

THE VERONESE CONSTRUCTION FOR FORMAL POWER SERIES AND GRADED ALGEBRAS

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ABSTRACT. Let $(a_n)_{n \geq 0}$ be a sequence of complex numbers such that its generating series satisfies $\sum_{n \geq 0} a_n t^n = \frac{h(t)}{(1-t)^d}$ for some polynomial $h(t)$. For any $r \geq 1$ we study the transformation of the coefficient series of $h(t)$ to that of $h^{(r)}(t)$ where $\sum_{n \geq 0} a_{nr} t^n = \frac{h^{(r)}(t)}{(1-t)^d}$. We give a precise description of this transformation and show that under some natural mild hypotheses the roots of $h^{(r)}(t)$ converge when r goes to infinity. In particular, this holds if $\sum_{n \geq 0} a_n t^n$ is the Hilbert series of a standard graded k -algebra A . If in addition A is Cohen-Macaulay then the coefficients of $h^{(r)}(t)$ are monotonely increasing with r . If A is the Stanley-Reisner ring of a simplicial complex Δ then this relates to the r th edgewise subdivision of Δ – a subdivision operation relevant in computational geometry and graphics – which in turn allows some corollaries on the behavior of the respective f -vectors.

1. INTRODUCTION AND STATEMENT OF RESULTS

In this paper we study for a rational formal power series of the form

$$f(t) := \sum_{n \geq 0} a_n t^n = \frac{h(t)}{(1-t)^d}, \quad a_n \in \mathbb{C} \text{ for } n \geq 0,$$

the transformation of the numerator polynomial when passing for some number $r \geq 1$ to the generating function $f^{(r)}(t) := \sum_{n \geq 0} a_{nr} t^n = \frac{h^{(r)}(t)}{(1-t)^d}$.

We are motivated by the following facts from commutative algebra. Let $A = \bigoplus_{n \geq 0} A_n$ be a standard graded k -algebra; that is A is finitely generated in degree 1 and $A_0 = k$. The Hilbert-Serre Theorem [10, see Chap. 10.4] asserts that its Hilbert-series $\text{Hilb}(A, t) = \sum_{n \geq 0} \dim_k A_n t^n$ is a rational function of the form $\text{Hilb}(A, t) = \frac{h(t)}{(1-t)^d}$ for some polynomial $h(t)$ such that $h(1) \geq 1$ and d the Krull dimension of A . The r th Veronese algebra of A is the standard graded k -algebra $A^{(r)} := \bigoplus_{n \geq 0} A_{nr}$ with Hilbert-series $\text{Hilb}(A^{(r)}, t) = \sum_{n \geq 0} \dim_k A_{nr} t^n = \text{Hilb}(A, t)^{(r)}$. Veronese algebras are well studied objects in commutative algebra and algebraic geometry. In particular, the limiting behavior of algebraic

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properties of $A^{\langle r \rangle}$ for large r has been considered in [1], [9] and more generally in [11]. Indeed as will be seen later the results from [1] and [9] relate to our own results.

Most results presented in this paper are very much in the spirit of results from [4] on the behavior of the h -vector and h -polynomial of barycentrically subdivided simplicial complexes. But even though the formulations of the theorems appear to be very similar the proofs are almost disjoint except for the use of Lemma 3.3 in the proof of Theorem 1.4.

Before we can state the first main result we have to define the following numbers. Here and in the sequel \mathbb{N} will denote the nonnegative integers and \mathbb{P} the strictly positive integers. For $d, r, i \in \mathbb{N}$ let

$$C(r, d, i) := \left| \left\{ (a_1, \dots, a_d) \in \mathbb{N}^d \mid \begin{array}{l} a_1 + \dots + a_d = i \\ a_j \leq r \text{ for } 1 \leq j \leq d \end{array} \right\} \right|,$$

for $d \geq 1$ and $C(r, 0, i) = \delta_{0,i}$. Note that it is easy to see that

$$C(r, d, i) = \sum_{\{\lambda \subseteq (r^d): |\lambda|=i\}} \binom{d}{m_1(\lambda), \dots, m_r(\lambda), d-l(\lambda)},$$

where, for a partition λ , $l(\lambda)$ denotes the number of parts of λ and $m_i(\lambda)$ the number of parts of λ that are equal to i .

Theorem 1.1. *Let $(a_n)_{n \geq 0}$ be a sequence of complex numbers such that for some $s, d \geq 0$ its generating series $f(t) := \sum_{n \geq 0} a_n t^n$ satisfies*

$$f(t) = \frac{h_0 + \dots + h_s t^s}{(1-t)^d}.$$

Then for any $r \in \mathbb{P}$ we have

$$f^{\langle r \rangle}(t) = \sum_{n \geq 0} a_{nr} t^n = \frac{h_0^{\langle r \rangle} + \dots + h_m^{\langle r \rangle} t^m}{(1-t)^d},$$

where $m := \max(s, d)$ and

$$h_i^{\langle r \rangle} = \sum_{j=0}^s C(r-1, d, ir-j) h_j,$$

for $i = 0, \dots, m$.

The following is a simple reformulation of Theorem 1.1 in the case of Hilbert-series of standard graded k -algebras.

Corollary 1.2. *Let A be a standard graded k -algebra of dimension d with Hilbert-series*

$$\text{Hilb}(A, t) = \frac{h_0 + \dots + h_s t^s}{(1-t)^d}.$$

Then for any $r \in \mathbb{P}$ we have

$$\text{Hilb}(A^{(r)}, t) = \frac{h_0^{(r)} + \cdots + h_m^{(r)} t^m}{(1-t)^d},$$

where $m := \max(s, d)$ and

$$h_i^{(r)} = \sum_{j=0}^s C(r-1, d, ir-j) h_j,$$

for $i = 0, \dots, m$.

As a consequence it follows that the first $d+1$ entries of the h -vector $h(A) = (h_0, \dots, h_s)$ of a standard graded algebra grow weakly when taking Veronese subalgebras in case $h_i \geq 0$ for $0 \leq i \leq s$. Note that this condition for example is satisfied if A is Cohen-Macaulay (see for example [2, Proposition 4.3.1]).

Corollary 1.3. *Let A be a standard graded k -algebra of dimension d with Hilbert-series*

$$\text{Hilb}(A, t) = \frac{h_0 + \cdots + h_s t^s}{(1-t)^d}$$

such that $h_i \geq 0$ for $i = 0, \dots, s$. Then for any $r \geq 1$ the Hilbert-series

$$\text{Hilb}(A^{(r)}, t) = \frac{h_0^{(r)} + \cdots + h_m^{(r)} t^m}{(1-t)^d}$$

of the r th Veronese algebra of A satisfies $h_i^{(r)} \geq h_i$ for $0 \leq i \leq d$. Moreover, if $r \geq d$ then $h_i^{(r)} > h_i$ for $1 \leq i \leq d-1$. In particular, all conclusions hold if A is Cohen-Macaulay.

Proof. By Corollary 1.2 it follows that $h_i^{(r)} = \sum_{j=0}^s C(r-1, d, ir-j) h_j$. Clearly, $C(r-1, d, ir-j) \geq 0$ for all r, d, i, j . Moreover, for $0 \leq i \leq d$ we have $C(r-1, d, ir-i) \geq 1$ since the sum

$$i(r-1) = \underbrace{(r-1) + \cdots + (r-1)}_{i \text{ times}} + \underbrace{0 + \cdots + 0}_{(d-i) \text{ times}}$$

is clearly among the sums counted by $C(r-1, d, i(r-1))$. This implies $h_i^{(r)} \geq h_i$ for $0 \leq i \leq d$. It is well known that for a standard graded algebra we have $h_0 = 1$. Now if $r \geq d$ then for $1 \leq i \leq d-1$ we have $ir \leq (d-1)r \leq d(r-1)$. Thus there is at least one sum representation of ir with d summands $\leq r-1$. Hence $C(r-1, d, ir) \geq 1$ and therefore $C(r-1, d, ir-0)h_0 \geq 1$ which then implies $h_i^{(r)} \geq h_i + 1$ for $1 \leq i \leq d-1$. \square

Note that for the Hilbert-series $\text{Hilb}(A, t) = \frac{h_0 + \cdots + h_d t^d}{(1-t)^d}$ of a standard graded algebra A it is well known that $h_0^{(r)} = h_0 = 1$ and $h_d^{(r)} = h_d$. Of course these identities also follow from Theorem 1.1.

Theorem 1.4. *For any $d \geq 2$ there are strictly negative real numbers $\alpha_1 \dots, \alpha_{d-2}$ such that for any sequence $(a_n)_{n \geq 0}$ of real numbers such that $a_0 = 1$ and $a_n \geq 0$ for large n whose generating series $f(t) = \sum_{n \geq 0} a_n t^n = \frac{h(t)}{(1-t)^d}$ for some polynomial $h(t)$ with $h(1) \neq 0$ there is a number $R > 0$ and sequences of complex numbers $(\beta_r^{(i)})_{r \geq 1}$, $1 \leq i \leq d$, such that :*

- (i) $\beta_r^{(i)}$ is real for $r > R$ and $1 \leq i \leq d$ and strictly negative for $r > R$ and $1 \leq i \leq d-1$.
- (ii) $\beta_r^{(i)} \rightarrow \alpha_i$ for $r \rightarrow \infty$ and $1 \leq i \leq d-2$.
- (iii) $\beta_r^{(d-1)} \rightarrow -\infty$ for $r \rightarrow \infty$.
- (iv) $\beta_r^{(d)} \rightarrow 0$ for $r \rightarrow \infty$.
- (v) $h_0^{(r)} + \dots + h_d^{(r)} t^d = \prod_{i=1}^d (1 - \beta_r^{(i)} t)$.

Corollary 1.5. *For any $d \geq 2$ there are strictly negative real numbers $\alpha_1 \dots, \alpha_{d-2}$ such that for any standard graded k -algebra of dimension d with Hilbert-series*

$$\text{Hilb}(A, t) = \frac{h_0 + \dots + h_s t^s}{(1-t)^d}$$

there are $R > 0$ and sequences of complex numbers $(\beta_r^{(i)})_{r \geq 1}$, $1 \leq i \leq d$, such that :

- (i) $\beta_r^{(i)}$ is real for $r > R$ and $1 \leq i \leq d$ and strictly negative for $r > R$ and $1 \leq i \leq d-1$.
- (ii) $\beta_r^{(i)} \rightarrow \alpha_i$ for $r \rightarrow \infty$ and $1 \leq i \leq d-2$.
- (iii) $\beta_r^{(d-1)} \rightarrow -\infty$ for $r \rightarrow \infty$.
- (iv) $\beta_r^{(d)} \rightarrow 0$ for $r \rightarrow \infty$.
- (v) $h_0^{(r)} + \dots + h_d^{(r)} t^d = \prod_{i=1}^d (1 - \beta_r^{(i)} t)$, for $r > R$.

In [1] and [9] it is shown that for a standard graded k -algebra A and r large enough the r th Veronese $A^{(r)}$ is Koszul (see [10, p. 450]). In turn it is known (see for example [14]) that this property implies that the numerator polynomial of the Hilbert-series has at least one real root. Thus in some sense the previous and the following corollary are inspired by this algebraic limiting results.

Corollary 1.6. *Let A be a standard graded k -algebra of dimension $d \geq 1$ with Hilbert-series*

$$\text{Hilb}(A, t) = \frac{h_0 + \dots + h_s t^s}{(1-t)^d}.$$

Then there is $R > 0$ such that for any $r > R$ we have:

- (i) $h_i^{(r)} \geq 1$, $0 \leq i \leq d-1$,
- (ii) $h_i^{(r)} = 0$ for $i \geq d+1$
- (iii) $h^{(r)}(t)$ has only real zeros.

In particular, for $r > R$ the sequence $(h_0^{(r)}, \dots, h_d^{(r)})$ is log-concave and unimodal.

Note that this results in particular implies that the unimodality conjectures by Stanley (see e.g. [16]) and Hibi (see [13]) on the h -vector of various classes of standard graded algebras are true one one takes sufficiently high Veronese subrings.

2. PROOF OF THEOREM 1.1

Proof of Theorem 1.1. Set $h(t) := \sum_{i=0}^s h_i t^i$ so that $f(t) = \frac{h(t)}{(1-t)^d}$. Let $\rho \in \mathbb{C}$ be a primitive r -th root of unity. Then ρ and all its powers ρ^j for any $j \in \mathbb{N}$ such that $j \not\equiv 0 \pmod{r}$ satisfy $\sum_{i=0}^{r-1} (\rho^j)^i = 0$. Hence:

$$\begin{aligned} \sum_{n \geq 0} a_{rn} t^{rn} &= \frac{1}{r} \sum_{i=0}^{r-1} f(\rho^i t) \\ &= \frac{1}{r} \sum_{i=0}^{r-1} \frac{h(\rho^i t)}{(1 - \rho^i t)^d} \\ &= \frac{\sum_{i=0}^{r-1} \left(\frac{1-t^r}{1-\rho^i t} \right)^d h(\rho^i t)}{r(1-t^r)^d} \\ &= \frac{\sum_{i=0}^{r-1} \left(\frac{1-(\rho^i t)^r}{1-\rho^i t} \right)^d h(\rho^i t)}{r(1-t^r)^d} \end{aligned}$$

But

$$\begin{aligned} \frac{1}{r} \sum_{\ell=0}^{r-1} \left(\frac{1-(\rho^\ell t)^r}{1-\rho^\ell t} \right)^d h(\rho^\ell t) &= \frac{1}{r} \sum_{\ell=0}^{r-1} \left((1 + \rho^\ell t + \cdots + (\rho^\ell t)^{r-1})^d h(\rho^\ell t) \right) \\ &= \frac{1}{r} \sum_{\ell=0}^{r-1} \left(\sum_{i \geq 0} C(r-1, d, i) (\rho^\ell t)^i \right) h(\rho^\ell t) \\ &= \frac{1}{r} \sum_{i \geq 0} C(r-1, d, i) \left(\sum_{\ell=0}^{r-1} (\rho^\ell t)^i \sum_{j=0}^s h_j (\rho^\ell t)^j \right) \\ &= \frac{1}{r} \sum_{j=0}^s \sum_{i \geq 0} C(r-1, d, i-j) h_j \sum_{\ell=0}^{r-1} (\rho^\ell t)^i \\ &= \sum_{i \geq 0} \left(\sum_{j=0}^s C(r-1, d, ir-j) h_j \right) t^{ri} \end{aligned}$$

and the result follows since if $i > s \geq d$ then $ir - j > sr - s \geq d(r-1)$ for all $0 \leq j \leq s$ while if $i > d > s$ then $ir - j > dr - d$ for all $0 \leq j \leq s$. \square

We now analyze the transformation described in Theorem 1.1 more closely. Consider the vectorspace R_d of all rational functions $\frac{h(t)}{(1-t)^d}$ for polynomials $h(t)$ of degree $\leq d$. We consider two bases of R_d . First the basis \mathcal{B}_d^1 consisting of all $\frac{t^i}{(1-t)^d}$ for $0 \leq i \leq d$. For the second basis we recall the definition of an Eulerian polynomial. For a number $i \geq 1$ we define $A_i(t) = \sum_{\sigma \in S_i} t^{\text{des}(\sigma)+1}$, where $\text{des}(\sigma)$ is the number of descents of the permutation σ . If we also set $A_0(t) = A_{-1}(t) = 1$ then the set \mathcal{B}_d^2 consisting of all $\frac{A_{i-1}(t)(1-t)^{d-i}}{(1-t)^d}$, $0 \leq i \leq d$, is a second basis of R_d . Using the fact that for $1 \leq i$ one has $A_i(1) \neq 0$ one easily checks that indeed \mathcal{B}_d^2 is a basis of R_d .

For a fixed $r \geq 1$ let $\Phi_r : R_d \rightarrow R_d$ be the map that sends $f(t)$ to $f^{(r)}(t)$. Note that in the basis \mathcal{B}_d^2 the map Φ_r sends $\sum_{n \geq 0} n^i t^n = \frac{A_i(t)}{(1-t)^{i+1}}$ for $0 \leq i < d$ to

$$\begin{aligned} \Phi_r\left(\sum_{n \geq 0} n^i t^n\right) &= \sum_{n \geq 0} (rn)^i t^n \\ &= r^i \sum_{n \geq 0} n^i t^n \\ &= r^i \frac{A_i(t)}{(1-t)^{i+1}} \end{aligned}$$

while $\Phi_r(A_{-1}(t)) = A_{-1}(t)$. In particular, this confirms that Φ_r is a map from R_d to R_d . Moreover, since Φ_r is easily seen to be linear it follows that in the basis \mathcal{B}_d^2 the map Φ_r is represented by the $(d+1) \times (d+1)$ diagonal matrix $\text{diag}(1, 1, r, \dots, r^{d-1})$. The preceding arguments and Theorem 1.1 imply the following lemma.

Lemma 2.1. *Let $\mathfrak{C}_{d,r} = (C(r-1, d, ir-j))_{0 \leq i,j \leq d}$. Then $\mathfrak{C}_{d,r}$ is the matrix representing the linear transformation Φ_r with respect to the basis \mathcal{B}_d^1 . In particular, $\mathfrak{C}_{d,r}$ is diagonalizable with eigenvalues 1 of multiplicity two and r, \dots, r^{d-1} of multiplicity one.*

Indeed we can give a factorization of $\mathfrak{C}_{d,r}$ which also clarifies its eigenspaces. Let $\mathfrak{L}_d = (l_{ij})_{0 \leq i,j \leq d}$ be the $(d+1) \times (d+1)$ matrix with entries l_{ij} defined by $A_{i-1}(t)(1-t)^{d-i} = \sum_{j=0}^d l_{j,i} t^j$.

Lemma 2.2. *For any $d, r \geq 1$ we have*

$$\mathfrak{C}_{d,r} = \mathfrak{L}_d \cdot \text{diag}(1, 1, r, \dots, r^{d-1}) \cdot \mathfrak{L}_d^{-1}.$$

The vector $\ell_i = (l_{0i}, \dots, l_{di})^t$ is an eigenvector of $\mathfrak{C}_{d,r}$ for the eigenvalue r^{i-1} for $1 \leq i \leq d$. Moreover:

- (i) $l_{d,i} = 0$ for $1 \leq i \leq d$ and $l_{d,0} = (-1)^d$.
- (ii) $l_{0,i} = 0$ for $2 \leq i \leq d$ and $l_{0,0} = l_{0,1} = 1$.
- (iii) $l_{j,d} \geq 1$, $1 \leq j \leq d-1$.

Proof. The matrix \mathfrak{L}_d describes the base change from \mathcal{B}_d^2 to \mathcal{B}_d^1 . Since by the arguments preceding Lemma 2.1 $\text{diag}(1, 1, r, \dots, r^{d-1})$ is the matrix of Φ_r with respect to the basis \mathcal{B}_d^2 and since by Theorem 1.1 $\mathfrak{C}_{d,r}$ is the matrix of Φ_r with respect to the basis \mathcal{B}_d^1 it follows that $\mathfrak{C}_{d,r} = \mathfrak{L}_d \text{diag}(1, 1, r, \dots, r^{d-1}) \mathfrak{L}_d^{-1}$. The preimage of the i th unit vector under \mathfrak{L}_d^{-1} is ℓ_i . Since the i th unit vector is an eigenvector for r^{i-1} of $\text{diag}(1, 1, r, \dots, r^{d-1})$ it follows by the first part of the lemma that ℓ_i is an eigenvector of $\mathfrak{C}_{d,r}$ for the eigenvalue r^{i-1} , $1 \leq i \leq d$.

Assertions (i) - (iii) are immediate consequences of the definitions. \square

The preceding lemma implies the following combinatorial identity for Eulerian numbers. For $d \geq 1$ and $0 \leq i \leq d$ let $A(d, i) = |\{\sigma \in S_d : \text{des}(\sigma) = i - 1\}|$.

Proposition 2.3. *Let $d, r \geq 1$. Then*

$$\sum_{j=0}^d C(r-1, d+1, ir-j) A(d, j) = r^d A(d, i)$$

for $i = 0, \dots, d$. In particular

$$\sum_{j=0}^d C(r-1, d+1, r-j) A(d, j) = r^d.$$

Clearly, Proposition 2.3 asks for a combinatorial proof.

3. PROOF OF THEOREM 1.4

Before we come to the proof of Theorem 1.4 we need some preparatory lemmas.

Lemma 3.1. *Let $f(t) = \frac{h_0 + \dots + h_s t^s}{(1-t)^d}$ for some $d, s \geq 0$. Write $f(t) = p(t) + \frac{h_1(t)}{(1-t)^d}$ for polynomials $p(t)$ and $h_1(t)$ where $h_1(t)$ is of degree $\leq d-1$. Then for any $r \geq s-d+1$ we have*

$$f(t)^{\langle r \rangle} = \frac{h^{\langle r \rangle}(t)}{(1-t)^d} = f_1(t)^{\langle r \rangle}$$

for some polynomial $h^{\langle r \rangle}(t)$ of degree $\leq d$ and $f_1(t) = \frac{p(0)(1-t)^d + h_1(t)}{(1-t)^d}$. Moreover, $h_0 = h^{\langle r \rangle}(0)$ for all r .

Proof. If $s \leq d$ then the result follows immediately from Theorem 1.1 so assume $s > d$. Then $f(t) = p_0 + p_1 t + \dots + p_{s-d} t^{s-d} + \frac{h_1(t)}{(1-t)^d}$ for a polynomial $h_1(t)$ of degree $\leq d-1$. Hence $f(t)^{\langle r \rangle} = p_0 + \frac{h_1(t)^{\langle r \rangle}}{(1-t)^d} = f_1(t)^{\langle r \rangle}$ for $r \geq s-d+1$. Since by Theorem 1.1 $h_1(t)^{\langle r \rangle}$ is a polynomial of degree $\leq d$ the assertion follows.

The last equality follows by evaluating the generating series at $t = 0$. \square

Remark 3.2. Let $f(t) = \sum_{n=0}^{\infty} a_n t^n = \frac{h(t)}{(1-t)^d}$. If $d = 0$ then for sufficiently large r we have $f(t)^{\langle r \rangle} = a_0$. If $d = 1$ then for sufficiently large r we have $f(t)^{\langle r \rangle} = \frac{h_0 + h_1 t}{1-t}$

with h_0, h_1 independent of r . In particular, the numerator polynomial of $f(t)^{\langle r \rangle}$ is real rooted with at most one root which is independent of r .

We recall the following lemma from [4].

Lemma 3.3 (Lemma 4.9 [4]). *Let $(g_n(t))_{n \geq 0}$ be a sequence of real polynomials of degree $d - 2$, $f(t)$ another real polynomial of degree $d - 2$ and $\rho > 1$, h_d real numbers such that:*

- ▷ $\lim_{n \rightarrow \infty} g_n(t)/\rho^n = 0$, where the limit is taken in \mathbb{R}^{d-1} .
- ▷ All the roots of the polynomial $f(t)$ are strictly negative and all coefficients of $f(t)$ are strictly positive.

Then there are real numbers α_i , $1 \leq i \leq d - 2$ and sequences $(\beta_i^{(n)})_{n \geq 0}$, $1 \leq i \leq d$ of complex numbers such that:

- (i) $\beta_i^{(n)}$, $1 \leq i \leq d$, are real for n sufficiently large.
- (ii) $\lim_{n \rightarrow \infty} \beta_i^{(n)} = \alpha_i$, $1 \leq i \leq d - 2$.
- (iii) $\lim_{n \rightarrow \infty} \beta_{d-1}^{(n)} = 0$.
- (iv) $\lim_{n \rightarrow \infty} \beta_d^{(n)} = -\infty$.
- (v) $\prod_{i=0}^{d-1} (t - \beta_i^{(n)}) = h_d + t g_n(t) + \rho^n t f(t) + t^d$.

Proof of Theorem 1.4. By Lemma 3.1 we may assume that $h(t)$ is of degree $\leq d$. Then Theorem 1.1 implies that for $f^{(r)}(t) = \frac{h^{(r)}(t)}{(1-t)^d}$ the polynomial $h^{(r)}(t)$ is again of degree $\leq d$.

Let $\ell_i = (\ell_{0i}, \dots, \ell_{di})^t$, $0 \leq i \leq d$ be the eigenvectors of $\mathfrak{C}_{d,r}$ as in Lemma 2.2. Let $\ell_i(t) = \frac{\sum_{j=0}^d \ell_{j,i} t^j}{(1-t)^d}$, $0 \leq i \leq d$ so $\mathcal{B}_d^2 = \{\ell_0(t), \dots, \ell_d(t)\}$. Let $f(t) = \alpha_0 \ell_0(t) + \dots + \alpha_d \ell_d(t)$ be the expansion of $f(t)$ in the basis \mathcal{B}_d^2 . Then by Lemma 2.2 we have $f(t)^{\langle r \rangle} = \alpha_0 \ell_0(t) + \alpha_1 \ell_1(t) + \alpha_2 r \ell_2(t) + \dots + \alpha_d r^{d-1} \ell_d(t)$. Let

$$g_r(t) \stackrel{\text{def}}{=} t^d h^{(r)} \left(\frac{1}{t} \right) - h_d - t^d - r^{d-1} \alpha_d t^d \tilde{\ell}_d \left(\frac{1}{t} \right)$$

where $\tilde{\ell}_d(t) \stackrel{\text{def}}{=} (1-t)^d \ell_d(t)$. Then $\deg(g_r) \leq d-1$, $g_r(0) = 0$ and $\lim_{r \rightarrow \infty} g_r(t)/r^{d-1} = 0$. Now $\tilde{\ell}_d(t) = A_{d-1}(t)$ is the $(d-1)$ st Eulerian polynomial. Thus $\ell_d(t) = t^d \ell_d(1/t)$. Hence if we set $\tilde{g}_r(t) \stackrel{\text{def}}{=} g_r(t)/t$, $\tilde{f}(t) \stackrel{\text{def}}{=} A_{d-1}(t)/t$ then $\tilde{f}(t)$ is a polynomial of degree $d-2$, which is real rooted by [6, p. 292, Ex. 3] and has strictly positive coefficients except for the constant, and all the hypotheses of Lemma 3.3 are satisfied.

Hence Lemma 3.3 becomes applicable and the result follows by passing to the reciprocal polynomials. \square

It remains to provide proofs of Corollaries 1.5 and 1.6. Corollary 1.5 is just a reformulation of Theorem 1.4 for Hilbert-series. So there is nothing to prove. But note that in general passing to Veronese subrings $A^{\langle r \rangle}$ for small r does not suffice to guarantee that $h^{(r)}(t)$ is real rooted, even in case $h(t)$ is of degree $d-1$ and has

strictly positive coefficients. For example, $f(t) = \frac{1+t+t^2+t^3+t^4}{(1-t)^5}$ is the Hilbert-series of the Stanley-Reisner ring of the boundary complex of the 4-simplex. However $f^{(2)}(t) = \frac{1+16t+31t^2+26t^3+6t^4}{(1-t)^5}$ whose numerator polynomial has only two real roots.

Proof of 1.6. If $d = 1$ then (iii) follows from (ii) which follows immediately from Lemma 3.1.

Let $d \geq 2$. Assertions (ii) and (iii) follow immediately from Lemma 3.1 and Corollary 1.5. From the proof of Theorem 1.4 we recall that the numerator polynomial of $\text{Hilb}(A^{(r)}, t)$ can be written as

$$(3.1) \quad h_d t^d + t^d g_r \left(\frac{1}{t} \right) + \alpha_d r^{d-1} A_{d-1}(t) + 1$$

for polynomials $g_r(t)$ and the Eulerian polynomial $A_{d-1}(t)$. Moreover we have $\lim_{r \rightarrow \infty} g_r(t)/r^{d-1} = 0$ and $A_{d-1}(t)$ has strictly positive coefficients except for the constant term which is 0. But this implies that for large enough r the polynomial $1/r^{d-1}(t^d g_r(1/t) + r^{d-1} A_{d-1}(t))$ has strictly positive coefficients. This then implies that for large enough r the polynomial in (3.1) has strictly positive coefficients except for possibly h_d , which proves (i). \square

4. VERONESE OF STANLEY-REISNER RINGS

For a simplicial complex Δ over ground set Ω and a field k we denote by $k[\Delta]$ its Stanley-Reisner ring. Recall that $k[\Delta] = k[x_\omega : \omega \in \Omega]/I_\Delta$, where I_Δ is the ideal generated by the monomials $\prod_{\omega \in A} x_\omega$ for $A \notin \Delta$. Assume that Δ is $(d-1)$ -dimensional. We denote by $f(\Delta) = (f_{-1}, \dots, f_{d-1})$ the f -vector of Δ ; that is f_i is the number of i -dimensional faces of Δ . Then it is well known that :

$$(4.1) \quad \begin{aligned} \text{Hilb}(k[\Delta], t) &= \frac{h_0 + \dots + h_d t^d}{(1-t)^d} \\ &= \frac{\sum_{i=0}^d f_{i-1} t^i (1-t)^{d-i}}{(1-t)^d} \end{aligned}$$

The r th Veronese of $k[\Delta]$ is a Stanley-Reisner ring only in extremal cases, but still it has turned out to be fruitful and meaningful to look for a simplicial complex $\Delta^{(r)}$ such that

$$(4.2) \quad \text{Hilb}(k[\Delta]^{(r)}, t) = \text{Hilb}(k[\Delta^{(r)}], t).$$

In [3] based on earlier ideas by Sturmfels [17], Brun and Römer consider the following situation. Set $S = k[x_1, \dots, x_n]$ the polynomial ring and let I_Δ be the Stanley-Reisner ideal of a simplicial complex Δ on ground set $[n]$. Then the r -th Veronese $(S/I_\Delta)^{(r)}$ can be described as a quotient of the polynomial ring $S(r)$ in the variables x_{i_1, \dots, i_n} indexed by numbers $0 \leq i_1, \dots, i_n$ such that $i_1 + \dots + i_n = r$. If $I(r)$ is the ideal in $S(r)$ such that $(S/I_\Delta)^{(r)} = S(r)/I(r)$ then Brun and Römer

[3, Section 6] describe a initial ideal of $I(r)$ which is the Stanley-Reisner ideal of a simplicial complex $\Delta(r)$ on vertex set $\Omega_r = \{(i_1, \dots, i_n) \in \mathbb{N}^n : i_1 + \dots + i_n = r\}$. By basic facts about initial ideals it follows that this $\Delta(r)$ fulfills (4.2). The simplicial complex $\Delta(r)$ turns out to be realizable as a subdivision of Δ . This subdivision is called r -th edgewise subdivision and as outlined in [3] has a long history in algebraic topology (see e.g. [8], [12]) and a shorter one in discrete geometry (see e.g. [7]).

Before we can describe edgewise subdivision we need some technical preparations. Consider \mathbb{R}^n together with its standard unit basis vectors $\mathbf{e}_1, \dots, \mathbf{e}_n$. By the obvious identification we can consider Δ as a simplicial complex over the ground set $\{\mathbf{e}_1, \dots, \mathbf{e}_n\} = \Omega_1$. Note that for $r \geq 1$ the elements of Ω_r are the points with integer coordinates in the the r -th dilation of the simplex spanned by Ω_1 . For a vector $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{R}^n$ its support $\text{supp}(\mathbf{a})$ is the set $\{i : a_i \neq 0\}$ of indices of non-zero coordinates. For $i \in [n]$ set $\mathbf{u}_i = \mathbf{e}_i + \mathbf{e}_{i+1} + \dots + \mathbf{e}_n$ and for $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{R}^n$ set $\iota(\mathbf{a}) := \sum_{i=1}^n a_i \mathbf{u}_i$.

The r -th edgewise subdivision of Δ is the simplicial complex $\Delta(r)$ on ground set Ω_r such that $A \subseteq \Omega_r$ is a simplex of $\Delta(r)$ if and only if

$$(\text{ESD1}) \quad \bigcup_{\mathbf{v} \in A} \text{supp}(\mathbf{v}) \in \Delta.$$

$$(\text{ESD2}) \quad \text{For all } \mathbf{v}, \mathbf{v}' \in A \text{ either } \iota(\mathbf{v} - \mathbf{v}') \in \{0, 1\}^n \text{ or } \iota(\mathbf{v}' - \mathbf{v}) \in \{0, 1\}^n.$$

Now the result by Brun and Römer [3] states.

Proposition 4.1 (Proposition 6.4 in [3]). *Let Δ be a simplicial complex on ground set $[n]$ and $I(r)$ be such that $(S/I_\Delta)^{(r)} = S(r)/I(r)$. Then there is a term order for which $I_{\Delta(r)}$ is the initial ideal of $I(r)$.*

In particular, $\Delta^{(r)} := \Delta(r)$ satisfies Equation (4.2). Clearly, a simplicial complex $\Delta^{(r)}$ satisfying Equation (4.2) is not uniquely defined. But $\Delta(r)$ appears to be a natural choice.

First we want to study the enumerative properties of $\Delta(r)$. A partial analysis can also be found in [7] but there the final formulas appear as alternating sums which is not fully satisfactory from the point of view of Enumerative Combinatorics. For enumerative purposes Equation (4.1) suggests to study a third basis of R_d . We denote by \mathcal{B}_d^3 the set of rational functions $\frac{t^i(1-t)^{d-i}}{(1-t)^d}$, $0 \leq i \leq d$. Indeed \mathcal{B}_d^3 will be crucial in the proof of the following proposition.

Proposition 4.2. *Let Δ be a simplicial complex of dimension $d-1$ with f -vector $f(\Delta) = (f_{-1}, \dots, f_{d-1})$ and $r \in \mathbb{P}$. If $\Delta^{(r)}$ is a simplicial complex such that $\text{Hilb}(k[\Delta]^{(r)}, t) = \text{Hilb}(k[\Delta^{(r)}], t)$ then its f -vector $f(\Delta^{(r)}) = (f_{-1}^{(r)}, \dots, f_{d-1}^{(r)})$ satisfies*

$$f_{i-1}^{(r)} = \sum_{\ell=i}^d \sum_{\substack{j_1 + \dots + j_i = \ell \\ j_1, \dots, j_i \geq 1}} \binom{r-1}{j_1-1} \binom{r}{j_2} \cdots \binom{r}{j_i} f_{\ell-1},$$

for $0 \leq i \leq d$.

Proof. We denote by $v_i(t) = \frac{t^i(1-t)^{d-i}}{(1-t)^d}$, $0 \leq i \leq d$, the elements of \mathcal{B}_d^3 . Then there are numbers $a_{i,\ell}^{(r)}$ ($0 \leq i, \ell \leq d$) such that

$$(4.3) \quad v_i(t)^{\langle r \rangle} = \sum_{\ell=0}^d a_{i,\ell}^{(r)} v_\ell(t).$$

Hence

$$(4.4) \quad \sum_{i=0}^{\infty} v_i(t)^{\langle r \rangle} x^i = \sum_{i=0}^{\infty} \sum_{\ell=0}^d a_{i,\ell}^{(r)} \frac{x^i t^\ell}{(1-t)^\ell}$$

$$(4.5) \quad = \sum_{\ell=0}^d \left(\sum_{i=0}^{\infty} a_{i,\ell}^{(r)} x^i \right) \frac{t^\ell}{(1-t)^\ell}$$

Next we derive a second expansion of the left hand side of (4.4). First we derive an expansion of $v_i(t)^{\langle r \rangle}$ as a formal power series. Clearly, for $i \geq 1$,

$$v_i(t) = \frac{t^i}{(1-t)^i} = \sum_{j=0}^{\infty} \binom{j-1}{i-1} t^j$$

Therefore, for $i \geq 1$,

$$v_i(t)^{\langle r \rangle} = \sum_{j=0}^{\infty} \binom{rj-1}{i-1} t^j.$$

For $i = 0$ we have $v_0^{\langle r \rangle}(t) = 1 = v_0(t)$.

The preceding expansion leads to the following identity.

$$\begin{aligned} \sum_{i=0}^{\infty} v_i(t)^{\langle r \rangle} x^i &= 1 + \sum_{i=1}^{\infty} \sum_{j=0}^{\infty} \binom{rj-1}{i-1} t^j x^i \\ &= 1 + \sum_{j=0}^{\infty} \left(x \sum_{i=1}^{\infty} \binom{rj-1}{i-1} x^{i-1} \right) t^j \\ &= 1 + \sum_{j=1}^{\infty} x(1+x)^{rj-1} t^j \\ (4.6) \quad &= 1 + t(1+x)^{r-1} \frac{x}{1-t(1+x)^r} \end{aligned}$$

Writing the formulas from (4.5) and (4.6) in terms of the variable $u = \frac{t}{1-t}$ or equivalently $t = \frac{u}{1+u}$ and comparing we obtain

$$1 + u(1+x)^{r-1} \frac{x}{1+u-u(1+x)^r} = \sum_{\ell=0}^d \left(\sum_{i=0}^{\infty} a_{i,\ell}^{(r)} x^i \right) u^\ell$$

Thus by

$$\frac{xu(1+x)^{r-1}}{1+u-u(1+x)^r} = \sum_{\ell=0}^{\infty} x(1+x)^{r-1} ((1+x)^r - 1)^\ell u^{\ell+1}$$

we obtain for $\ell \geq 1$

$$\sum_{i=0}^{\infty} a_{i,\ell}^{(r)} x^i = x(1+x)^{r-1} ((1+x)^r - 1)^{\ell-1}$$

and $\sum_{i=0}^{\infty} a_{i,0}^{(r)} x^i = 1$. Hence for any $\ell \geq 0$ and $i \geq 0$

$$(4.7) \quad a_{i,\ell}^{(r)} = \sum_{\substack{j_1+\dots+j_\ell=i \\ j_1,\dots,j_\ell \geq 1}} \binom{r-1}{j_1-1} \binom{r}{j_2} \cdots \binom{r}{j_\ell}$$

From this we conclude the following equalities which imply the assertion

$$\begin{aligned} \sum_{i=0}^d f_{i-1}^{(r)} v_i(t) &= \text{Hilb}(k[\Delta^{(r)}], t) = \text{Hilb}(k[\Delta]^{(r)}, t) \\ &= \text{Hilb}(k[\Delta], t)^{(r)} = \sum_{i=0}^d f_{i-1} v_i(t)^{(r)} \\ &= \sum_{i=0}^d f_{i-1} \sum_{\ell=0}^d a_{i,\ell}^{(r)} v_\ell(t) \\ &= \sum_{\ell=0}^d \left(\sum_{i=0}^d f_{i-1} a_{i,\ell}^{(r)} \right) v_\ell(t) \\ &= \sum_{\ell=0}^d \left(\sum_{i=\ell}^d f_{i-1} \sum_{\substack{j_1+\dots+j_\ell=i \\ j_1,\dots,j_\ell \geq 1}} \binom{r-1}{j_1-1} \binom{r}{j_2} \cdots \binom{r}{j_\ell} \right) v_\ell(t). \end{aligned}$$

The last equality follows from (4.7) and the fact that $a_{i,\ell}^{(r)} = 0$ for $i < \ell$. \square

As an immediate consequence of Theorem 1.4 we also get the following result.

Proposition 4.3. *For any $d \geq 2$ there are real numbers $\alpha'_1, \dots, \alpha'_{d-2}$ such that for any $(d-1)$ -dimensional simplicial complex Δ there are $R > 0$ and sequences of complex numbers $(\beta_r^{(i)})_{r \geq 1}$, $1 \leq i \leq d$, such that :*

- (i) $\beta_r^{(i)}$ is real for $r > R$ and $1 \leq i \leq d$ and strictly negative for $r > R$ and $1 \leq i \leq d-1$.
- (ii) $\beta_r^{(i)} \rightarrow \alpha'_i$ for $r \rightarrow \infty$ and $1 \leq i \leq d-2$.
- (iii) $\beta_r^{(d-1)} \rightarrow -\infty$ for $r \rightarrow \infty$,
- (iv) $\beta_r^{(d)} \rightarrow -1$ for $r \rightarrow \infty$,
- (v) $f_{-1}^{(r)} + \cdots + f_{d-1}^{(r)} t^d = \prod_{i=1}^d (1 - \beta_r^{(i)} t)$, for $r > R$.

In particular, for $r > R$ the f -vector of $\Delta^{(r)}$ is log-concave and unimodal.

In [4, Theorem 3.1] it is shown that for a simplicial complex with non-negative h -vector the h -polynomial of its first barycentric subdivision is real rooted. The example preceding the proof of Corollary 1.6 shows that such a result is not true for edgewise subdivision. More precisely, it shows that the h -polynomial of the second edgewise subdivision of the boundary of the 4-simplex is of degree 4 but has only two real roots.

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