2nd Marburger Arbeitsgemeinschaft Mathematik – MAM 2

The Toral Rank Conjecture

Organizers: Ilka Agricola, Manuel Amann, Giovanni Bazzoni, Oliver Goertsches,
Bernhard Hanke, Sönke Rollenske, and Leopold Zoller

Marburg, March 14th to 18th, 2022

The Toral Rank Conjecture (TRC), formulated in 1985 by Steve Halperin, asserts that whenever a compact torus T^r acts almost freely, i.e., with only finite stabilizers, on a compact Hausdorff space X, the inequality $\dim H^*(X;\mathbb{Q}) \geq \dim H^*(T^r;\mathbb{Q}) = 2^r$ holds. It is one of the most prominent open questions in the theory of topological transformation groups.

Schedule of Talks

Monday	Tuesday	Wednesday	Thursday	Friday
9:00 – 10:45 Registration & Coffee	9:00 – 10:00 10:10 – 10:55 Stegemeyer	9:00 – 10:30 Bazzoni	10:00 – 11:00 Coffee break	9:00 – 10:30 Gritschacher
10:45 Introduction		10:30 – 11:00 Coffee break		10:30 – 11:00 Coffee break
11:00–12:30 Serrano	10:55 – 11:30 Coffee break	11:00 – 12:30 Corro	11:00 – 12:30 Zoller	11:00 – 12:30 Rollenske
	11:30 – 12:30 Tralle			
12:30–14:00 Lunch	12:30–14:45 Lunch and coffee break	12:30 – Lunch and coffee break	12:30 – 14:00 Lunch	12:30 – Lunch and departure
14:00–15:30 Russo			14:00 – 15:30 van Steirteghem	
15:30–16:00 Coffee break	14:45 – 15:45 16:00 – 17:00 Alquezar/Frenck		15:30 – 16:00 Coffee break	
16:00–17:10 Kupper			16:00 – 17:00 Gritschacher	
	19:00 – Conference dinner			

All talks will take place at the Department of Mathematics and Computer Science on Lahnberge (bus stop Hans-Meerwein-Str.). Please see the conference homepage for how to get there.

Registration: SR VII (05D01). From the main entrance: please follow the signs at the walls.

Talks: HS IV (04A30). From the main entrance: please follow the yellow arrows on the floor.

(1) Compact Lie group actions

Give an introduction to properties of compact Lie group actions. Define orbits, stabilizers, isotropy types. Explain the slice theorem and the notion of regular orbits. Define the rank of a space, and give a first formulation of the TRC.

References: [21], [24], [15, Beginning of Section 7.3], [17, Appendix B]

Speaker: Jordi Daura Serrano

Contact person: Manuel Amann

(2) The Euler characteristic of the fixed point set

In this talk we will start exploring topological consequences of torus actions. Kobayashi proved in 1958 that the Euler characteristic of the fixed point set M^T of an action of a torus T on a compact manifold M equals the Euler characteristic of M.

References: [22]

Speaker: Giovanni Russo

Contact person: Oliver Goertsches

(3) Equivariant cohomology

The purpose of this talk is to give an introduction to equivariant cohomology from the topological point of view, i.e., via the Borel model. The main result of this talk will be the Borel localization theorem, which relates the equivariant cohomology of a T-action on a space X to that of the fixed point set X^T.

References: [5], [9], [17, Appendix C], [19]

Speaker: Philippe Kupper

Contact person: Manuel Amann

(4) Fibrations and spectral sequences

Give an introduction to spectral sequences, and prove some applications: 1) give an alternative proof of $\chi(X) = \chi(X^T)$, 2) understand when $H_G(X)$ is finitely generated, 3) show that $H_G(X)$ is finite-dimensional if and only if the G-action on X is almost free.

References: [4], [19], [16] (translated to the Borel model)

Speaker: Maximilian Stegemeyer

Contact person: Ilka Agricola

(5) The TRC for (c-)symplectic and Hard Lefschetz manifolds

The purpose is to investigate cohomologically Hamiltonian actions via the tools of equivariant cohomology and prove the TRC for Hard-Lefschetz Manifolds (or more generally Lefschetz-type c-symplectic spaces). Besides the TRC, other interesting aspects include the fact that a c-hamiltonian action on a Hard-Lefschetz space is equivariantly formal, which generalizes the corresponding result on Hamiltonian actions on symplectic manifolds. Interestingly, there is an example showing that the corresponding statement is false in the c-symplectic setting ([2, Rem. 1.6.4]).

References: [4], [15, Section 7.3.3], [23], [2]

Speaker: Alex Tralle

Contact person: Oliver Goertsches

(6) Rational homotopy theory I

Give an introduction to rational homotopy theory. In particular, introduce the concept and basic properties of Sullivan (minimal) models (it is convenient to restrict to the case of simply connected spaces).

References: [15, 13, 14]

Speaker: Georg Frenck / Carlos Alquezar

Contact person: Giovanni Bazzoni

(7) Rational homotopy theory II

Continuation of the previous talk. The goal should be to construct the Sullivan minimal model of a fibration. Also discuss the model of a pullback fibration.

References: [15, 13, 14]

Speaker: Georg Frenck / Carlos Alquezar

Contact person: Giovanni Bazzoni

(8) Variants I (=Rational homotopy theory III)

(Briefly) discuss the results on realization of algebraic models (through finite CW-complexes/manifolds). Explain the algebraic reformulation of the conjecture [15, Prop. 7.17] and compute the toral rank in examples.

References: [15]

Speaker: Giovanni Bazzoni

Contact person: Leopold Zoller

(9) Rationally elliptic and homogeneous spaces

Compute the toral rank of rationally elliptic spaces and interactions with the homotopy Euler characteristic. Prove the TRC for homogeneous spaces. Depending on time one can consider more generally the case of 2-step spaces [26].

References: [3], [15, Sections 7.3.1, 7.3.2, Proposition 7.23]

Speaker: Diego Corro

Contact person: Ilka Agricola

(10) Lower bounds for the toral rank

Construct the Hirsch–Brown model of an action, and use it to deduce linear lower bounds on the Betti numbers to prove the TRC for tori of dimension ≤ 3. Explain further obstructions to free torus actions given by results on the length of the lower-deg-filtration.

References: [4], [5, Theorems 4.4.3, 4.4.5] [6], [25]

Speaker: Leopold Zoller

Contact person: Leopold Zoller

(11) The TRC and the Buchsbaum–Eisenbud–Horrocks conjecture

Give a brief introduction to the concept of free resolutions and the Buchsbaum–Eisenbud–Horrocks Conjecture. Explain the connections between this conjecture and the TRC which hold under certain formality conditions.

References: [27], [8], [20]

Speaker: Bart van Steirteghem

Contact person: Sönke Rollenske

(12) Counterexamples to generalized TRCs

There are different ways to generalize the TRC by dropping different conditions which come naturally with the original geometric setting. In continuation of the last talk discuss the counterexamples from [20] related to free resolutions and the Buchsbaum–Eisenbud–Horrocks Conjecture, in particular, as well as their topological non-realizability. Also illustrate how the assertion from the TRC can fail to hold when the torus bundles are no longer principal (see [7]).

References: [27], [20], [7]

Speaker: will be incorporated in talks 8 and 12

Contact person: Sönke Rollenske

(13) Variants II: p-tori.

Explain what is known on cohomological consequences of the existence of a free action of a p-torus $(\mathbb{Z}_p)^r$ on a space X.

References: [10, 11, 18, 1]

Speaker: Simon Gritschacher

Contact person: Bernhard Hanke

(14) Variants III: Lie algebras

Formulate the Lie-Algebra-Version of the TRC and explain its connection to the previous variants. Prove the conjecture for 2-step nilpotent Lie algebras.

References: [12]

Speaker: Sönke Rollenske

Contact person: Sönke Rollenske

References

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