangulations for $n = 8, k = 1, 2, 3$.

$M\}$. Recall that a Gröbner-basis G of an ideal I is a subset $G \subseteq I$ such that $\mathrm{in}_{\preceq}(G) = \mathrm{in}_{\preceq}(I)$. It is well known that $\mathrm{in}_{\preceq}(I)$ and I share their Hilbert-series. In particular, their Krull-dimension and multiplicity coincide (see [1]).

2. Type-A k-Triangulations

The definition of type-A k-triangulations takes advantage of geometric properties of the convex n-gon, $n \geq 3$, which we define as the convex hull of the n-th roots of unity $\{1 = \xi_0, \ldots, \xi_{n-1}\}$ in $\mathbb{C} \cong \mathbb{R}^2$ numbered in clockwise order. A diagonal between the ith and jth root of unity is the line segment $\partial_{i,j} := \{\lambda\xi_i + (1-\lambda)\xi_j \big| \lambda \in (0,1)\} \subset \mathbb{R}^2$ which we may identify with the set $\{i, j\}$ and often denote as ij. Let $\Omega_n = \{ij \subset [n],\, i \neq j\}$ be the set of diagonals of the n-gon. A $(k+1)$-crossing is a $(k+1)$-set of diagonals mutually intersecting in the relative interior of the n-gon. As a consequence, diagonals in the set $\Gamma_{n,k} := \{ij \big| n - k \leq |j - i| \leq k\}$ cannot be part of a $(k+1)$-crossing. For $n \geq 2k + 1$ and $k \geq 1$ one defines $\Delta_{n,k}$ as the simplicial complex on vertex-set Ω_n and the minimal nonfaces being the set of all $(k+1)$-crossings. When $k = 1$, the facets of this complex correspond to triangulations of the n-gon. It is easy to see that $\Delta_{n,k}$ can be decomposed as a join $\Delta_{n,k} =: \Delta^*_{n,k} \star 2^{\Gamma_{n,k}}$, where the complex $\Delta^*_{n,k}$ is the set of all faces of $\Delta_{n,k}$ contained in $\Omega_{n,k} := \Omega_n \setminus \Gamma_{n,k}$. Following [20], we call facets of both complexes $\Delta_{n,k}$ and $\Delta^*_{n,k}$ (type-A)-GENERALIZED k-TRIANGULATIONS. The prefix type-A is motivated by the theory of cluster-complexes where the complex $\Delta_{n,1}$ is shown to be the cluster-complex of an arbitrary cluster-algebra of type A_n, while the complex $\mathcal{D}_{n,1}$ to be defined in the next section is the cluster-complex of type B_n cluster-algebras. For more information we refer the reader to [12] and [13].

Both complexes $\Delta_{n,k}$ and $\Delta^*_{n,k}$ have been the central objects of several papers. In the sequel we will list some more recent results, most of them originating in work of Dress, Koolen and Moulton [11], and refer the reader to [26] for prior developments.

Theorem 1 ([11],[27]). *The complex $\Delta^*_{n,k}$ is pure of dimension $k(n - 2k - 1) - 1$.*

Theorem 2 ([20]). *The number of facets of $\Delta_{n,k}$ resp. $\Delta^*_{n,k}$ is given by the following determinant:*

$$\det \begin{pmatrix} C_{n-2} & C_{n-3} & \cdots & C_{n-k} & C_{n-k-1} \\ C_{n-3} & C_{n-4} & \cdots & C_{n-k-1} & C_{n-k-2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ C_{n-k-1} & C_{n-k-2} & \cdots & C_{n-2k+1} & C_{n-2k} \end{pmatrix} = \prod_{1 \leq i \leq j \leq n-2k-1} \frac{i + j + 2k}{i + j}$$

where $C_n = \frac{1}{n+1}\binom{2n}{n}$ is the nth Catalan-Number.

Recently Krattenthaler [22] gave an alternative proof of Theorem 2 in the context of Fomin's growth diagrams (see also [7] and [31] for related work).

Theorem 3 ([8]). *The geometric realization of the simplicial complex $\Delta^*_{n,k}$ is a piecewise linear sphere.*

Since every boundary complex of a simplicial polytope is a piecewise linear sphere, Theorem 3 has raised the question whether $\Delta^*_{n,k}$ is polytopal, which is commonly believed to have a positive answer.

The question of polytopality takes its appeal not only from geometry, but also from enumerative combinatorics. An affirmative answer would reveal certain properties of the h-vector of $\Delta_{n,k}$ as they are stated in the famous g-theorem by Billera, Lee, McMullen and Stanley, see [17]. A consequence of special interest in enumerative combinatorics is the unimodality ($h_0 \leq h_2 \leq h_{\lfloor \frac{d}{2} \rfloor} = h_{\lfloor \frac{d+1}{2} \rfloor} \geq \ldots h_d$) of the h-vector.

In special cases the polytopality has been ascertained:

Proposition 4. [8] *The complex $\Delta^*_{n,k}$ is the boundary-complex of a simplicial polytope for $2k+1 \leq n \leq 2k+3$ and for $k = 1$, If $k = 1$, this polytope is the well known associahedron, for $n = 2k+1$ it is the (-1)-sphere, for $n = 2k+2$ it is the k-simplex and for $n = 2k+3$ it is a cyclic polytope.*

Recall that the cyclic polytope $\mathcal{C}_d(n)$ is the convex hull of n different points on the moment curve $M_d := \{(1, t, t^2, \ldots, t^d),\ t \in \mathbb{R}\}$. It is well known that the combinatorics of its boundary-complex does not depend on the choice of the points. Polytopal realizations of the associahedron can be found in [16], [23] and [24].

Finally, there is an unexpected relation of $\Delta_{n,k}$ and the ideal $P_{n,k}$ of Pfaffians of degree $k+1$ of a generic $n \times n$-skew symmetric matrix in the indeterminates x_{ij}, $1 \leq i < j \leq n$. For that we identify the variable in the Stanley-Reisner ring of $\Delta_{n,k}$ indexed with the diagonal $\{i, j\}$ with the variable x_{ij}.

Theorem 5 ([21]). *For $2 \leq 2k+2 \leq n$ there is a term order $\preceq$ for which $\mathrm{in}_{\preceq}(P_{n,k}) = I_{\Delta_{n,k}}$.*

3. Main results and type-B k-triangulations

For the type-B case we identify line-segments in Ω_{2n} which can be transformed into each other by a 180°-rotation of the $2n$-gon. We will write $\bar{M}$ for a rotated set $M \subset \Omega_{2n}$ and $\bar{d}$ for the rotated $d \in \Omega_{2n}$.

Line-segments with $d = \bar{d}$, i.e. those that cross the origin in the $2n$-gon, will be called diameters. Let $\mathcal{F}_n$ be the set Ω_{2n} after identification, then for $1 \leq k \leq n-1$ the complex of type-B generalized k-triangulations $\mathcal{D}_{n,k}$ is the simplicial complex of subsets of $\mathcal{F}_n$ not containing any $(k+1)$-crossing in the above sense.

When $k = 1$, the facets of this complex correspond to type-B triangulations of the $2n$-gon as defined in [33].

Let $\mathcal{F}_{n,k}$ be the set of classes in $\Omega_{2n,k}$ and $\Theta_{n,k}$ the set of classes in $\Gamma_{2n,k}$ with respect to the identification mentioned above. Again it is easy to see that $\mathcal{D}_{n,k}$ can be written as a join $\mathcal{D}_{n,k} =: \mathcal{D}^*_{n,k} \star 2^{\Theta_{n,k}}$, where the complex $\mathcal{D}^*_{n,k}$ is defined on the vertex-set $\mathcal{F}_{n,k}$. Facets of $\mathcal{D}^*_{n,k}$ will be called type-B generalized k-triangulations. See Figure 2 for some examples.

We will also consider the complex $\Delta^{*\text{symm}}_{2n,k}$ that we define to be the subcomplex of $\Delta^*_{2n,k}$ which is generated by the facets F of $\Delta_{2n,k}$ with $F = \bar{F}$.

We are now in the position to state our main results.

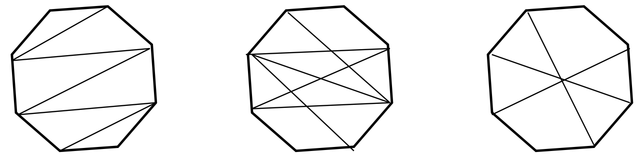

FIGURE 2. Generalized type-B k-triangulations for $n = 4, k = 1, 2, 3$.

Theorem 6. *For all $1 \leq k \leq n-1$ each facet of $\Delta^{*symm}_{2n,k}$ contains exactly k diameters.*

As an instant consequence of the preceding theorem and Theorem 1 we get:

Corollary 7. *The complex $\mathcal{D}^*_{n,k}$ is pure of dimension $k(n-k)-1$.*

Concerning polytopality we prove a type-B analogue of Proposition 4:

Proposition 8. *The complex $\mathcal{D}^*_{n,k}$ is the boundary-complex of a simplicial polytope for $k \in \{1, n-2, n-1\}$. If $k = 1$, this polytope is the cyclohedron ([33]), for $k = n-1$ it is the k-simplex and for $k = n-2$ it is a cyclic polytope.*

The cyclohedron was first introduced by Stanley in [32]. For an explicit realization as a polytope see [18]. This supplies us with the formulation of the following conjecture.

Conjecture 9. *The complex $\mathcal{D}^*_{n,k}$ is polytopal for all $1 \leq k \leq n-1$.*

We cannot prove Conjecture 9 in its full strength. But as a step towards the conjecture we prove the following theorem.

Theorem 10. *For $1 \leq k \leq n-1$ the complex $\mathcal{D}^*_{n,k}$ is a $\bmod 2$-homology-sphere.*

For a prime p, we call a simplicial complex Δ a $\bmod p$-homology-sphere if for each face $\sigma \in \Delta$ we have

$$H_\star(\mathrm{link}_\Delta(\sigma), \mathbb{Z}_2) \cong H_\star(S^{k(n-k)-1-\dim\sigma-1}, \mathbb{Z}_2).$$

In particular, for $\sigma = \emptyset$ we then have

$$H_\star(\mathrm{link}_\Delta\emptyset, \mathbb{Z}_2) = H_\star(\Delta, \mathbb{Z}_2) \cong H_\star(S^{k(n-k)-1}, \mathbb{Z}_2)$$

.

Theorem 10 provides us with the prerequisites in order to obtain bounds on the number of generalized type-B triangulations. While the upper bound is an immediate consequence of [28],[29] the proof of the lower bound can be found in Section 7.

Theorem 11. *For all $1 \leq k \leq n-1$ the number $T(n,k)$ of type-B generalized k-Triangulations of the $2n$-gon satisfies*

- *the lower bound*

$$T(n,k) \geq \det\left[\binom{2n-i-j}{n-i}\right]_{i,j=1,\ldots,k} = \det\left[\binom{2(n-k)}{n-k+i-j}\right]_{i,j=1,\ldots,k} = \prod_{h=1}^{n-k}\prod_{i=1}^{k}\prod_{j=1}^{n-k}\frac{h+i+j-1}{h+i+j-2} \tag{3.1}$$

- *the upper bound given by the number of facets of the $k(n-k)$ dimensional cyclic polytope with $n(n-k)$ vertices*

$$T(n,k) \leq \begin{cases} 2\cdot\binom{(n-k)^2+\frac{k(n-k)-1}{2}}{(n-k)^2} & \text{if } k(n-k) \text{ odd,} \\ 2\cdot\binom{(n-k)^2+\frac{k(n-k)}{2}}{(n-k)^2}+\binom{(n-k)^2-1+\frac{k(n-k)}{2}}{(n-k)^2-1} & \text{if } k(n-k) \text{ even.}\end{cases}$$

Remark 12. *If we fix $l = (n-k)$ and let n and therefore k go to infinity, the quotient of the lower and the upper bound converges to $\frac{2!\cdot 3!\cdots(l-1)!\cdot l^2!}{2\cdot l!\cdot(l+1)!\cdots(2l-1)!\cdot(l/2)^{l^2}}$.*

It has been shown in [33] that $T(n,1) = \binom{2n-2}{n-1}$, which means that the lower bound is met in this case, which is also true when $\mathcal{D}_{n,n-1}$ is the simplex of dimension $(n-1)$. Proposition 8 yields that the lower bound and the upper bound coincide in the case $k = n-2$. This and results in Section 7 lead us to the next conjecture.

Conjecture 13. *The inequality (3.1) is an equality.*

Finally, we provide a description of the Stanley-Reisner ideal of $\mathcal{D}_{n,k}$ in terms of submatrices. This will lead to a partially conjectural connection of $\mathcal{D}_{n,k}$ and the ideal of minors of degree $k+1$ of a generic $n \times n$ matrix, analogous to Theorem 5. In this case the identification of the set of variables used in the Stanley-Reisner ideal of $\mathcal{D}_{n,k}$ and the set of variables of a generic $n \times n$ matrix is not as obvious as in the Pfaffian case. Therefore, for the description of $I_{D_{n,k}}$ in terms of the entries of a generic $n \times n$ matrix, we apply the following bijections between the ground set of $D_{n,k}$ and the index pairs (i,j) for $1 \leq i,j \leq n$.

$$\Psi: \quad \{\{a_1 < b_1\},\{a_2 < b_2\}\} \mapsto \begin{cases} ((a_i+1) \bmod n, b_i \bmod n) & \text{if } b_i - a_i \leq n \\ ((b_i+1) \bmod n, a_i \bmod n) & \text{if } b_i - a_i > n,\end{cases}$$

The map Ψ will be studied in more detail in the proof of Theorem 14. Let $1 \leq k < n$. For nonempty subsets $A, B \subset [n]$ with $\sharp A = \sharp B = (k+1)$ we define $M(A,B) := (x_{ij})_{i\in A, j\in B}$ to be the corresponding row- and column-selected submatrix. In the following when we consider the integers modulo some number m then we always choose $\{1,\ldots,m\}$ as the system of representatives of numbers modulo m.

$$N(A,B) := \{(a_{(i+l) \bmod (k+1)}, b_i), i = 1,\ldots,(k+1)\} \tag{3.2}$$

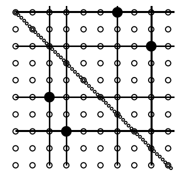

FIGURE 3. $N(A,B)$ for $n = 10$, $k = 3$, $A = \{1,3,6,8\}$, $B = \{3,4,7,9\}$

where $(k+1) \geq \ell$ is chosen such that the following two conditions hold:

$$a_{i+\ell} \quad > \quad b_i \text{ for all } i = 1, \ldots, (k+1) - l \tag{3.3}$$

$$a_{j+\ell-1} \quad \leq \quad b_j \text{ for one } j \in \{1, \ldots, (k+1) - l + 1\} \text{ or } l = 0. \tag{3.4}$$

For $k+1 = 1$ we let $N(A,B) = \{(a_1, b_1)\}$. (See Figure 3 for an example.) The set $N(A,B)$ consists of all indices of matrix entries on the longest diagonal of $M(A,B)$ that lies strictly below the main-diagonal of X augmented by the indices of the entries on the complementary diagonal.

Using the identification of vertices and variables provided by the map Ψ we obtain the following description of the Stanley-Reisner ideal of $\mathcal{D}^*_{n,k}$.

Theorem 14. *The Stanley-Reisner ideal of $\mathcal{D}^*_{n,k}$ is generated by all monomials*

$$m_{A,B} = \prod_{(i,j) \in N(A,B)} x_{ij} \text{ such that } \sharp A = \sharp B = (k+1).$$

The preceding result allows us to connect the complex $\mathcal{D}^*_{n,k}$ to the determinantal ideal $I_{n,k}$. For $1 \leq k < n$ the ideal $I_{n,k}$ is defined as follows. Let $S := \mathbb{K}[x_{11}, \ldots, x_{nn}]$ be the polynomial ring over the field $\mathbb{K}$ and consider the x_{ij} as entries in the $(n \times n)$-matrix X of indeterminates. Then $I_{n,k}$ is the ideal generated by the $(k+1)$-minors, i.e. by the formal determinants of all square-submatrices of size $(k+1) \times (k+1)$ of X. The determinantal ideal $I_{n,k}$ is one of the classical objects in commutative algebra which offers numerous links to the theory of invariants, group representation theory and combinatorics. For more details see [5] and [2].

In Section 7 we will establish a term order $\preceq$ for which $I_{D_{n,k}} \subseteq \text{in}_{\preceq}(I_{n,k})$ (see Theorem 29). Indeed, we verify that for $I = I_{D_{n,k}}$ and $J = \text{in}_{\preceq}(I_{n,k})$ all assumptions of the following lemma are satisfied except for (ii). In [21] we were able to verify all assumptions of the lemma for ideals of Pfaffians leading to Theorem 5. We are grateful to Ezra Miller for pointing out to us that various generalizations and disguises of Lemma 15 appear in the literature. In particular, We would like to mention [6, Lemma 4.2] which together with Exercise 8.13 from [25] leads to the seemingly most general version (see also the notes to Chapter 8 in [25] for further references). We refer to the version of the result from [21] since its formulation suits our needs best.

Lemma 15 ([21])**.** *Let $T = k[x_1, \ldots, x_\ell]$ be the polynomial ring in ℓ variables. Suppose that $I \subseteq J$ are monomials ideals in T such that the following hold:*

(i) $\dim(T/I) = \dim(T/J)$.
(ii) $e(T/I) = e(T/J)$.
(iii) $I = I_\Delta$ *for a pure simplicial complex* Δ *on ground set* $[\ell]$.

Then $I = J$.

Thus in order to establish equality between $I = I_{D_{n,k}}$ and $J = \mathrm{in}_{\preceq}(I_{n,k})$ it remains to verify condition (ii). Even though we are not able to prove this fact, we can deduce one inequality from the fact that $I \subseteq J$.

Lemma 16. *Let Δ be a pure simplicial complex over the vertex-set $\{x_1, \ldots, x_n\}$ and let J be a monomial ideal in $S = \mathbb{K}[x_1, \ldots, x_n]$. If the Stanley-Reisner ideal I_Δ is contained in J and their Krull-dimensions coincide, then for the multiplicities holds*

$$e(S/I_\Delta) \geq e(S/J).$$

Proof. We consider the polarization J^{pol} of J, i.e. we replace every occurrence of $x_i^{\alpha_i}$ in a generator from the unique minimal set of monomial generators of J by a product of additional variables $x_i^{(1)}, \ldots, x_i^{(\alpha_i)}$ and then consider the ideal generated by them in the ring S', which we may write as $k[x_1, \ldots, x_{n+m}]$ after a suitable renaming of the variables. The ideal J^{pol} is squarefree, therefore exists a simplicial complex Γ, such that for the Stanley-Reisner ideal I_Γ holds: $J^{pol} = I_\Gamma$. Now we take ideal generated by I_Δ in S'; it corresponds to a squarefree ideal I', that can be interpreted as the Stanley-Reisner ideal of the complex $\Delta \star 2^{\{x_{n+1}, \ldots, x_{n+k}\}}$. (Where we write $2^{\{x_{n+1}, \ldots, x_{n+k}\}}$ for the simplex with vertex-set $\{x_{n+1}, \ldots, x_{n+k}\}$). All in all we obtain for both Stanley-Reisner ideals:

$$I_{\Delta \star 2^{\{x_{n+1}, \ldots, x_{n+k}\}}} \subset I_\Gamma.$$

We can conclude for the corresponding simplicial complexes:

$$\Gamma \subset \Delta \star 2^{\{x_{n+1}, \ldots, x_{n+k}\}} \text{ and } \dim \Gamma = \dim \left(\Delta \star 2^{\{x_{n+1}, \ldots, x_{n+k}\}}\right).$$

Therefore the number of faces of maximal dimension of Γ is smaller or equal than the number of facets of $\Delta \star 2^{\{x_{n+1}, \ldots, x_{n+k}\}}$, and therefore smaller or equal to the number of facets of Δ. Since Δ is a pure complex, this number is the multiplicity of Δ which equals the multiplicity of S/I_Δ. $\square$

For the special cases $k \in \{1, n-2, n-1\}$ we indeed can establish assumption (ii) of Lemma and hence show that $I_{D_{n,k}} = \mathrm{in}_{\preceq}(I_{n,k})$. Moreover, the term-order $\preceq$ satisfies the following conditions:

(S) The initial ideal $\mathrm{in}_{\preceq}(I_{n,k})$ is the Stanley-Reisner ideal of the simplicial complex $\mathcal{D}_{n,k}$ which decomposes into a join $\mathcal{D}_{n,k}^* \star 2^\Theta$ of a triangulation $\mathcal{D}_{n,k}^*$ of a sphere and a full simplex 2^Θ on ground set Θ.

(M) The cardinality of the set Θ is the absolute value of the a-invariant of the quotient of the polynomial ring by $I_{n,k}$.

Term-orders and ideals that satisfy these two condition have recently appeared in several places in the literature (see [6] for some original results in this direction and an exhaustive survey of the previous known instances). For the type-A case of our situation these are the results from [21]. Initial ideals which satisfy (S) and (M) will be called spherical initial ideals.

Thus the a verification of Conjectures 9 and 13 would imply:

Conjecture 17. *For the term-order $\preceq$ defined in Section 7 the initial ideal of $I_{n,k}$ is a spherical initial ideal.*

With the help of Lemma 15 we get the following equivalent formulations of Conjecture 13.

Lemma 18. *The following statements are equivalent:*

- *The $(k+1)$-minors form a Gröbner-bases for the term-order $\preceq$.*
- *The Stanley-Reisner Ideal of $\mathcal{D}_{n,k}$ coincides with $\mathrm{in}_{\preceq}(I_{n,k})$.*
- *The number of generalized type-B triangulations is counted according to Conjecture 13.*

4. Proof of Theorem 6

To determine the dimension and ascertain the pureness of $\mathcal{D}_{n,k}$ we symmetrize an approach of Jonsson [20]. The original formulation in terms of triangulations was present in a preprint version [19] of [20] but was replaced by a formulation in terms of polyominos. Indeed, all the results mentioned below are special cases of results from [20], but explicitly stated only in the preprint [19].

In this section we will often consider the complexes $\Delta_{n,k}$ and $\Delta_{n-1,k}$ in parallel. It turns out that in the proofs it is convenient to number the vertices of the n-gon in the definition of $\Delta^{\star}_{n,k}$ from 0 to $n-1$ and the vertices of the $(n-1)$-gon in the definition of $\Delta_{n-1,k}$ by 1 to $n-1$.

By a subtle observation of Jonsson [19] there exist sets of diagonals B, B_1 in the n-gon resp. $(n-1)$-gon such that

$$\mathrm{link}_{\Delta^*_{n,k}} B \cap \Omega_{n,k} \quad \cong \quad \mathrm{link}_{\Delta^*_{n-1,k}} B_1 \cap \Omega_{n-1,k}. \tag{4.1}$$

Recall that the link of a complex Δ with vertex-set V with respect to a face $\sigma \in \Delta$ is defined as

$$\mathrm{link}_{\Delta}(\sigma) := \{\tau \in \Delta \big| \tau \cup \sigma \in \Delta, \tau \cap \sigma = \emptyset\}.$$

The definition of the sets B, B_1 is easier with the help of the set $\gamma_{n,k} := \Gamma_{n,k} \setminus \Gamma_{n,k+1}$.

Lemma 19 ([19, 20])**.** *For every facet F in $\Delta^*_{n,k}$ exists a unique set $B := B_1 \cup B_0$ with*

$$B_1 := \{1b_1, 2b_2, \ldots, kb_k\}$$

and

$$B_0 := \{0b_1, 1b_2, \ldots, k-1b_k\},$$

such that

$$k+1 \leq b_1 < b_2 < \cdots < b_k \leq (n-1) \text{ and } B \subset F \cup \gamma_{n,k}.$$

If we additionally define

$$\begin{aligned} Z_B &:= \{ij \big| b_i < j < b_{i+1}, i = 0, \ldots, k\} \\ K_B &:= \{ij \big| k+1 \leq i < b_1, b_k < j \leq n-1\} \end{aligned}$$

we get that

$$\begin{aligned} Z_B \cap F &= \emptyset \\ K_B \cap F &= \emptyset. \end{aligned}$$

The isomorphism in (4.1) is defined with the help of a partition of the vertex-sets as follows:

Lemma 20 ([19, 20]). *For a given facet F of $\Delta^*_{n,k}$ let B_0, B_1, K_B, Z_B be defined as in Lemma 19. We get the partitions*

$$\begin{array}{rcl} \Omega_{n,k} & = & B \cup Z_B \cup K_B \cup S_0 \cup S_1 \cup S_2 \\ \Omega_{n-1,k} & = & B_1 \cup K_B \cup S_0' \cup S_1 \cup S_2 \\ V_{n,k} & = & S_0 \cup S_1 \cup S_2 \\ V_{n-1,k} & = & S_0' \cup S_1 \cup S_2. \end{array}$$

where the latter sets are the vertex-sets $V_{n,k}$ of $\operatorname{link}_{\Delta^*_{n,k}}(B \cap \Omega_{n,k})$ and $V_{n-1,k}$ of $\operatorname{link}_{\Delta^*_{n-1,k}}(B_1 \cap \Omega_{n-1,k})$ by defining

$$\begin{array}{rcl} S_0 & := & \{(i-1)j \big| i \in [1,k], b_i < j \leq n-k-2+i\} \\ S_0' & := & \{ij \big| i \in [1,k], b_i < j \leq n-k-2+i\} \\ S_1 & := & \{ij \big| 1 \leq i \leq k, i+k+1 \leq j < b_i\} \\ S_2 & := & \{ij \big| k+1 \leq i, j \leq n-1\} \cap \Omega_{n,k}) \backslash K_B. \end{array}$$

Lemma 21 ([19, 20]). *The mapping*

$$\begin{gathered} \varphi_B : S_0 \cup S_1 \cup S_2 \cup B \longrightarrow S_0' \cup S_1 \cup S_2 \cup B_1 \\ ij \mapsto \begin{cases} (i+1)j & \text{if } ij \in B_0 \cup S_0 \\ ij & \text{if } ij \in B_1 \cup S_1 \cup S_2 \end{cases} \end{gathered}$$

induces an isomorphism

$$\operatorname{link}_{\Delta^*_{n,k}} B \cap \Omega_{n,k} \cong \operatorname{link}_{\Delta^*_{n-1,k}} B_1 \cap \Omega_{n-1,k}.$$

An immediate consequence of the foregoing lemmas is

Lemma 22. *For a given facet F and its unique set B, the vertex-set $V^{symm}_{2n,k}$ of $\operatorname{link}_{\Delta^{symm}_{2n,k}}(B \cup \bar{B}) \cap \Omega_{2n,k}))$ can be written as*

$$V^{symm}_{2n,k} \quad = \quad (S_0 \cup S_1 \cup S_2) \cap (\bar{S}_0 \cup \bar{S}_1 \cup \bar{S}_2).$$

Again it turns out to be convenient to let $\Delta^{*\mathrm{symm}}_{2n,k}$ be coming from a $2n$-gon numbered from 0 to $2n-1$ and $\Delta^{*\mathrm{symm}}_{2n-2,k}$ be defined on the $2n-2$-gon with numbering of the vertices from 1 to $n-1$ and from $n+1$ to $2n-1$.

We define the effect of

$$\varphi : V^{symm}_{2n,k} \cup B \cup \bar{B} \to \Omega_{2n-2,k}$$

on the diagonal ij with $i < j$ by the table in Figure 4, each cell corresponding to the intersection of the sets in the respective row and column, this intersection being empty for cells marked with Φ. The mapping φ can be seen in action in Figure 5.

	S_0	S_1	S_2	B_0	B_1
$\bar{S}_0$	$(i+1)(j+1)$	$i(j+1)$	$i(j+1)$	$(i+1)(j+1)$	$i(j+1)$
$\bar{S}_1$	$(i+1)j$	ij	ij	$(i+1)j$	ij
$\bar{S}_2$	$(i+1)j$	ij	ij	$(i+1)j$	ij
$\bar{B}_0$	$(i+1)(j+1)$	$i(j+1)$	$i(j+1)$	$(i+1)(j+1)$	Φ
$\bar{B}_1$	$(i+1)j$	ij	ij	Φ	ij

FIGURE 4. Definition of φ.

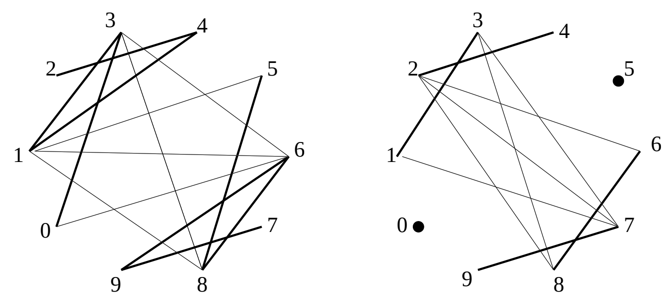

FIGURE 5. Action of φ on a facet of $\Delta^{*\mathrm{symm}}_{5,2}$, elements of $B \cup \bar{B}$ printed bold.

Lemma 23.

(1) *The mapping φ preserves rotational symmetry, i.e. $\varphi(\bar{e}) = \overline{\varphi(e)}$ for all diagonals e in the domain of φ.*
(2) *Each diameter in $\Omega_{2n-2,k}$ is the image of a diameter in $V^{symm}_{2n,k} \cup B \cup \bar{B}$.*
(3) *For a fixed facet F of $\Delta^{*symm}_{2n,k}$, each diameter in $\varphi(F)$ is the image of a diameter in F.*
(4) *If a subset $\sigma \subset V^{symm}_{2n-2,k} \cup B \cup \bar{B}$ does not contain a $(k+1)$-crossing, this is also true for $\varphi(\sigma)$.*
(5) *If a subset $\sigma \subset \Omega_{2n-2,k}$ does not contain a $(k+1)$-crossing, this is also true for $\varphi^{-1}(\sigma)$.*

Proof.

(1) This is clear from the symmetry of the table in Figure 4.
(2) Let $d = i(i+n)$ be a diameter in $\Omega_{2n-2,k}$ and $d' = (i-1)(n+i-1) \in \Omega_{2n,k}$. We need to discriminate several cases:
   (A) $i \notin \{1, \dots, k\}$. Then we have $d \in S_2 \cap \bar{S}_2$ and $\varphi(d) = d$.
   (B) $i \in \{1, \dots, k\}$
      (B1) If $b_i < n+i-1$ then $d' \in S_0 \cap \bar{S}_0$ and $\varphi(d') = d$.
      (B2) If $b_i = n+i-1$ then $d' \in B_0 \cap \bar{B}_0$, and $\varphi(d') = d$.
      (B3) If $b_i = n+i$, then $d \in B_1 \cap \bar{B}_1$ and $\varphi(d) = d$.
      (B4) If $b_i > n+i$, then $d \in S_1 \cap \bar{S}_1$ and $\varphi(d) = d$.
(3) Let F be a facet in $\Delta^{symm}_{2n,k}$ and $d = i(i+n)$ a diameter in $\varphi(F)$. Whenever we have $\varphi(e) = d$ for a non-diameter $e \in \Omega_{2n,k}$, then we also have $\varphi(\bar{e}) = d$ and we know $e \in S_0 \cup B_0 \setminus (\bar{S}_0 \cup \bar{B}_0)$. We know there is an i such that

$e, \bar{e}$ are of the form $e = (i-1)(i+n), \bar{e} = i(i+n-1)$. Any diagonal of the $2n$-gon, except for $d' := (i-1)(n+i-1)$, which crosses d, crosses at least one of the diagonals $e, \bar{e}$. Since d, d' do not cross $e, \bar{e}$, one of d and d' is contained in F.

If $e \in B_0$, we have $i - 1 \in [0, k-1]$ and $n + i = b_i$. Then $d \in B_1 \cap \bar{B}_1$ and as a consequence $d \in F$ and $\varphi(d) = d$.

If $e \in S_0$, we have $i - 1 \in [0, k-1]$ and $b_i < n + i$. But b_i can not be smaller than $n + i - 1$: Since $\bar{e} \notin S_0 \cup B_0$, we have $n + i - 1 < b_{i+1}$, but this means $\bar{e} \in Z_B \cap F = \emptyset$ (see Lemma 19).

As a consequence we have $b_i = n + i - 1$, meaning that $d' \in B_0 \cap \bar{B}_0$ and by this $d' \in F$ and $\varphi(d') = d$.

(4) Let $\sigma \subset V^{symm}_{2n-2,k} \cup B \cup \bar{B}$ be a maximal subset not containing a $(k+1)$-crossing. Assume $\tau = \varphi(\sigma)$ comprises a $(k+1)$-crossing E, which we choose in a fashion such that the number of elements in $\varphi(B_0)$ is maximal compared to other $(k+1)$-crossings in τ.

Since $\varphi(B_0) = \varphi(B_1)$ as one easily shows, we get $\varphi(\sigma \setminus B_0) = \varphi(\sigma)$. As a consequence we have two non-crossing segments $ix, jy \in \sigma \setminus B_0$ such that $\varphi(ix), \varphi(jy) \in E$. W.l.o.g. we can conclude that $i = j \leq k-1, ix \in S_0 \setminus B_0$ and $iy \notin S_0 \cup B_0$. This implies $y \leq b_i$ since otherwise $iy \in Z_B$ and $x > b_{i+1}$ as well as $\varphi(ix) \in \{(i+1)x, (i+1)(x+1)\}$.

We define sets E', E'' by

$$\begin{aligned} E' &:= (E \setminus \varphi(ix)) \cup (i+1)b_{i+1} \\ E'' &:= (E \setminus \varphi(ix)) \cup (i+1)(b_{i+1}+1). \end{aligned}$$

If $x > b_{i+1} + 1$, both E and E' are $(k+1)$-crossings. But at least one of E' and E'' contains an element of $\varphi(B)$ which is not in E in contradiction to the choice of E.

That leaves us with $b_{i+1} + 1$ as the only possible value for x.

In this case E' is a $(k+1)$-crossing different from E, consequently $(i + 1)b_{i+1}$ must not be an element of $\varphi(B)$.

The set E'' is a new $(k+1)$-crossing with more elements from $\varphi(B)$ if $\varphi(ix) = (i+1)(x+1)$, because else we have $E'' = E$.

Thus we need to have

(1) $(i+1)b_{i+1} \notin \varphi(B)$
(2) $\varphi(ix) = (i+1)x$
(3) $x = b_{i+1} + 1$.

From (1) we can conclude that $ib_{i+1} \in B_0 \cap (\bar{B}_0 \cup \bar{S}_0)$. A consequence of (2) and (3) is $ix \in S_0 \setminus (\bar{B}_0 \cup \bar{S}_0)$.

All in all this leads to $n \leq b_{i+1} \leq n+k-1, i \geq b_{b_{i+1}-n}$ and $b_{i+1} + 1 > n + k - 1$. We find that $b_{i+1} = n + k - 1$ and thus $i \geq b_{k-1} \geq k + 1$ which finally yields a contradiction.

(5) First we show that two crossing diagonals still cross after the application of φ: Let ix and jy be crossing diagonals, while $\varphi(ix), \varphi(jy)$ do not cross. W.l.o.g. we can assume $j = i+1$, $ix \in S_0 \cup B_0, (i+1)y \in B_1 \cup S_1 \cup S_2, y > x$. As a consequence we get $x \geq b_{i+1}$ and, for the case $(i+1)y \in B_1 \cup S_1$, that $y \leq b_{i+1}$, contradicting $y > x$. If $(i+1)y$ was an element of S_2, we had $i + 1 = k$ and from $y > x \geq b_k$ we conclude that $(i+1)y \in K_B \not\subset S_2$.

Now we assume that in contrast to the assertion a subset $\sigma \subset \varphi(\Omega_{2n,k})$ does not contain a $(k+1)$-crossing and $\varphi^{-1}(\sigma)$ does. Then with the result above $\varphi(\varphi^{-1}(\sigma)) = \sigma$ contains a $(k+1)$-crossing as well, contradicting the assumption.

□

With Lemma 23 we are ready for the proof of Theorem 6:

Proof of Theorem 6. For arbitrary k we proceed by induction over n, the base case being $n = 2k+2$. Here $\Omega_{2k+2,k}$ exclusively consists of diameters, all mutually intersecting, so that each facet contains exactly k of them. Now for $n > 2k+2$, choose a facet F from $\Delta^{*\mathrm{symm}}_{2n,k}$. Let B as in Lemma 22 and φ be defined on the corresponding partition. Then we know by Lemma 23 (4), that $\varphi(F)$ does not contain a $(k+1)$-crossing. Adding another diameter d to $\varphi(F)$ would produce a $(k+1)$-crossing, since otherwise according to Lemma 23, (2), $\varphi^{-1}(d)$ exists and $\varphi^{-1}(\varphi(F) \cup \{d\})$ would be a proper superset of F and be free of $(k+1)$-crossings, contradicting the facet-property of F. According to our assumption, $\varphi(F)$ contains exactly k diameters, each of them is the image of a diameter in F as stated in (3) of the foregoing lemma. This means that F comprises at least k diameters and since all diameters are mutually intersecting there are exactly k diameters. □

5. Proof of Theorem 10

For a simplicial complex Δ and a finite group G we call Δ a G-complex if G acts simplicially on the vertex-set of Δ, i.e. $g\sigma := \{gv \big| v \in \sigma\} \in \Delta$ for all $\sigma \in \Delta$ and $g \in G$. We call a simplicial G-complex regular if for each subgroup $U \leq G$ of G and any choice of elements $g_0, \ldots, g_n \in U$ we have that if $\{v_0, \ldots, v_n\}$ and $\{g_0v_0, \ldots, g_nv_n\}$ are both simplices in Δ, there exists an element g in U such that $gv_i = g_iv_i$ for all $0 \leq i \leq n$.

Recall that the barycentric subdivision of a simplicial complex Δ is a simplicial complex on vertex-set $\Delta \setminus \{\emptyset\}$ whose simplices are the subsets of $\Delta \setminus \{\emptyset\}$ that are totally ordered with respect to inclusion. Clearly, if Δ is a G-complex then its barycentric subdivision $\mathrm{sd}(\Delta)$ is a G-complex as well. Recall, for a simplicial complex its barycentric subdivision is the simplicial complex on ground set $\Delta \setminus \{\emptyset\}$ whose simplices are the inclusionwise chains of Δ. We will make use of the following theorem.

Theorem 24 ([3])**.** *Let Δ be a simplicial G-complex. If for all $g \in G$, $\sigma \in \Delta$ and all $\sigma \in \Delta$ we have $gv = v$ for all vertices $v \in \sigma \cap g(\sigma)$, then the barycentric subdivision $sd(\Delta)$ is a regular G-complex.*

For a G-complex Δ its fix-complex Δ^G is the simplicial complex consisting of those simplices $\sigma \in \Delta$ that are elementwise fixed by G. For a regular G-complex its fix-complex sometimes inherits topological properties from the complex.

Theorem 25 ([3])**.** *Let $G = \mathbb{Z}_p$ be the cyclic group of prime order p. If Δ is a d-dimensional regular simplicial G-complex such that $H_i(\Delta, \mathbb{Z}_p) \cong H_i(S^d, \mathbb{Z}_p)$ for all $i \leq d$ then there is an $\ell \leq d$ such that $H_i(\Delta^G, \mathbb{Z}_p) \cong H_i(S^\ell, \mathbb{Z}_p)$ for all $i \leq d$.*

Note, that in contrast to our definition, in [3] a simplicial complex with the homological properties required in Theorem 25 is called a mod p-homology-sphere.

We will apply Theorem 25 to a suitable subdivison of $\Delta^*_{2n,k}$. Since by Theorem 3 a geometric realization of $\Delta^*_{2n,k}$ is a sphere, it follows immediately that it is a $\bmod p$-homology-sphere for all primes p.

For the proof of Theorem 10 we construct a subdivision $S_{2n,k}$ of the complex $\Delta^*_{2n,k}$, such that the group $\mathbb{Z}_2$ acts simplicially and regularly and $\mathcal{D}^*_{n,k}$ is isomorphic to the fix-complex $S^{\mathbb{Z}_2}_{2n,k}$. The assertion then is a consequence of Theorem 25 and Corollary 7. Figure 6 shows $\Delta^*_{6,1}$ and the construction of $T_{6,1}$. The bold facets are the faces of the fix-complex and correspond to faces of $\mathcal{D}^*_{3,1}$.

The complex $\Delta^*_{2n,k}$ carries a natural $\mathbb{Z}_2$-action which is induced by sending a diagonal d to its image $\bar{d}$ under 180°-rotation. From now on we identify $\Delta^*_{2n,k}$ with its embedding into some $\mathbb{R}^m$ which is piecewise linearly extended from the embedding of its vertices. In particular, the barycenter of the edge $\{d, d'\}$ will be $\frac{1}{2}(d + d')$. We set

$$\mathcal{R}_{2n,k} := \{\frac{1}{2}(d+\bar{d})\big|d \in \Omega_{2n,k}, d \neq \bar{d}\}, \Omega'_{2n,k} := \Omega_{2n,k} \cup \mathcal{R}_{2n,k}.$$

The set $\Omega'_{2n,k}$ will serve as the ground set of our subdivision. For $\sigma \in \Delta^*_{2n,k}$ we let $\sigma^{symm} := \{d \in \Omega_{2n,k}\big|d \in \sigma \text{ and } \bar{d} \in \sigma, d \neq \bar{d}\}$ be the symmetric part of σ without the diameters and set $2l := \sharp\sigma^{symm}$. We choose an l-element subset $D_\sigma := \{d_1, \ldots, d_l\}$ from σ^{symm} such that $d_i \neq \bar{d}_j$ for all $1 \leq i < j \leq l$. For each d_i and each $\varepsilon \in \{0,1\}^l$ we set

$$d_i^{\varepsilon_i} := \begin{cases} d_i \text{ if } \varepsilon_i = 1 \\ \bar{d}_i \text{ if } \varepsilon_i = 0 \end{cases}$$

and $D^\varepsilon_\sigma := \{d_i^{\varepsilon_i}, i = 1, \ldots, l\}$. Finally we define for $\varepsilon \in \{0,1\}^l$ the simplex

$$\sigma_\varepsilon := (\sigma \setminus \sigma^{symm}) \cup \{\frac{1}{2}(d+\bar{d})\big|d \in D_\sigma\} \cup D^\varepsilon_\sigma.$$

We let $T_{2n,k}$ be the simplicial complex which is generated by all σ_ε for $\sigma \in \Delta^*_{2n,k}$ and $\varepsilon \in \{0,1\}^l$.

By $\mathrm{conv}(\{D^\varepsilon_\sigma\big|\varepsilon \in \{0,1\}^l\}) = \mathrm{conv}(\sigma^{symm})$, it follows that $T_{2n,k}$ is a subdivision of $\Delta^*_{2n,k}$. Since as mentioned before $\Delta^*_{2n,k}$ is a a $\bmod 2$-homology-sphere the same is true for $T_{2n,k}$.

Clearly, $\mathbb{Z}_2$ acts simplicially on $T_{2n,k}$.

Since each σ_ϵ contains at most one of d and $\bar{d}$ for any $d \in \Omega_{2n,k}$, we get for all $\sigma \in T_{2n,k}$ and all $g \in \mathbb{Z}_2$ that

$$gv = v \text{ for all } v \in g\sigma \cap \sigma. \tag{5.1}$$

Now by a suitable subdivision we give $T_{2n,k}$ the structure of a regular $\mathbb{Z}_2$-complex.

We replace all $\sigma \in T_{2n,k}$ by the join $\sigma_1 * \sigma_2$, where σ_1 is the barycentric subdivision of $\sigma \cap \Omega_{2n,k}$ and $\sigma_2 = \sigma \cap \mathcal{R}_{2n,k}$. Note, that here we adopt the convention that the join of a simplex with the empty set is the simplex. We write $S_{2n,k}$ for the resulting simplicial complex. By construction, $S_{2n,k}$ is a subdivision of $T_{2n,k}$ and therefore of $\Delta^*_{2n,k}$.

To show regularity it suffices to show the following. If $\sigma := \{v_0, \ldots, v_n\}$ and $\tau := \{v_0, \ldots, v_r, \bar{v}_{r+1}, \ldots, \bar{v_n}\}$ are two simplices of $S_{2n,k}$ then either $\bar{v_i} = v_i$ for $r+1 \leq i \leq n$ (resp. $\sigma = \tau$) or $\bar{v}_i = v_i$ for $0 \leq i \leq r$ (resp. $\sigma = \bar{\tau}$).

Since $\mathcal{R}_{2n,k}$ is pointwise fixed by $\mathbb{Z}_2$ we can ignore the vertices from $\mathcal{R}_{2n,k}$ and assume that σ, τ are simplices in the barycentric subdivision of $T'_{2n,k} := \{\rho \in$

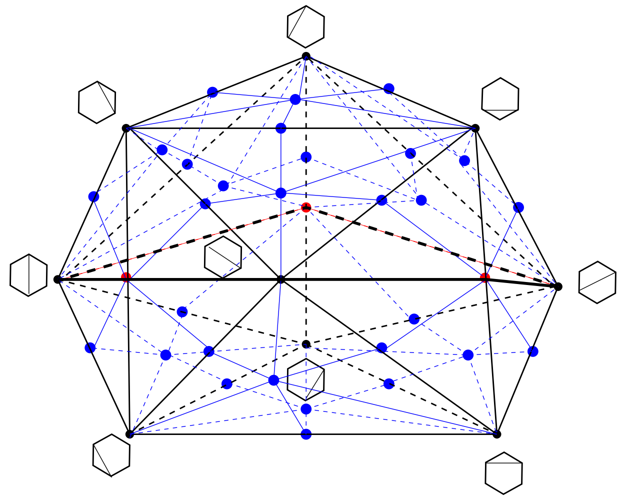

FIGURE 6. Construction in the proof of Theorem 10

$T_{2n,k} \mid \rho \subseteq \Omega_{2n,k}\}$. By (5.1) the complex $T_{2n,k}$ and therefore $T'_{2n,k}$ satisfy the assumption of Theorem 24. The theorem implies that the barycentric subdivision of $T'_{2n,k}$ is a regular $\mathbb{Z}_2$-complex. From this it follows that either $\sigma = \tau$ or $\sigma = \bar{\tau}$.

By construction,

$$\phi : \{d, \bar{d}\} \mapsto \frac{1}{2}(d + \bar{d})$$

induces an isomorphism between $\mathcal{D}^*_{n,k}$ and the fixed complex $S^{\mathbb{Z}_2}_{2n,k}$. With Theorem 25 we get that

$$H_i(\mathcal{D}^*_{n,k}, \mathbb{Z}_2) \cong H_i(S^{k(n-k)-1}, \mathbb{Z}_2).$$

Now let τ be a face of $\mathcal{D}^*_{n,k}$. Since $\mathcal{D}^*_{n,k}$ is pure of dimension $k(n-k)-1$, we have $\dim \operatorname{link}_{\mathcal{D}^*_{n,k}}(\tau) = k(n-k) - 1 - \dim \tau - 1$.

The isomorphism from above yields

$$\operatorname{link}_{\mathcal{D}^*_{n,k}}(\tau) \cong \operatorname{link}_{S^{\mathbb{Z}_2}_{2n,k}}(\phi(\tau))$$

and since $S^{\mathbb{Z}_2}_{2n,k}$ is a mod 2-homology-sphere, the assertion follows.

6. Proof of Theorem 14 and Proposition 8

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The statements of Theorem 14 and Proposition 8 seem to be unrelated. Nevertheless, we provide their proofs in a joint section, since the proof of Proposition 8 makes use of a bijection established in the proof of Theorem 14.

Proof of Theorem 14. Let $V = [n] \times [n]$. It is easy though tedious to check that the mappings $\Psi : \mathcal{F}_n \to V$ and $\Phi : V \to \mathcal{F}_n$ given by

$$\Psi : \quad \{\{a_1 < b_1\}, \{a_2 < b_2\}\} \mapsto \begin{cases} ((a_i + 1) \bmod n, b_i \bmod n) & \text{if } b_i - a_i \leq n \\ ((b_i + 1) \bmod n, a_i \bmod n) & \text{if } b_i - a_i > n, \end{cases}$$

$$\Phi : \quad (a, b) \mapsto \begin{cases} \{\{(a-1) \bmod 2n, b\}, \{a - 1 + n, b + n\}\} & \text{if } a \leq b \\ \{\{a - 1 + n, b\}, \{a - 1, b + n\}\} & \text{if } a > b \end{cases}$$

$$\begin{pmatrix} \star & \star & \star & \star & 2 & 1 \\ 11 & \star & \star & \star & \star & 12 \\ 10 & 9 & \star & \star & \star & \star \\ \star & 8 & 7 & \star & \star & \star \\ \star & \star & 6 & 5 & \star & \star \\ \star & \star & \star & 4 & 3 & \star \end{pmatrix}$$

FIGURE 7. Numbering of the vertices of $\mathcal{D}^*_{6,4}$ as in the proof of Proposition 8.

are well-defined mutually inverse bijections.

Furthermore, Ψ maps the set $F_{n,k}$ to $[n]\times[n]\backslash\{(i,j) \mid j = (i+l) \bmod n$ for one $l < k\}$ and for a $(k+1)$ crossing $K \subset \mathcal{F}_{n,k}$ there are $(k+1)$-sets $A, B \subset [n]$ such that $\Psi(K) = N(A,B)$. Conversely, any subset of V of the type $N(A,B)$ is mapped on a $(k+1)$-crossing by Φ. This already proves Theorem 14. $\square$

Proof of Proposition 8. If $k = 1$ then $D^*_{n,k}$ is the cyclohedron, which is well known (see for example [33]) to be the boundary complex of a polytope. In the case $k = n-1$ the complex $D^*_{n,k}$ is the boundary complex of the simplex on the set of diameters. it remains to prove the proposition for $n \geq 3$ the complex $\mathcal{D}^*_{n,n-2}$ is isomorphic to the boundary-complex of the $(2n-4)$-dimensional cyclic polytope with $2n$ vertices $\mathcal{C}_{2n-4}(2n)$. We use the following characterization of the boundary-complex by Gale: Let C_{2n} the cycle-graph with $2n$-vertices numbered in clockwise order and for $\sigma \subset [2n]$ let $C_{2n}(\sigma)$ be the subgraph induced by the vertices in σ.

Theorem 26 (see [15])**.** *Identifying each vertex $x_i = (i, i^2, i^3, \ldots, i^d) \in \mathbb{R}^d$ of $\mathcal{C}_{2n-4}(2n)$ with vertex i of the graph C_{2n} for $i = 1, \ldots, 2n$ yields: The subset $\sigma \subset [n]$ is a face of the boundary-complex of $\mathcal{C}_d(2n)$ if and only if $|\sigma| + \omega(\sigma) \leq d$, where $\omega(\sigma)$ is the number of odd-sized connected components of $C_{2n}(\sigma)$.*

The correspondence from Theorem 26 translates into the following classification of the minimal nonfaces of $\mathcal{C}_{2n-4}(2n)$.

Corollary 27. *The minimal nonfaces of $\mathcal{C}_{2n-4}(2n)$ are in bijection with the $(n-1)$-subsets $\eta \subset [2n]$, for which $C_{2n}(\eta)$ consists of exactly $(n-1)$ connected components.*

As noted in the paragraph above, we view $\mathcal{D}^*_{n,n-2}$ via the bijection Ψ from the proof of Theorem 14 as a simplicial complex on vertex-set $[n] \times [n]\backslash\{(i,j) \mid j = (i+\ell) \bmod n$ for a $\ell < n-2\}$ with nonfaces $N(A,B)$ for the $(n-1)$-element subsets A, B of $[n]$, . If seen as a set of entries in an $n \times n$-matrix this vertex set is a set of matrix elements such that there are 2 entries in each row and column.

We number the vertices such that for each $i \in \{1, \ldots, 2n\}$ the vertex with number i shares a row or a column with the vertices numbered by $i-1, i+1 (\bmod 2n)$. Recall that $k \bmod k = k$ (See Figure 7 for an example.)

We now show that each numbering φ of the vertices with this property identifies nonfaces of $\mathcal{D}^*_{n,n-2}$ with nonfaces of $\mathcal{C}_{2n-4}(2n)$ and vice versa.

For all $A, B \subset [n]$ with $|A| = |B| = n-1$ the set $N(A,B)$ contains exactly $n-1$ elements and in each row and column of $[n] \times [n]$ there is at most one element of $N(A,B)$. Hence the subgraph $C_{2n}(\varphi(N(A,B))$ does not contain neighbouring vertices of C_{2n} and thus induces exactly $(n-1)$ connected components.

Conversely, if we have an $(n-1)$-subset τ of the vertex-set of C_{2n} such that $C_{2n}(\tau)$ has exactly $(n-1)$ connected components, then elements from $\varphi^{-1}(\tau)$ are spread over $(n-1)$ different rows and columns. Let $A = \{a_1, \ldots, a_{n-1}\}$ be the set of row-indices and $B = \{b_1, \ldots, b_{n-1}\}$ the set of column-indices.

Both sets $\varphi^{-1}(\tau)$ and $N(A,B)$ have the property that no two of their elements share a row or a column. Together with the fact that in each row and column of the matrix $[n] \times [n]$ there are only two vertices of $\mathcal{D}_n^{n-2}$ we get that the two sets coincide whenever they have nonempty intersection.

Define $i_A, i_B \leq n$ to be the unique elements of $[n] \setminus A$ and $[n] \setminus B$. First, we examine the set $\varphi^{-1}(\tau)$. If $i_A \neq (i_B+2) \bmod n$ we have $((i_B+2) \bmod n, (i_B+1) \bmod n) \in \varphi^{-1}(\tau)$, since it is the unique entry of the matrix in row $(i_B+2) \bmod n$. If $i_A = (i_B+2) \bmod n$, we have that $((i_B+1) \bmod n, (i_B-1) \bmod n)$ is the unique entry in row $(i_B+1) \bmod n$ and thus an element of $\varphi^{-1}(\tau)$.

On the other hand the set $N(A,B)$ is of the form $\{(a_{(j+\ell) \bmod n-1}, b_j) \mid j = 1, \ldots, n-1\}$ for some $0 \leq \ell \leq 2$.

$\ell = 0$: We have $\ell = 0$ if and only if $i_A = 1$ and $i_B = n$, since only in this case $a_j = j+1 > j = b_j$ holds for all $j = 1, \ldots, n-1$. As a consequence we get that $((i_B+2) \bmod n, (i_B+1) \bmod n) = (2,1) = (a_1, b_1) \in N(A,B) \cap \varphi^{-1}(\tau)$.

$\ell = 2$: We have $\ell = 2$ if and only if $i_B \leq i_A - 2$, since only in this case there is a $j \in \{i_B, \ldots, i_A - 2\} \subset \{1, \ldots, n-2\}$ such that $a_{j+1} \leq b_j$ and we always have $a_{i+2} \geq i+2 > i+1 \geq b_i$ for all $i \in \{1, \ldots, n-3\}$.

For $i_B < i_A - 2$ we have $a_{i_B+2} = i_B + 2$ und $b_{i_B} = i_B + 1$ and as a consequence $(i_B+2, i_B+1) \in N(A,B) \cap \varphi^{-1}(\tau)$. For $i_B = i_A - 2$ we can conclude that $a_{i_B+1} = i_B + 1$ and $b_{i_B-1} = i_B - 1$. Thus $(i_B+1, i_B-1) \in N(A,B) \cap \varphi^{-1}(\tau)$.

$\ell = 1$: By the previous argumentation we have $\ell = 1$ for the remaining cases, that is $i_B > i_A - 2$ and $(i_A, i_B) \neq (1, n)$. If $i_B < n-1$ (and consequently $i_A \neq (i_B+2) \bmod n$) we have $(i_B+2, i_B+1) = (a_{i_B+1}, b_{i_B}) \in N(A,B) \cap \varphi^{-1}(\tau)$. If $i_B = n-1$ and $i_A \neq (i_B+2) \bmod n = 1$ we have $(1,n) = (a_1, b_{n-1}) \in N(A,B) \cap \varphi^{-1}(\tau)$.

For $i_B = n$, and $i_A \neq (i_B+2) \bmod n = 2$ we get

$$(2,1) = (a_2, b_1) \in N(A,B) \cap \varphi^{-1}(\tau).$$

Whenever $i_B = n-1$, and $i_A = (i_B+2) \bmod n = 1$ we conclude

$$(n, n-2) = (a_{n-1}, b_{n-2}) \in N(A,B) \cap \varphi^{-1}(\tau).$$

Finally, if $i_B = n$, and $i_A = (i_B+2) \bmod n = 2$ we know

$$(1, n-1) = (a_1, b_{n-1}) \in N(A,B) \cap \varphi^{-1}(\tau).$$

□

7. Construction of the term-order

The main goal of this section is to prove Theorem 29 which then together with Theorem 14 proves the desired inclusion of ideals.

Definition 28. Let

$$\begin{array}{rcl} \varphi : [n] \times [n] & \to & \mathbb{N} \\ (i,j) & \mapsto & \left[(2-i) \cdot n + (j-1) \cdot (n-1) - 1 \bmod n^2\right] + 1, \end{array}$$

$$\begin{pmatrix} 5 & 9 & 13 & 17 & 21 \\ 25 & 4 & 8 & 12 & 16 \\ 20 & 24 & 3 & 7 & 11 \\ 15 & 19 & 23 & 2 & 6 \\ 10 & 14 & 18 & 22 & 1 \end{pmatrix}$$

FIGURE 8. The mapping φ for $n = 5$

and order the entries in $X = (x_{ij})_{1 \le i,j \le n}$ according to

$$x_{ij} \preceq x_{kl} :\Leftrightarrow \varphi(i,j) \le \varphi(k,l).$$

It is easily checked that ϕ is a bijection from $[n] \times [n]$ to $[n^2]$.

We now define an order on the set of monomials, which will be the graded, reverse-lexicographic continuation of $\preceq$. We denote each monomial $t = \prod_{(i,j)\in[n]^2} x_{ij}^{\gamma_{ij}} \in S$ as

$$\mathbf{x}^\beta := \prod_{l=0}^{n^2-1} x_{\varphi^{-1}(n^2-l)}^{\beta_{l+1}},$$

where $\beta \in \mathbb{N}^{n^2}$. For example, we write $\mathbf{x}^{(3,0,\ldots,2)}$ for $x_{nn}^2 \cdot x_{21}^3$.

Then $x^\alpha \prec x^\beta$ if and only if the sum of the entries in α is smaller than the sum of the entries in β or if both sums are equal, $\alpha_i > \beta_i$, where i is the largest index in which both vectors differ.

The essential property of $\preceq$ is the content of the following theorem. For its formulation we use the notation $N(A,B)$ from Equation (3.2), see also Theorem 14.

Theorem 29. *For subsets $A, B \subset [n]$, $\#A = \#B = k+1$ we have*

$$\mathrm{lm}_\preceq(\det M(A,B)) = \prod_{(i,j)\in N(A,B)} x_{ij}. \tag{7.1}$$

For the proof we need several lemmas. In the following, let $A, B \subset [n]$ be nonempty subsets of equal sizes.

Lemma 30. *If x_{a_i,b_j} divides the leading monomial of $\det(M(A,B))$, we have that*

$$\mathrm{lm}_\preceq(detM(A \setminus \{a_i\}, B \setminus \{b_j\}) \cdot x_{a_i,b_j} = \mathrm{lm}_\preceq(det(M(A,B)).$$

Proof. We apply Laplace-expansion for the j-ths row of $M(A,B)$ and get

$$\det(M(A,B)) = \sum_{p=1}^{(k+1)} (-1)^{j+p} \cdot x_{a_p,b_j} \cdot \det(M(A\backslash\{a_i\}, B\backslash\{b_j\})). \tag{7.2}$$

We can conclude

$$\mathrm{lm}_\preceq(\det(M(A,B)) = \mathrm{lm}_\preceq\Big(x_{a_i,b_j} \det(M(A\backslash\{a_i\}, B\backslash\{b_j\})\Big),$$

since $x_{a_i,b_j} \det(M(A\backslash\{a_i\}, B\backslash\{b_j\}$ is the only summand in Equation (7.2) which is a multiple of x_{a_i,b_j}. The assertion is a consequence of the fact that $\mathrm{lm}_\preceq(mm') = \mathrm{lm}_\preceq(m)\mathrm{lm}_\preceq(m')$ for all monomials m, m'. □

Lemma 31. *Let $(a_i, b_j) \in N(A,B)$. Then $N(A \setminus \{a_i\}, B \setminus \{b_j\}) \subset N(A,B)$.*

Proof. Since $(a_i, b_j) \in N(A,B)$ we know that $i = (j+\ell) \bmod (k+1)$ for some $\ell \le k$. We get that

$$a_{p+\ell} \quad > \quad b_p, \text{ for all } p = 1, \dots k+1-\ell$$

and there is at least one q such that

$$a_{q+\ell-1} \quad \le \quad b_q,\ q \in \{1, \dots k+1-\ell+1\}.$$

Let $A' := A \backslash \{a_{j+\ell}\} =: \{a'_1 < \cdots < a'_k\}$, $B' := B \backslash \{b_j\} =: \{b'_1 < \cdots < b'_k\}$. We will show that

$$N(A', B') = N(A,B) \backslash \{(a_{(j+\ell) \bmod (k+1)}, b_j)\}.$$

(Case 1) $j \le k+1-\ell$: In this case $\{(a'_{p+\ell}, b'_p) \big| p = 1, \dots, k+1\} = N(A,B) \backslash \{(a_{p+j}, b_j)\}$. This implies

$$a'_{p+\ell} > b'_p, \text{ for all } p = 1, \dots, (k+1) - \ell - 1. \tag{7.3}$$

We show that ℓ is either zero or minimal such that (7.3) holds. We get $N(A', B') = \{(a'_{p+\ell}, b'_p) \big| p = 1, \dots, (k+1)\}$. We distinguish the following cases:

(SubCase a) If $j < q-1$ we have $b'_{q-1} = b_q \ge a_{q+\ell-1} = a'_{q-1+\ell-1}$, thus ℓ is minimal in Equation (7.3).

(SubCase b) If $j = q-1$ then $b_j = b_{q-1}$ and $a_{j+\ell} = a_{q+\ell-1}$ are being removed, so that $b'_{q-1} = b_q$ and $a'_{q+\ell-2} = a_{q+\ell-2}$. Since $b_q \ge a_{q+\ell-1}$ we have $b_q \ge a_{q+\ell-2}$ and therefore $b'_{q-1} \ge a'_{q-1+\ell-1}$. We conclude that ℓ is minimal in Equation (7.3).

(SubCase c) If $j = q$ it follows that $b'_q = b_{q+1}$ and $a_{q+\ell-1} = a'_{q+\ell-1}$. Since $b_q \ge a_{q+\ell-1}$ it holds that $b_{q+1} \ge a_{q+\ell-1}$ and in consequence we get $b'_q \ge a'_{q+\ell-1}$. Again we find that ℓ is minimal in Equation (7.3).

(SubCase d) If $j > q$ then $(k+1) - \ell \ge j > q$. Thus $b'_q = b_q \ge a_{q+\ell-1} = a'_{q+\ell-1}$, and thereby ℓ is minimal in Equation (7.3).

(Case 2) $(k+1) \ge j > (k+1) - \ell$: Then row $a_{(j+\ell) \bmod (k+1)}$ is being deleted. Since $(j+\ell) \bmod (k+1) < p + \ell$ for $p = 1, \dots, (k+1)$ we get $b'_p = b_p$, $p = 1, \dots, (k+1) - \ell$ and $a'_{p+\ell} = a_{p+\ell+1}$, $i = 1, \dots, (k+1) - \ell - 1$. We conclude

$$a'_{p+\ell-1} = a_{p+\ell} \quad > \quad b_p = b'_p, \text{ for all } p = 1, \dots (k+1) - \ell - 1$$

and there is at least one q such that

$$a'_{q+\ell-2} = a_{q+\ell-1} \quad \le \quad b_q = b'_q,\ q \in \{1, \dots k+1-\ell+1\}.$$

This implies

$$N(A', B') = \{(a'_{p+\ell-1}, b'_p) \big| p = 1, \dots, (k+1) - \ell - 1\}) = N(A,B) \backslash \{(a_{j+\ell}, b_j)\}.$$

□

For the following lemma, let $\Delta^- := \{x_{ij} \big| i > j\}$ and $\Delta^+ := \{x_{ij} \big| i \le j\}$. We say that an indeterminate x_{ij} is (weakly) to the right and (weakly) above x_{kl}, if $i \le k$ and $j \ge l$. The rectangle spanned by x_{ij} and x_{kl} is the set $R(x_{ij}, x_{kl}) := \{x_{qr} : i \le q \le k, l \le r \le j\}$.

Let a, b be elements of X such that a is (weakly) to the right and (weakly) above b. We list two simple and important properties of our order $\preceq$:

$$\begin{pmatrix} w_{1,1} & \cdots & w_{1,i} & \star & \dots & \star \\ \vdots & & \vdots & \vdots & & \vdots \\ w_{i-1,1} & \cdots & w_{i-1,i} & \star & \cdots & \star \\ v & \cdots & x_{a_i,b_i} & \star & \cdots & \star \\ \vdots & & \vdots & \vdots & & \vdots \\ v & \cdots & v & \star & \cdots & \star \end{pmatrix}$$

FIGURE 9. $a_{(i+\ell)\,\mathrm{mod}(k+1)} > b_i$ and $\ell = 0$

(E1): If $a, b \in \Delta^+$, then $R(a,b)$ is contained in Δ^+ and the upper right corner is the maximal element of R with respect to $\preceq$. The same is true for Δ^-.

(E2): If $x \in \Delta^-$ and $y \in \Delta^+$ and x and y share a column or row, we have $x \succeq y$.

Lemma 32. *There is an element $(i,j) \in N(A,B)$ such that x_{ij} divides the leading monomial of* $\det(M(A,B))$.

Proof. Let $x_{a_{(i+\ell)\,\mathrm{mod}(k+1)},b_i} := \min_{\preceq}\{x_{i,j} \big| (i,j) \in N(A,B)\}$.

(Case 1) $a_{(i+\ell)\,\mathrm{mod}(k+1)} > b_i \dots$

(SubCase a) $\ell = 0$

From the definition of $N(A,B)$ we get $i + \ell \leq (k+1)$. In this case we have that $x_{a_{(i+\ell)\,\mathrm{mod}(k+1)},b_i}$ lies in Δ^- and to the right and above of x_{a_{k+1},b_1}. Because of (**E1**) we know that

$$V := \{x_{a_p,b_m} \text{ such that } p \geq i \text{ and } m \leq i\} \subset \Delta^-.$$

It holds that $v \leq x_{a_{(i+\ell)\,\mathrm{mod}(k+1)},b_i}$ for all $v \in V$ (see Figure 9). Assume $\det(M(A,B))$ has a term t that is larger with respect to $\preceq$ than the product of all x_{ij} such that $(i,j) \in N(A,B)$. Assume further that t does not contain any of those x_{ij} as a factor. Then t also contains a factor from V.

If $i = 1$ this assumption gives us a contradiction immediately, since then t cannot have a factor from the first column of $M(A,B)$. Let $i > 1$. Since t needs to have a factor in every row and in every column of $M(A,B)$. This means that t comprises i factors from $W := \{x_{a_p,b_m}$ such that $p < i + \ell = i$ and $m \leq i\}$. Because W stretches over i columns and $i-1$ rows, there is a row of $M(A,B)$ containing two factors of t. This yields the desired contradiction.

(SubCase b) $\ell > 0$

According to the definition of $N(A,B)$ it holds that $i + \ell \leq (k+1)$. Analogous to the first case, set

$$V := \{x_{a_p,b_m} \text{ such that } p \geq i + \ell \text{ and } m \leq i\}.$$

Again because of (**E1**) we know $v \leq x_{a_{(i+\ell)\,\mathrm{mod}(k+1)},b_i}$ for all $v \in V$. Since $\ell > 0$ there is a $j \leq (k+1) - \ell + 1$ such that $a_{j+\ell-1} \leq b_j$ and so we know $x_{a_{j+\ell-1},b_j} \in \Delta^+$ (see Figure 10).

We consider the following subcases:

$j \leq i$: Here x_{a_i,b_i} lies to the right and below of $x_{a_{j+\ell-1},b_j}$ and is therefore an element of Δ^+ as well. This together with (**E1**) is the

$$\begin{pmatrix} \star & v' & v' & \star & \star & \star \\ \star & v' & v' & \star & \star & \star \\ \star & x_{a_{j+\ell-1},b_j} & v' & \star & \star & \star \\ \star & \star & \star & \star & \star & \star \\ v & v & x_{a_{i+\ell},b_i} & \star & \star & \star \\ v & v & v & \star & \star & \star \end{pmatrix} \begin{pmatrix} \star & \star & \star & \star & \star & \star \\ \star & \star & \star & \star & \star & \star \\ \star & \star & \star & \star & \star & \star \\ v & x_{a_{i+\ell},b_i} & \star & \star & v' & v' \\ v & v & \star & \star & v' & v' \\ v & v & \star & \star & x_{a_{j+\ell-1},b_j} & v' \end{pmatrix}$$

FIGURE 10. $a_{(i+\ell)\bmod(k+1)} > b_i$ and $(k+1) \geq \ell > 0$

reason why the rectangle

$$V' := \{x_{a_p,b_m} \text{ such that } p \leq j+\ell-1 \text{ and } m \leq i\}$$

is contained in Δ^+. By (**E2**) we know that $x_{a_1,b_i} \prec x_{a_{(i+\ell)\bmod(k+1)},b_i}$. Again (**E1**) tells us that $v' \prec x_{a_1,b_i} \prec x_{a_{(i+\ell)\bmod(k+1)},b_i}$ for all $v' \in V'$.

Assume $\det(M(A,B))$ had a monomial t that was larger w.r.t. $\preceq$ than the product of all $x_{i,j}$ such that $(i,j) \in N(A,B)$ and none of those x_{ij} was a factor of t. Then t would not contain a factor from $V \cup V'$. The set $V \cup V'$ stretches over $i-j+1$ columns of $M(A,B)$ inside of which all except for $i+\ell-(j+\ell-1)-1 = i-j$ rows are covered. This means, that t had two factors in a row or, if $i = j$, no factor in a certain column, which is a contradiction.

$j > i$: Here we have that $x_{a_{i+\ell},b_m}$ lies to the right and above $x_{a_{j+\ell-1},b_j}$ and is therefore contained in Δ^+. Property (**E1**) tells us that the rectangle

$$V' := \{x_{a_p,b_m} \text{ such that } i+\ell \leq p \leq j+\ell-1 \text{ and } m \geq j\}$$

is a subset of Δ^+. Because of (**E2**) we know $x_{a_{i+\ell},b_{k+1}} \prec x_{a_{i+\ell},b_i}$ and again (**E1**) tells us that $v' \preceq x_{a_{(i+\ell)\bmod(k+1)},b_i}$ for all $v' \in V'$. Again we assume that $\det(M(A,B))$ had a term t that was larger than the product of all x_{ij} such that $(i,j) \in N(A,B)$ and none of those x_{ij} was a factor of t. Again t could not contain a factor from $V \cup V'$. Since $V \cup V'$ comprises $j+\ell-1-(i+\ell)+1 = j-i$ rows of $M(A,B)$ inside of which all except of $j-i-1$ columns are covered. That means that t had two factors in a column, which is again a contradiction.

(Case 2) $a_{(i+\ell)\bmod(k+1)} > b_i$ and $(k+1) \geq \ell > 0$.

According to the definition of $N(A,B)$ we have $i+\ell > (k+1)$ in this case and there is a $j \leq (k+1)-\ell+1$ such that $a_{j+\ell-1} < b_j$. Thus we know $x_{a_{j+\ell-1},b_j}, x_{a_{i+\ell},b_i} \in \Delta^+$ and $x_{a_{i+\ell},b_i}$ is to the right and above $x_{a_{j+\ell-1},b_j}$. For the enclosed rectangle

$$V := \{x_{a_p,b_m} \text{ such that } j+\ell-1 \geq p \geq (i+\ell)\bmod(k+1) \text{ and } j \leq m \leq i\},$$

we know that $v \leq x_{a_{(i+\ell)\bmod(k+1)},b_i}$ for all $v \in V$ (see Figure 11).

Again we assume $\det(M(A,B))$ had a t that was larger with respect to our term-order than the product of all x_{ij} such that $(i,j) \in N(A,B)$ and that did not contain one of the x_{ij} as a factor. In consequence t could not have a factor from V. For $i > j$ the set V stretches over $j+\ell-1-(i+\ell)\bmod(k+1)+1 = j+\ell-(i+\ell-(k+1)) = (k+1)-i+j$ rows of $M(A,B)$ and

$$\begin{pmatrix} \star & \star & \star & \star & \star & \star \\ \star & v & v & v & x_{a_{i+\ell},b_i} & \star \\ \star & v & v & v & v & \star \\ \star & x_{a_{j+\ell-1},b_j} & v & v & v & \star \\ \star & \star & \star & \star & \star & \star \\ \star & \star & \star & \star & \star & \star \end{pmatrix}$$

FIGURE 11. $a_{(i+\ell) \bmod (k+1)} > b_i$ and $(k+1) \geq \ell > 0$

covers $i-j+1$ columns of it, leaving only $(k+1)-(i-j+1) = (k+1)-i+j-1$ uncovered. For $i = j$ we know that V consists of the full ith column of $M(A, B)$. That means that t contains either two or no factor in a column, a contradiction!

□

Proof of Theorem 29. For $k = 0$ the statement is clear. For $k > 0$ we proceed via an induction. The leading monomial of $\det M(A, B)$ contains according to Lemma 32 a factor $x_{a_i,b_j} \in N(A, B)$. A consequence of Lemma 30 is the fact, that

$$\mathrm{lm}_{\preceq}(\det M(A, B)) = x_{a_i,b_j} \cdot \mathrm{lm}_{\preceq}(\det M(A', B')),$$

where $A' = A \backslash \{a_i\}$, $B' = B \backslash \{b_j\}$. The inductive assumption together with Lemma 31 yields the theorem. □

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