Dirac operators, holonomy and string theory

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Chapter 1

Summary

1. Introduction

Since Paul Dirac's formulation (1928) of the field equation for a quantized electron in flat Minkowski space, Dirac operators on Riemannian manifolds have become a powerful tool for the treatment of various problems in geometry, analysis and theoretical physics. Meanwhile, since the fifties the French school around M. Berger has developed the idea that manifolds should be subdivided into different classes according to their holonomy group. If they are not of general type, the name *special (integrable) geometries* has become customary for these. Already at early stage there were hints that parallel spinor fields induce special geometries, but this idea was not further investigated. In the early seventies, A. Gray generalized the classical holonomy concept to the effect that he introduced a classification principle for *non-integrable special Riemannian geometries* ([**Gra71**]) and studied the defining differential equations of each class. The connection between these two lines of development in mathematical physics became clear in the eighties in the context of twistor theory and the study of small eigenvalues of the Dirac operator and was mainly developed by the Berlin school around Th. Friedrich ([**Fri80**],[**BFGK91**]). In the homogeneous case, integrable geometries correspond to symmetric spaces, whose classification by E. Cartan had been a milestone in the differential geometry of the 20th century. The much richer class of homogeneous reductive spaces—which is inaccessible to any kind of classification—has been studied intensively since the mid sixties and has turned out to be a main source of examples for non-integrable geometries.

The interest in non integrable geometries was revived in the past years through recent developments in string theory. Firstly, integrable geometries (Calabi-Yau manifolds, Joyce manifolds etc.) are exact solutions of the Strominger model (1986, [**Str86**]), however with vanishing B-field. If one deforms these vacuum equations and looks for models with non trivial B-field, a new approach proposed in 2000-2002 by Friedrich, Ivanov, Papadopoulos and others ([**FI02**], [**DI01**], [**IP01**]) implies that solutions can be constructed geometrically from non integrable geometries. In this way, manifolds *not* belonging to the field of algebraic geometry (integrable geometries) become accessible to interesting models in theoretical physics. Starting from a problem in string theory, B. Kostant introduced a purely algebraic object known as "Kostant's cubic Dirac operator" ([**Kos99**]) in representation theory.

The habilitation thesis at hand starts with the decisive observation ([**Agr02**],[**Agr03**]) that Kostant's operator may be interpreted as a usual Dirac operator which is induced by a non standard connection on a homogeneous naturally reductive space. In particular, this Dirac operator satisfies a remarquably simple formula for its square which is a direct generalization of Parthasarathy's formula on symmetric spaces ([**Par72**]). This established the link between Kostant's algebraic considerations and recent models in string theory; in particular, it makes homogeneous naturally reductive spaces to key examples for string theory and allowed us the derivation of strong vanishing theorems on them. For representation theory, this opened the possibility to realize infinite-dimensional representations in kernels of twisted Dirac operators on homogeneous spaces, as it had been carried out on symmetric spaces in the seventies ([**Par72**], [**Wol74**], [**AS77**]). The success of this "deformed" Dirac operator in the homogeneous case made it a natural question to look for possibilities to carry over these results to non homogeneous situations. This was achieved in two joint articles with Thomas Friedrich

([**AF04a**], [**AF04b**]) by giving an appropriate generalisation of the non standard connection and the formula for the square of its Dirac operator as well as the Casimir operator on manifolds with non integrable geometries. At the same time, they lay the foundations for a much larger research program, namely, the systematic investigation of metric connections with torsion defined on the frame bundle or solely on the spinor bundle. The article [**AF03**] studies a situation of this last type; there, we discuss the solution space of the Killing equation for a spin connection defined by a three- and a four-form. Finally, the article [**AT04**] deals with the geodesics of metric connections with vectorial torsion.

2. Background and motivation

Mathematical motivation. From classical mechanics, it is a well-known fact that symmetry considerations can simplify the study of geometric problems—for example, Noether's theorem tells us how to construct first integrals like momentum from invariance properties of the underlying mechanical system. In fact, beginning in the seventies of the 19th century, the view formed that the principle organizing geometry ought to be the study of its symmetry groups. In his inaugural lecture at the University of Erlangen, which later became known as the "Erlanger Programm", Felix Klein said in 1872:

> "Es ist eine Mannigfaltigkeit und in derselben eine Transformationsgruppe gegeben; man soll die der Mannigfaltigkeit angehörigen Gebilde hinsichtlich solcher Eigenschaften untersuchen, die durch die Transformationen der Gruppe nicht geändert werden[1]."

Hence, the classical symmetry approach in differential geometry was based on the *isometry group* of a manifold, that is, the group of all transformations transforming the given manifold into itself. By the mid fifties of the 20th century, a second intrinsic group associated to a Riemannian manifold turned out to be deeply related to its fundamental properties like curvature and parallel objects. This so-called *holonomy group* determines how a vector can change under parallel transport along a closed loop inside the manifold (only in the flat case will the transported vector coincide with the original one). Berger's theorem (1955) classifies all possible holonomy groups of a simply connected, irreducible and nonsymmetric Riemannian manifold (M, g). It can be either $\mathrm{SO}(n)$ in the generic case or

$\dim M$	$4n$	$2n$	$2n$	$4n$	7	8	(16)
$\mathrm{Hol}(M)$	$\mathrm{Sp}(n)\mathrm{Sp}(1)$	$\mathrm{U}(n)$	$\mathrm{SU}(n)$	$\mathrm{Sp}(n)$	G_2	$\mathrm{Spin}(7)$	$(\mathrm{Spin}(9))$
name	quaternionic Kähler	Kähler	Calabi-Yau	hyper-Kähler	par.	par.	(par.)
par. objects	–	$\nabla J = 0$	$\nabla J = 0$	$\nabla J = 0$	$\nabla \omega^3 = 0$	–	–
curvature	$\mathrm{Ric} = \lambda g$	–	$\mathrm{Ric} = 0$	$\mathrm{Ric} = 0$	$\mathrm{Ric} = 0$	$\mathrm{Ric} = 0$	–

Manifolds having one of these holonomy groups would then be called manifolds with special (integrable) holonomy, or *special (integrable) geometries* for short. We put the case $n = 16$ and $\mathrm{Hol}(M) = \mathrm{Spin}(9)$ into parentheses, because Alekseevski and Brown/Gray showed independently that such a manifold is necessarily symmetric ([**Ale68**], [**BG72**]). The point is indeed that Berger proved that the groups on his list were the only possibilities, but he was not able to show whether they actually occured as holonomy groups of compact manifolds. It took another thirty years to find out that—with the exception of Spin(9)—this is indeed the case: The existence of metrics with holonomy $\mathrm{SU}(m)$ or $\mathrm{Sp}(m)$ on compact manifolds follows from Yau's solution of the Calabi conjecture (1978), compact manifolds with holonomy G_2 or $\mathrm{Spin}(7)$ were constructed by D. Joyce (1996, see [**Joy00**]). Already in the sixties it had been observed that the existence of a spinor which was parallel

[1] "Let a manifold and in this a transformation group be given; the objects belonging to the manifold ought to be studied with respect to those properties which are not changed by the transformations of the group." – quoted from F. Klein, *Das Erlanger Programm*, Ostwalds Klassiker der exakten Wissenschaften Band 253, Verlag H. Deutsch, Frankfurt a. M., 1995, p. 34.

with respect to the Levi-Civita connection implied the vanishing of the Ricci curvature ([**Bon66**]) and restricted the holonomy group of the manifold ([**Hit74**], [**McKW89**]), but the difficulties to construct explicit compact manifolds with special Ricci-flat metrics inhibited any further research on the deeper meaning of this result. There was progress in this direction only in the homogeneous case: in some sense, symmetric spaces are the "integrable" geometries inside the much larger class of homogeneous reductive spaces. Consider for a noncompact semisimple Lie group G and a maximal compact subgroup K such that $\operatorname{rank} G = \operatorname{rank} K$ its associated symmetric space G/K. Then the Dirac operator can be twisted by a finite dimensional irreducible unitary representation τ of K, and it is shown by Parthasarathy, Wolf, Atiyah and Schmid that, for suitable τ, most of the discrete series representations of G can be realized on the L^2-kernel of this twisted Dirac operator ([**Par72**], [**Wol74**], [**AS77**]). The crucial step herein is to relate the square of the Dirac operator with the Casimir operator Ω_G of G; for trivial τ, the corresponding formula is

$$D^2 \;=\; \Omega_G + \frac{1}{8}\,\mathrm{scal}\,.$$

Meanwhile a quest for suitable generalizations of the classical holonomy concept was started. In 1971, A. Gray introduced the notion of *weak holonomy* ([**Gra71**]), "one of his most original concepts" and "an idea much ahead of its time"(N. Hitchin in [**Hit01**]). This concept turned out to yield interesting non-integrable geometries in dimensions $n \leq 8$ and $n = 16$. In particular, manifolds with weak holonomy $\mathrm{U}(n)$ and G_2 became known as *nearly Kähler* and *nearly parallel* G_2 manifolds, respectively. But whereas the metrics of the compact Ricci-flat integrable geometries cannot be realized in any explicit way, there are many well-known homogeneous reductive examples of non-integrable geometries ([**Gra70**], [**Fer87**], [**BFGK91**], [**FKMS97**], [**BG99**], [**Fin03**] and many others). The connection to Dirac operators emerged shortly after Friedrich showed in 1980 ([**Fri80**]) his seminal inequality for the first eigenvalue λ_1 of the Dirac operator on a compact Riemannian manifold M^n of non negative curvature,

$$(\lambda_1)^2 \;\geq\; \frac{nR_0}{4(n-1)}\,, \tag{1}$$

where R_0 denotes the minimum of the scalar curvature. In this estimate, equality occurs precisely if the corresponding eigenspinor ψ satisfies the stronger Killing equation

$$\nabla^{\mathrm{LC}}_X\psi \;=\; \pm\frac{1}{2}\sqrt{\frac{R_0}{n(n-1)}}X\cdot\psi \;=:\; \mu X\cdot\psi,$$

and this is in turn linked with the existence of a non-integrable geometry on M^n; for example, a compact 6-dimensional connected simply connected hermitian manifold is nearly Kähler if and only if it admits a Killing spinor with real Killing number μ ([**Gru90**]). The connection to the then emerging twistor theory was established by A. Lichnerowicz, who showed that on a compact manifold the space of twistor spinors coincides—up to a conformal change of the metric—with the space of Killing spinors ([**Lich88**]).

Physical motivation. String theory (see for example [**LT89**]) is a physical theory aiming at describing nature at small distances ($\simeq 10^{-25}$ m). The concept of pointlike elementary particles is given up and replaced by one-dimensional objects as building blocks of matter—the so-called strings. Particles are then resonance states of strings and can be described together with their interactions up to very high energies (small distances) without internal contradictions. Besides gravitation, it incorporates many other gauge interactions and, hence, is an excellent candidate for a more profound description of matter than the standard model of elementary particles. Quantization of superstrings is only possible in the critical dimension 10, M-theory is a non-perturbative description of superstrings with "geometrized" coupling and lives in dimension 11. By dimensional reduction, one obtains predictions for the physical 4- (or 3-)dimensional world, hence the interest in manifolds of the form

$M^k \times N^l$ with $(k,l) = (4,6),(4,7),(3,7),(3,8)$. Since its early days, string theory has been intricately related with some branches of algebraic geometry. This is due to the fact that the integrable, Ricci-flat geometries with a spinor field parallel with respect to the Levi-Civita connection are exact solutions of the Strominger model for a string vacuum with vanishing B-field and constant dilaton. This is a rich and active area of mathematical research, leading to interesting developments like the discovery of mirror symmetry. As early as 1986, Strominger gave a consistent model for superstrings with non-vanishing B-field, and described hypothetical solutions as complex manifolds. But the lack of exact solutions with $B \neq 0$ remained a serious problem.

As one of the many interesting mathematical problems appearing in the context of string theory, the physicists Ramond and Pengpan observed empirically that there was an infinite set of irreducible representations of Spin(9) which partitioned into triplets $\mathcal{S} = \cup_i\{\mu^i, \sigma^i, \tau^i\}$, where the representations in each triplet were related in a remarkable way. For example, the infinitesimal character value of the Casimir operator is constant on the triplet, and $\dim\mu^i + \dim\sigma^i = \dim\tau^i$ if numerated appropriately. These triplets are used to describe massless supermultiplets, for example, $N = 2$ hypermultiplets in $(3+1)$ dimensions with helicity U(1) or $N = 1$ supermultiplets in eleven dimensions, where SO(9) is the light-cone little group ([**BRX02**]). In order to explain this fact, B. Kostant introduced an element in the tensor product of the Clifford algebra and the universal enveloping algebra of a Lie group and derived a remarkable formula for its square ([**GKRS98**], [**Kos99**]). The triplet structure of representations observed for Spin(9) is then due to the fact that the Euler characteristic of $F_4/\mathrm{Spin}(9)$ is three, hence the name "Euler multiplets" has become common for describing this effect. In the next section, we will give a new interpretation of Kostant's cubic operator, leading to new perspectives in the construction of non-integrable models. In his article on "Superstrings with torsion" ([**Str86**]), A. Strominger describes the basic model in the common sector of type II superstring theory as a 6-tuple $(M^n, g, \nabla, H, \Phi, \Psi)$ consisting of a spin manifold M^n, a Riemannian metric g, a 3-form H, a dilaton function Φ and a spinor field Ψ. Then the field equations can be written in the following form (here and in the sequel, ∇^g denotes the Levi-Civita connection):

$$\mathrm{Ric}_{ij} - \frac{1}{4}H_{imn}H_{jmn} + 2\nabla_i^{\mathrm{LC}}\partial_j\Phi \;=\; 0, \quad \delta(e^{-2\Phi}H) \;=\; 0,$$

$$(\nabla^g_X + \frac{1}{4}X \lrcorner H)\psi \;=\; 0, \quad (d\Phi - \frac{1}{6}H)\cdot\psi \;=\; 0\,.$$

If one introduces a new metric connection ∇ such that its torsion is given by the 3-form H,

$$\nabla_X Y \;:=\; \nabla^g_X Y + \frac{1}{2}H(X,Y,-),$$

one sees that the third equation is equivalent to $\nabla\Psi = 0$. Similarly, the other equations can be rewritten in terms of ∇. For constant dilaton Φ, they take the particularly simple form ([**FI02**])

$$\mathrm{Ric}^\nabla \;=\; 0, \quad \delta^g(H) \;=\; 0, \quad \nabla\Psi \;=\; 0, \quad H\cdot\Psi \;=\; 0\,, \tag{2}$$

and the second equation ($\delta^g(H) = 0$) then follows from the first equation ($\mathrm{Ric}^\nabla = 0$). For M compact, I showed in [**Agr03**, Theorem 4.1] that a solution of all equations is necessarily such that $H = 0$, i. e. an integrable Ricci-flat geometry with classical holonomy given by Berger's list. Besides the well-established Calabi-Yau manifolds, Joyce manifolds with classical holonomy G_2 or Spin(7) thus became of interest in recent times (see [**AW01**], [**CKL01**]). From a geometrical and mathematical point of view, this result means that it is important to study weaker problems first, i. e. the non compact case or partial solutions. A first step is the investigation of metric connections with totally skew-symmetric torsion and their Dirac operators, parallel spinors etc. Friedrich and Ivanov proved that several non-integrable geometric structures (almost contact metric structures, nearly Kähler, weak G_2-structures) admit a unique connection ∇ preserving it with totally skew-symmetric torsion ([**FI02**]). Hence, these geometries admit invariant connections with a uniqueness property which do not coincide with the Levi-Civita connection. This is a first step towards an understanding of non-integrable geometries purely by its holonomy properties. In fact, time was just ready for a new look at the intricate relationship between holonomy, special geometries and differential forms: Fino, Chiossi and Salamon introduced and studied the (slightly more restrictive) concept of *intrinsic torsion* ([**Fin98**], [**CS02**]). Swann tried *weakening holonomy* ([**S00**]), and N. Hitchin characterized non-integrable geometries as critical points of some linear functionals on differential forms ([**Hit01**]). In particular, he obtained a generalization of Calabi-Yau-manifolds ([**Hit00**]) and a new, previously unknown special geometry in dimension 8 ("weak PSU(3)-structures"), which deserves further investigation. Furthermore, the Italian school (Chiossi, Salamon, Fino, Dotti and others) devoted over the past years a lot of effort to the explicit construction of homogeneous examples of non-integrable geometries with special properties in small dimensions (see for example [**Fin03**], [**FG03**], [**FPS02**], [**S01**] and the litterature cited therein), hence making it possible to test the different concepts on explicit examples.

The first non-integrable geometry that raised the interest of string theorists was the squashed 7-sphere with its weak G_2-structure, although the first steps in this direction were still marked by confusion about the different holonomy concepts. A good overview about G_2 in string theory is the survey article by M. Duff ([**Duf02**]). It includes speculations about possible applications of weak Spin(9)-structures in dimension 16 ([**Fr01**]), which a priori are of too high dimension to be usually considered in physics. In dimension three, it is well known (see for example [**SSTP88**]) that the Strominger equation $\nabla\Psi = 0$ can basically only be solved on a compact Lie group with its biinvariant metric, and that the torsion of the invariant connection ∇ coincides with the Lie bracket. In dimension four, the Strominger model leads to a HKT structure, i. e. a hyperhermitian structure that is parallel with respect to ∇ ([**IP01**], [**DI01**]), and—in the compact case—the manifold is either a Calabi-Yau manifold or a Hopf surface. Hence, the first interesting dimension for further mathematical investigations is five.
Obviously, besides the basic correspondence outlined here, there is still a more detailed connection between special geometries and detailed properties of physical models constructed from them to be understood. Some weak geometries have been rederived by physicists looking for partial solutions by numerical analysis of ODE's and heavy special function machinery ([**GKMW01**]).

3. Overview of results

So far, there has been only very little investigation of homogeneous manifolds yielding (partial) solutions to the string equations, mainly because most mathematicians working in the field have a strong background in algebraic geometry. Yet, previous experience from weak special geometries leads to the expectation that homogeneous solutions should abound, since they yield huge classes of (completely accessible !) examples of such geometries. It is thus natural to study the homogeneous non-symmetric case first if one is interested in *geometric* constructions (as opposed to numerical solutions like those mentioned above). In his article [**Kos99**], B. Kostant introduced a purely algebraic object called "cubic Dirac operator" in order to explain the occurrence of multiplets of representations as observed by Ramond and Pengpang (see Section 2). The square of his algebraic object fulfilled a remarkable identity, which formally looked like the classical Parthasarathy formula in the symmetric case. My key observation in [**Agr03**] was that one can introduce a metric connection on certain homogeneous spaces whose torsion (viewed as a $(0,3)$-tensor) is a 3-form such that the associated Dirac operator has Kostant's algebraic object as its symbol. More precisely, consider a Riemannian reductive homogeneous space $M = G/H$ with Lie algebra decomposition $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$, and denote the induced scalar product on $\mathfrak{m}$ by $\langle\, ,\, \rangle$. By a theorem of Wang, there is a one-to-one correspondence between the set of G-invariant metric affine connections and the set of $\mathrm{Ad}\,(H)$-equivariant linear mappings $\Lambda_{\mathfrak{m}} : \mathfrak{m} \to \mathfrak{so}(\mathfrak{m})$. Consider the one-parameter family of connections ∇^t defined by $(t \in \mathbb{R})$

$$\Lambda_{\mathfrak{m}}(X)Y \;:=\; t \cdot [X, Y]_{\mathfrak{m}},$$

where we split the commutator into its $\mathfrak{m}$ and $\mathfrak{h}$ part, $[X,Y] = [X,Y]_{\mathfrak{m}} + [X,Y]_{\mathfrak{h}}$. It joins the canonical connection ($t = 0$) to the Levi-Civita connection ($t = 1/2$) and has torsion

$$T^t(X,Y,Z) \ := \ \big\langle T^t(X,Y), Z \big\rangle \ = \ (2t-1) \langle [X,Y]_{\mathfrak{m}}, Z \rangle \,.$$

Hence T^t is totally skew-symmetric if and only if M belongs to the large class of spaces which are *naturally reductive with respect to* G, i. e. the metric satisfies for all $X, Y, Z \in \mathfrak{m}$

$$\langle [X,Y]_{\mathfrak{m}}, Z \rangle + \langle Y, [X,Z]_{\mathfrak{m}} \rangle \ = \ 0 \,.$$

The canonical connection can alternatively be defined as the unique invariant connection with self-parallel torsion, $\nabla^0 T^0 = 0$. We have computed all basic geometric data of these connections. After lifting ∇^t to the spinor bundle, one finds for its Dirac operator D^t the remarkable expression

$$D^t \psi \ = \ \sum_{i=1}^{n} Z_i \cdot Z_i(\psi) + t \cdot H \cdot \psi \,. \tag{3}$$

Here, $Z_1, \ldots, Z_n$ denotes an orthonormal basis of $\mathfrak{m}$, $Z_i(\psi)$ is the usual directional derivative, and H is (up to omission of a factor depending on t) the threefold product inside the Clifford algebra induced by the 3-form T^t,

$$H \ := \ \frac{3}{2} \sum_{i<j<k} \langle [Z_i, Z_j]_{\mathfrak{m}}, Z_k \rangle \, Z_i \cdot Z_j \cdot Z_k .$$

It is this element H which inspired to B. Kostant the name "cubic Dirac operator". Our analysis shows that it is a normal Dirac operator associated to some non-standard connection and that *every* Dirac operator coming from a metric connection with skew-symmetric torsion will have this form, i. e. being "cubic" is not a special property of the family considered here. In analogy to the classical Parthasarathy formula in the symmetric case, we computed the square of D^t in the present case. Define the "$\mathfrak{m}$-part" of the Jacobi identity as

$$\mathrm{Jac}_{\mathfrak{m}}(X,Y,Z) \ := \ [X,[Y,Z]_{\mathfrak{m}}]_{\mathfrak{m}} + [Y,[Z,X]_{\mathfrak{m}}]_{\mathfrak{m}} + [Z,[X,Y]_{\mathfrak{m}}]_{\mathfrak{m}} \,.$$

Then we have the following theorem:

Theorem 3.1 (General Kostant-Parthasarathy formula [**Agr02**, Théorème 2.1], [**Agr03**, Theorem 3.2])**.** *For $n \geq 5$, the square of D^t satisfies the identity*

$$\begin{aligned}
(D^t)^2 \psi \ &= \ \Omega_{\mathfrak{g}}(\psi) + \frac{1}{2}(1-3t) \sum_{i,j,k} \langle [Z_i, Z_j]_{\mathfrak{m}}, Z_k \rangle \, Z_i \cdot Z_j \cdot Z_k(\psi) \\
&- \ \frac{1}{2} \sum_{i<j<k<l} \big\langle Z_i, (9t^2-1) \mathrm{Jac}_{\mathfrak{m}}(Z_j, Z_k, Z_l) \big\rangle \cdot Z_i \cdot Z_j \cdot Z_k \cdot Z_l \cdot \psi \\
&+ \ \frac{1}{8} \sum_{i,j} Q_{\mathfrak{h}}([Z_i, Z_j], [Z_i, Z_j]) \psi + \frac{3}{8} t^2 \sum_{i,j} Q_{\mathfrak{m}}([Z_i, Z_j], [Z_i, Z_j]) \psi \,.
\end{aligned}$$

It is immediate that this formula allows considerable simplifications if $t = 1/3$, and this is in fact the connection used by B. Kostant in his work.

Theorem 3.2 (Kostant-Parthasarathy formula for $D^{1/3}$ [**Agr02**, Théorème 2.2], [**Agr03**, Theorem 3.3])**.** *For $n \geq 5$ and $t = 1/3$, the general formula for $(D^t)^2$ reduces to*

$$(D^{1/3})^2 \psi \ = \ \Omega_{\mathfrak{g}}(\psi) + \frac{1}{8} \Big[\mathrm{scal}^{1/3} + \frac{1}{9} \sum_{i,j} Q_{\mathfrak{m}}([Z_i, Z_j], [Z_i, Z_j]) \Big] \psi \,. \tag{4}$$

This formula is the differential geometric version of the algebraic identity [**Kos99**, Thm 2.13]. It coincides with Parthasarathy's classical formula from 1972 in case M is symmetric. In fact, S. Slebarski had already noticed independently that the parameter value $t = 1/3$ had distinguished properties (see Theorem 1 and the introduction of [**Sle87a**], as well as [**Goe99**]). Although his articles [**Sle87a**] and [**Sle87b**] contain several attempts to generalize Parthasarathy's formula for

D^2, none of them seems to come close to Kostant's result. We shall call the connection $\nabla^{1/3}$ the *Kostant-Slebarski connection.* Notice that the scalar appearing in equation (4) can be expressed in purely geometric terms, yielding

$$(D^{1/3})^2\psi \;=\; \Omega_{\mathfrak{g}}(\psi) + \frac{1}{8}\mathrm{Scal}^g + \frac{1}{16}||T||^2, \tag{5}$$

where Scal^g denotes the Riemannian scalar curvature. As a corollary, one obtains immediately from Theorem 3.1 that the first order differential operator

$$\mathcal{D}\psi \;:=\; \sum_{i,j,k} \langle [Z_i, Z_j]_{\mathfrak{m}}, Z_k\rangle \, Z_i \cdot Z_j \cdot Z_k(\psi)$$

is G-invariant. It exists only on non symmetric homogeneous spaces and certainly deserves further investigation. If G is compact, the scalar appearing on the right hand side of equation (4) can be rewritten in representation theoretic terms as $||\varrho_{\mathfrak{g}}||^2 - ||\varrho_{\mathfrak{h}}||^2$ and has the property of being *always* positive, even if the Riemannian scalar curvature of M becomes negative. Hence, Theorem 3.2 implies an eigenvalue estimate for the Dirac operator $D^{1/3}$ in the tradition of Friedrich's estimate from 1980 (eq. (1)).

Corollary 3.1 ([**Agr02**, Corollaire 2.3], [**Agr03**, Corollary 3.1]). *If the operator $\Omega_{\mathfrak{g}}$ is non-negative, the first eigenvalue $\lambda_1^{1/3}$ of the Dirac operator $D^{1/3}$ satisfies the identity*

$$\left(\lambda_1^{1/3}\right)^2 \;\geq\; ||\varrho_{\mathfrak{g}}||^2 - ||\varrho_{\mathfrak{h}}||^2 \tag{6}$$

Equality occurs if and only if there exists an algebraic spinor that is fixed under the lift of the isotropy representation.

Examples of equality are for example the Aloff-Wallach metrics on $\mathrm{SU}(3)/S^1$. In fact, the fixed spinors are precisely the parallel spinors of the canonical connection. Notice that a Killing spinor need not be a fixed spinor under the lifted isotropy representation or vice versa, though it can happen that both notions coincide. Looking back at the string equations for constant dilation (eq. (2)), one sees that the Dirac operator from equation (3) is particularly suitable to study parallel spinors with $H \cdot \psi = 0$:

Theorem 3.3 ([**Agr03**, Thm 4.3]). *If the operator $\Omega_{\mathfrak{g}}$ is non-negative and ∇^t is not the Levi-Civita connection (i. e., $t \neq 1/2$), there cannot exist any spinor satisfying the equations $\nabla^t\psi = 0$ and $H \cdot \psi$ simultaneously.*

It had been observed in Section 2 that for the Levi-Civita connection, the existence of a parallel spinor implies the vanishing of the Riemannian Ricci curvature ([**Bon66**]). Curiously, for other connections, the existence of a parallel spinor and Ricci flatness rather seem to be obstructions to each other. Following the lines of the Riemannian proof, one gets for the general case intricate conditions linking the Ricci tensor to the torsion form. We used them to prove the following result:

Proposition 3.1 ([**Agr03**, Prop. 4.1]). *If the canonical connection ∇^0 is Ricci flat and admits a parallel spinor, the exterior derivative of its torsion T^0 satisfies $(X \lrcorner dT^0) \cdot \psi = 0$ for all vectors X in $\mathfrak{m}$. In dimension $n = 4, 5, 6$ and 7, this last condition cannot hold for algebraic reasons.*

We end our remarks on vanishing results with the precise statement of a general theorem already mentioned before:

Theorem 3.4 ([**Agr03**, Theorem 4.1]). *Let M^n be a compact Riemannian manifold with metric $\langle\, , \rangle$ and a metric connection ∇ with totally skew symmetric torsion T. Suppose that there exists a spinor field ψ such that all the equations*

$$\mathrm{Ric}^{\nabla} \;=\; 0, \quad \nabla\Psi \;=\; 0, \quad T \cdot \Psi \;=\; 0$$

hold. Then $T = 0$ and ∇ is the Levi-Civita connection.

The article [**Agr03**] ends with a thorough discussion of an example, namely, the naturally reductive metrics on the 5-dimensional Stiefel manifold $V_{4,2} = \mathrm{SO}(4)/\mathrm{SO}(2)$. This contact manifold (see [**Bla02**] as a general reference for these) was known before to carry a Sasakian metric with two Killing spinors ([**Fri80**]); from the point of view of parallel spinors, all contact metrics on $V_{4,2}$ turned out to be equally interesting.

Before describing the contents of the next papers, let us begin with some general remarks on torsion. The notion of torsion of a connection was invented by Elie Cartan, and appeared for the first time in a short note at the Académie des Sciences de Paris in 1922 (see [**Car22**]). Although it contains no formulas, Cartan observes that such a connection may or may not preserve geodesics, and turns his attention first to those who actually do so. In this sense, E. Cartan was the first to investigate this class of connections. At that time, it was not yet customary—as it became later in the second half of the 20th century—to assign to a Riemannian manifold only its Levi-Civita connection. Rather, Cartan demands (see [**Car24b**]):

> *Étant donné une variété plongée dans l'espace affine (ou projectif, ou conforme etc.), attribuer à cette variété la connexion affine (ou projective, ou conforme etc.) qui rende le plus simplement compte des relations de cette variété avec l'espace ambiant.*

He then goes on to explain in very general terms how the connection should be *adapted* to the geometry under consideration. We believe that this point of view should be taken into account in Riemannian geometry, too. The canonical connection of a naturally reductive Riemannian space is a first example (see [**Agr03**]). Moreover, we know many non integrable geometric structures on Riemannian manifolds admitting a unique metric connection preserving the structure and with non vanishing skew-symmetric torsion (see [**FI02**], [**Fri03b**]). Following Cartan as well as the idea that torsion forms are candidates for the so called B-field in string theory, the geometry of these connections deserves systematic investigation. Basically, there were up to now no general results concerning the holonomy group of connections with torsion. The question whether or not a connection of that type admits parallel tensor fields differs radically from the corresponding problem for the Levi-Civita connection. In particular, one is interested in the existence of parallel spinor fields, since they are interpreted in string theory as supersymmetries of the model.

We give a short review of the 8 classes of geometric torsion tensors. Consider a Riemannian manifold (M^n, g). In a point, the difference between its Levi-Civita connection ∇^g and any linear connection ∇ is a $(2,1)$-tensor A,

$$\nabla_X Y \;=\; \nabla^g_X Y + A(X,Y), \quad X, Y \in TM\,.$$

The vanishing of the symmetric or the antisymmetric part of A has an immediate geometric interpretation. The connection ∇ is torsion-free if and only if A is symmetric. The connection ∇ has the same geodesics as the Levi-Civita connection ∇^g if and only if A is antisymmetric. Following Cartan, we study the algebraic types of the torsion tensor for a metric connection. Denote by the same symbol the $(3,0)$-tensors derived from A, T by contraction with the metric. We identify TM with TM^* via the metric from now on. Let $\mathcal{T}$ be the $n^2(n-1)/2$-dimensional space of all possible torsion tensors,

$$\mathcal{T} \;=\; \{T \in \otimes^3 TM \mid T(X,Y,Z) = -T(Y,X,Z)\} \;\cong\; \Lambda^2 TM \otimes TM\,.$$

On the other side, a connection ∇ is metric if and only if A belongs to the space

$$\mathcal{A}^g \;:=\; TM \otimes \Lambda^2 TM \;=\; \{A \in \otimes^3 TM \mid A(X,V,W) + A(X,W,V) \;=\; 0\}\,.$$

Proposition 3.2 ([**Car25**, p.51], [**TV83**]). *For $n \geq 3$, the space $\mathcal{T}$ of possible torsion tensors splits under $\mathrm{O}(n,\mathbb{R})$ into the sum of three irreducible representations, $\mathcal{T} \cong \mathbb{R}^n \oplus \Lambda^3(\mathbb{R}^n) \oplus \mathcal{T}'$, as does $\mathcal{A}^g$. Furthermore, the formulas*

$$\begin{aligned} T(X,Y,Z) &= A(X,Y,Z) - A(Y,X,Z)\,, \\ 2\,A(X,Y,Z) &= T(X,Y,Z) - T(Y,Z,X) + T(Z,X,Y)\;. \end{aligned}$$

define an equivariant bijection $\mathcal{A}^g \to \mathcal{T}$.

The eight classes of linear connections are now defined by the possible parts of their torsions T in these components. If one looks at the class of linear *metric* connections, then these are also uniquely determined by their torsion. The nice lecture notes by Tricerri and Vanhecke [**TV83**] use a similar approach in order to classify homogeneous spaces by the algebraic properties of the torsion of the canonical connection. They construct homogeneous examples of all classes, and study their "richness". The described decomposition shows that a natural class of non-torsion free metric connections are those with skew-symmetric torsion form $T \in \Lambda^3(\mathbb{R}^n)$; the articles [**AF04a**], [**AF04b**] and [**AF03**] deal with these as well as spinorial connections defined by exterior forms of higher degree. Observe that we can characterize these connections geometrically as follows:

Corollary 3.2. *A connection ∇ on M is metric and geodesics preserving precisely if its torsion T lies in $\Lambda^3(TM)$. In this case, $2A = T$ holds,*

$$\nabla_X Y \ = \ \nabla^g_X Y \ + \ \frac{1}{2}\, T(X,Y,-),$$

and the ∇-Killing vector fields coincide with the Riemannian Killing vector fields.

[**AT04**] is devoted to the geodesics of metric connections whose torsion is given by a vector, i. e. lies in the first component of the decomposition in Proposition 3.2. We shall henceforth speak of connections with *vectorial torsion.*

In our joint article [**AF04a**], we first study the linear case in very general terms, i.e., euclidian space. We associate with any exterior form $T \in \Lambda^k(\mathbb{R}^n)$ its covariant derivative ∇^T acting on spinor fields $\psi : \mathbb{R}^n \to \Delta_n$ by the formula

$$\nabla^T_X \psi \ := \ \nabla^g_X \psi \ + \ (X \lrcorner T) \cdot \psi \, .$$

For a 3-form $T \in \Lambda^3(\mathbb{R}^n)$, the spinorial covariant derivative ∇^T is induced by a linear metric connection with torsion tensor $2\,T$,

$$\nabla^T_X Y \ := \ \nabla^g_X Y \ + \ 2\,T(X,Y,-)\, .$$

Definition 3.1. ([**AF04a**, Dfn 2.1 and 2.2]). Let T be an exterior form on $\mathbb{R}^n$. The Lie algebra $\hat{\mathfrak{g}}_T$ is the subalgebra of the Clifford algebra $\mathfrak{cl}(\mathbb{R}^n)$ generated by all elements $X \lrcorner T$, where $X \in \mathbb{R}^n$ is a vector. The Lie algebra

$$\mathfrak{h}_T \ := \ \big[\hat{\mathfrak{g}}_T,\, \hat{\mathfrak{g}}_T\big] \ \subset \ \mathfrak{cl}(\mathbb{R}^n)$$

is called the *(infinitesimal) holonomy algebra* of the exterior form T.

The Lie algebra $\hat{\mathfrak{g}}_T$ is invariant under the action of the isotropy group G_T, but has no relationship with the Lie algebra $\mathfrak{g}_T$ of G_T. If $k + \binom{k-1}{2} \equiv 0 \bmod 2$ (for example, $k = 3$), $\hat{\mathfrak{g}}_T$ lies in $\mathfrak{so}(n) \subset \mathfrak{cl}(\mathbb{R}^n)$ ([**AF04a**, Prop. 2.1]). In our examples, $\mathfrak{h}_T$ did always coincide with $\hat{\mathfrak{g}}_T$; for 3-forms, we were able to show thar $\hat{\mathfrak{g}}_T$ is a semisimple Lie algebra ([**AF04a**, Theorem 3.1]). Moreover, it cannot preserve a non degenerate 2-form or a spinor, hence there are no parallel spinors for $T \neq 0$ ([**AF04a**, Prop. 3.3, Thm 3.2]). On the other side, many representations of a compact, semisimple Lie algebra occur as the holonomy algebra of some 3-form, for example the adjoint representation can be realized in this way. We introduce an obstruction for a Lie algebra representation to be the holonomy algebra of some 3-form and show on an example how it may be used to rule out some representations. In particular, the unique, irreducible 16-dimensional representation of the algebra $\mathfrak{spin}(9)$ cannot be the holonomy algebra of some 3-form.

In the next step, we generalize the algebraic results to the case of a Riemannian spin manifold (M^n, g, T) with a metric connection ∇. Consider the one-parameter family of linear metric connections with torsion defined by

$$\nabla^s_X Y \ := \ \nabla^g_X Y + 2s\,T(X,Y,-)$$

and its lift to the spinor bundle S of M. In [**FI02**, Thm 3.1, 3.3], Friedrich and Ivanov derived a formula for the square of the Dirac operator D^s associated with ∇^s. This formula for $(D^s)^2$ has the disadvantage of still containing a first order differential operator as well as several 4-forms, which are difficult to treat algebraically. Inspired by the homogeneous case—where the Kostant-Slebarski connection is related to the canonical connection by a 1/3-shift, we were looking for an alternative comparison of $(D^s)^2$ with the Laplace operator of some *other* connection $\nabla^{s'}$ from the same family. It turns out that the Laplacian for the parameter s should be linked to the Dirac operator for the parameter $s/3$, the remainder being a zero order operator. Similar formulas can be found in [**Bis89**].

Theorem 3.5 ([**AF04a**, Thm 5.2])**.** *The spinor Laplacian Δ^s and the square of the Dirac operator $D^{s/3}$ are related by*

$$(D^{s/3})^2 \;=\; \Delta^s + s\, dT + \frac{1}{4}\,\mathrm{Scal}^g - 2s^2\,||T||^2 .$$

By integrating the latter formula on a compact manifold M^n, we obtain:

Theorem 3.6 ([**AF04a**, Thm 5.3])**.** *Let (M^n, g, T) be a compact, Riemannian spin manifold of non positive scalar curvature, $\mathrm{Scal}^g \leq 0$, and suppose that the 4-form dT acts on spinors as a non positive endomorphism. If there exists a solution $\psi \neq 0$ of the equation*

$$\nabla^T_X \psi \;=\; \nabla^g_X \psi \,+\, (X \lrcorner T)\cdot \psi \;=\; 0\ ,$$

then the 3-form and the scalar curvature vanish, $T = 0 = \mathrm{Scal}^g$, and ψ is parallel with respect to the Levi-Civita connection.

Theorem 3.6 applies, in particular, to Calabi-Yau or Joyce manifolds. These are compact, Ricci-flat Riemannian manifolds in dimensions $n = 6, 7$ with one parallel spinor field. Let us perturb the connection ∇^g by a 3-form such that dT is non positive on spinors (for example, a closed form). Then the new connection ∇^T does not admit ∇^T-parallel spinor fields. Physically speaking, this means that exact vacuum solutions of the Strominger model cannot be deformed in this way into non-vacuum solutions. Nilmanifolds and their compact quotients $M^n = \mathrm{G}/\Gamma$ are a second family of examples where the theorem applies. A further family of examples arises from certain naturally reductive spaces and a torsion form T being proportional to the torsion form of the canonical connection, see [**Agr03**].

The integral formula from Theorem 3.6 can also be applied to the determination of the possible values of s for which parallel spinors can exist. In the generic case, the existence of a ∇^s-parallel spinor restricts the possible parameter s via a polynomial equation. Moreover, our integral formulas prove that, on a compact manifold, basically only three parameters are possible ([**AF04a**, Thm 6.1]). In case that the torsion form is associated with a special non integrable geometry, the connection ∇^s with a parallel spinor is sometimes unique. A result of that type requires additional informations concerning the underlying geometry. We prove it for 5-dimensional Sasakian manifolds equipped with their canonical connection ([**AF04a**, Prop. 6.2]).

The last part of [**AF04a**] is devoted to examples. We construct on the Aloff-Wallach manifold $N(1,1) = \mathrm{SU}(3)/S^1$ a two-parameter family of metrics that admits two inequivalent cocalibrated G_2-structures. It is well known (see [**BFGK91**]) that $N(1,1)$ carries two distinguished metrics, namely an Einstein and a 3-Sasakian one (which, of course, is automatically Einstein). For this family, we investigate the torsion forms of their unique connections as well as other geometric data. Our approach is different from the usual one (see [**CMS96**]). First we construct 3-forms with parallel spinors on $N(1,1)$. The underlying G_2-structure is cocalibrated and many of the geometric data are encoded into the torsion 3-form we started with. Moreover, we are interested not only in the type of the G_2-structure, but mainly in the geometry of the unique connection preserving this structure. One result is that we were able to find a metric (which is none of the ones mentioned above) and a 3-form T such that ∇^T and ∇^{-T} admit parallel spinors, hence illustrating in a non trivial situation that several values are indeed possible in [**AF04a**, Thm 6.1]. The same method is then applied in order to construct spinorial connections defined by 4-forms and admitting parallel spinor fields. It turns out that some of these connections are closely related to the 3-Sasakian structure of $N(1,1)$:

Theorem 3.7 ([**AF04a**, Thm 8.1, 8.2])**.** 1) *Any 3-Sasakian manifold in dimension seven admits a $\mathbb{P}^2$-parameter family of metric connections with skew-symmetric torsion and parallel spinors. The holonomy group of these connections is a subgroup of G_2.*

2) *Any* 3*-Sasakian manifold in dimension seven admits a* $\mathbb{P}^2$*-parameter family of spinorial connections defined by* 4*-forms and with parallel spinors. The spinorial holonomy group of these connections is a subgroup of* $\mathrm{GL}(7,\mathbb{R})$.

In [**AF03**], we introduce a second order differential operator Ω for a Riemannian manifold (M^n, g, ∇) equipped with a metric connection with skew-symmetric torsion T. If we denote by $(D^{1/3})^2$ the square of the Dirac operator corresponding to the connection with torsion form $T/3$, this so-called *Casimir operator* of M^n differs from $(D^{1/3})^2$ by a zero order term. The parameter shift by 1/3 is the same than previously used in the general Kostant-Parthasarathy formula (Theorem 3.1 in this summary) as well as the Weitzenböck formula (Theorem 3.5). Our operator Ω is constructed in such a way to coincide with the Casimir operator of a naturally reductive space in the homogenous situation, hence motivating its name. If the torsion form T is ∇-parallel, the formula for Ω simplifies to

$$\Omega \;=\; (D^{1/3})^2 \,-\, \frac{1}{16}\left(2\,\mathrm{Scal}^g \,+\, ||T||^2\right),$$

which can be directly compared to equation (5) and Theorem 3.2. Even for a naturally reductive homogeneous space, this definition has one big advantage: $D^{1/3}$, Scal^g and $||\mathrm{T}||^2$ are geometrically invariant objects, whereas the Casimir operator $\Omega_{\mathfrak{g}}$ in its usual sense still heavily relies on the concrete realization of the homogeneous space M as a quotient.
The kernel of Ω contains all ∇-parallel spinors, which makes it a useful tool for the study of the space of all parallel spinors. From the integral formulas in [**AF04a**], one obtains criteria when Ω is a non-negative operator, a question which is difficult to answer even in the homogeneous case ([**AF04b**, Prop. 3.3]). If the torsion form is ∇-parallel, the Casimir operator Ω and the square of the Dirac operator $(D^{1/3})^2$ commute with the endomorphism T ([**AF04b**, Prop. 3.4]),

$$\Omega \circ T \;=\; T \circ \Omega\,, \quad (D^{1/3})^2 \circ T \;=\; T \circ (D^{1/3})^2.$$

Hence, T and D^g (the usual Riemannian Dirac operator) act in the kernel of $D^{1/3}$ if M is additionnally compact. One further consequence is an eigenvalue estimate linking the spectrum of the endomorphism T and the minimum of the scalar curvature. We give a detailed description of the Casimir operator on 5-dimensional Sasakian manifolds, 6-dimensional nearly Kähler manifolds and 7-dimensional cocalibrated G_2-manifolds.

The article [**AF03**] deals with a special supergravity model. In July 2003, G. Papadopoulos (Cambridge) raised the question of finding solutions to the equation

$$\nabla^g_X \Psi \,+\, \frac{1}{4}\cdot(X \lrcorner T)\cdot\Psi \,+\, \frac{1}{144}\cdot(X \lrcorner F \,-\, 8\cdot X\wedge F)\cdot\Psi \;=\; 0\,,$$

where T is a 3-form as before and F a 4-form flux on the Riemannian spin manifold M (see [**Duf02**], [**FP02**]). This is a highly overdetermined system of first order partial differential equations. 11-dimensional space-time solutions are interesting and the models with a maximal number of supersymmetries have been classified (see [**FP02**]). The Kaluza-Klein reduction of $\mathcal{M}$-theory (see [**Ali01**], [**BJ03**], [**BDS01**] and [**WNW85**]) yields that dimensions $4 \leq n \leq 8$ are of interest, too, possibly with some additional algebraic constraints, like $T\cdot\Psi = 0$ or $F\cdot\Psi = 0$.
The aim of [**AF03**] is to present a geometric method for solving the equation under consideration. The main idea of our approach is easy to explain. We start with a Riemannian manifold admitting a spinor field Ψ of some special type. For this, there are many possibilities. The spinor field may be a Riemannian Killing spinor (see [**Fri80**]) on some irreducible Einstein space,

$$\nabla^g_X \Psi \;=\; \lambda\cdot X\cdot\Psi\,.$$

The spinor field may be a Kählerian Killing spinor (see [**Kir93**]) defined on some special Kähler manifold. In odd dimensions, we can start with an η-Einstein-Sasakian manifold and its contact Killing spinor (see [**FK00**]). On a reductive space, the spinor field may be an invariant spinor of the isotropy representation. In any case, triples $(M^n,\, g,\, \Psi)$ of the type we need have been studied very intensively in mathematics since more then 20 years. In particular, the dimensions $n \leq 8$ and the corresponding special geometries play a crucial role. Let us moreover assume that there exists a "canonical" family (T_ω, F_ω) of forms on M^n depending on some parameter ω. We consider the system of $n \cdot 2^{[n/2]}$ algebraic equations in the parameters ω,

$$\Big(\lambda \cdot X + \frac{1}{4} \cdot (X \lrcorner \mathrm{T}_\omega) + \frac{1}{144}(X \lrcorner \mathrm{F}_\omega - 8 \cdot X \wedge \mathrm{F}_\omega)\Big) \cdot \Psi = 0 .$$

Furthermore, we can add the equations $T_\omega \cdot \Psi = 0$ or $F_\omega \cdot \Psi = 0$. We solve the corresponding equations with the help of standard math computer programs. In this way, we obtain families of 3- and 4-forms solving the equation on the manifold we started with. The first interesting dimension in presence of a 4-flux is seven. Applying the method just described, we can prove:

Theorem 3.8 ([**AF03**, Thm 2.1])**.** *Let M^7 be a 7-dimensional 3-Sasakian manifold and fix a Riemannian Killing spinor Ψ, as well as two real numbers p and q. Then there exists a 7-dimensional family of torsion forms T and flux forms F defined by the contact structures such that*

$$\nabla^g_X \Psi + \frac{1}{4} \cdot (X \lrcorner T) \cdot \Psi + p \cdot (X \lrcorner F) \cdot \Psi + q \cdot (X \wedge F) \cdot \Psi = 0 .$$

The condition $\mathrm{F} \cdot \Psi = 0$ restricts to a subfamily of dimension six. If $4\,p - 3\,q \neq 0$, the condition $\mathrm{T} \cdot \Psi = 0$ defines again a 6-dimensional subfamily. If $4\,p - 3\,q = 0$, then $\mathrm{T} \cdot \Psi = (14/3) \cdot \Psi$ for any torsion form in the family. Both constraints together imply that the spinor field Ψ is necessarily zero.

From the geometric point of view, there is an interesting case, namely $4\,p - 3\,q = 0$. This is *not* the ratio of the parameters p, q appearing in supergravity.
The G_2-case has been already investigated in $\mathcal{M}$-theory compactifications to dimension four (see [**BDS01**]). In both cases, the underlying metric has to be Einstein (see [**BG99**]). On the Aloff-Wallach space $N(1,1)$, we were able to construct families of non Einstein metrics equipped with torsion forms, flux forms and Killing spinors:

Theorem 3.9. *For every metric $g_{s,y}$ on $N(1,1) = \mathrm{SU}(3)/S^1$ and every pair $(p,q) \in \mathbb{R}^2$, there exists a 10-dimensional affine space of forms (T,F) such that the spinor field Ψ_3 satisfies the Killing spinor equation*

$$\nabla_X \Psi := \nabla^g_X \Psi + \frac{1}{4}(X \lrcorner \mathrm{T})\Psi + p\,(X \lrcorner \mathrm{F})\Psi + q\,(X \wedge \mathrm{F})\Psi = 0.$$

Furthermore, the additional condition $\mathrm{F} \cdot \Psi_3 = 0$ singles out a 9-dimensional affine subspace. For $4\,p - 3\,q \neq 0$, the set of forms satisfying $\mathrm{T} \cdot \Psi_3 = 0$ is again a 9-dimensional affine subspace, but its intersection with forms such that $\mathrm{F} \cdot \Psi_3 = 0$ is empty. For $4\,p - 3\,q = 0$, there are no 3-forms such that $\mathrm{T} \cdot \Psi = 0$.

(Ψ_3 is some special spinor field on $N(1,1)$ described in Section 7 of [**AF04a**]). The paper ends with the discussion of the special coupling $4p = 3q$ in dimension 7.

The article [**AT04**] returns to the 8 classes of metric connections with torsion described in Proposition 3.2. Besides metric connections whose torsion is a 3-form, the simplest class consists of those connections which have vectorial torsion. Explicitly, they may be written as

$$\nabla_X Y = \nabla^g_X Y + g(X,Y)V - g(V,Y)X$$

for some fixed vector field V on M^n. The case of a surface ($n = 2$) is special in as much that any metric connection has to be of this type. In fact, classical topics of surface theory like the Mercator projection which maps loxodromes on the sphere to straight lines in the plane can be understood in a different light with their help.
For some reasons, these connections have not attracted as much attention in the past as we believe they deserve. Correspondingly, an overview over the existing literature (that we are aware of) is quickly given. In [**TV83**], Tricerri and Vanhecke were led to the study of such connections in the context of the classification problem of homogeneous structures on manifolds. They showed that if M is connected, complete, and simply connected and V is parallel, i. e. $\nabla V = 0$, then (M,g) has to be isometric to hyperbolic space. Vicente Miquel studied in [**Miq82**] and [**Miq01**] the growth of

geodesic balls of such connections, but did not investigate the detailed shape of geodesics. In any way, a curve $\gamma(t)$ is a geodesic of ∇ if it satisfies the differential equation

$$\nabla^g_{\dot\gamma}\dot\gamma + g(\dot\gamma,\dot\gamma)V - g(V,\dot\gamma)\dot\gamma \ = \ 0\,.$$

Taking the scalar product of this equation with $\dot\gamma$ yields $g(\nabla^g_{\dot\gamma}\dot\gamma,\dot\gamma)=0$, that is, $\dot\gamma$ has constant length $E>0$, which reflects of course just the fact that ∇ was metric. Hence, the geodesic equation can be written

$$\nabla^g_{\dot\gamma}\dot\gamma + E^2\, V - g(V,\dot\gamma)\dot\gamma \ = \ 0\,. \tag{7}$$

In fact, there are qualitatively two cases to be distinguished. If $\dot\gamma$ is parallel to V at the origin, $\dot\gamma(0) = \alpha\cdot V(\gamma(0))$, we conclude that $\nabla^g_{\dot\gamma(0)}\dot\gamma(0) = 0$ and $\gamma(t)$ coincides locally with a classical geodesic of the Levi-Civita connection. In particular, a ∇-geodesic which stays parallel to V for all times is exactly a ∇^g-geodesic. Generic ∇-geodesics are those for which $\dot\gamma$ is never parallel to V; their shape will be qualitatively very different from that of their Levi-Civita "cousins". Periodic geodesics are necessarily non-generic. More precisely, one shows:

Proposition 3.3 ([**AT04**, Cor. 2.1])**.** *For a Killing vector field V, any periodic ∇-geodesic is automatically a Levi-Civita geodesic and, up to a constant, an integral curve of V.*

One problem with the study of geodesics for connections with vectorial torsion is the lack of good integrals of motion. Isometries leaving V invariant generate symmetries of the ∇-geodesics, but no invariants of Noether type. Yet, invariants of motion must exist in some special situations, as the following example due to Cartan shows:

Example 3.1 (Cartan's example)**.** In [**Car23**, § 67, p. 408–409], Cartan describes the two-dimensional sphere with its flat metric connection, and observes (without proof) that "on this manifold, the straight lines are the *loxodromes*, which intersect the meridians at a constant angle. The only straight lines realizing shortest paths are those which are normal to the torsion in every point: *these are the meridians*[2]".

This suggests that there exists a class of metric connections on surfaces of revolution whose geodesics admit a generalization of Clairaut's theorem, yielding loxodromes in the case of the flat connection. Furthermore, it is well known that the Mercator projection maps loxodromes to straight lines in the plane (i. e., Levi-Civita geodesics of the euclidian metric), and that this mapping is conformal. Theorem 3.10 provides the right setting to the understanding of both effects.

Theorem 3.10 ([**AT04**, Thm 3.1])**.** *Let σ be a function on the Riemannian manifold (M,g), ∇ the metric connection with vectorial torsion defined by $V=-\mathrm{grad}(\sigma)$, and consider the conformally equivalent metric $\tilde g = e^{2\sigma}g$. Then:*

1. *Any ∇-geodesic $\gamma(t)$ is, up to a reparametrisation τ, a $\nabla^{\tilde g}$-geodesic, and the function τ is the unique solution of the differential equation $\ddot\tau + \dot\tau\dot\sigma = 0$, where we set $\sigma(t) := \sigma\circ\gamma\circ\tau(t)$;*
2. *If X is a Killing field for the metric $\tilde g$, the function $e^\sigma g(\dot\gamma, X)$ is a constant of motion for the ∇-geodesic $\gamma(t)$.*

Cartan's example is obtained (and generalized) by taking $\sigma := -\ln r(s)$ on the surface of revolution generated by rotating the parametrised curve $\alpha = (r(s), h(s))$. Besides for the sphere, the result is of particular interest for the catenoid: since it is a minimal surface, the Gauss map to the sphere is a conformal mapping, hence it maps loxodromes to loxodromes. Thus, Beltrami's theorem ("If a portion of a surface S can be mapped LC-geodesically onto a portion of a surface S^* of constant Gaussian curvature, the Gaussian curvature of S must also be constant", see for example [**Kre91**, §95]) does *not* hold for metric connections with vectorial torsion—the sphere is a a surface of constant Gaussian curvature, but the catenoid is not. In the last Section, we treat the euclidian plane with an

[2]Sur cette variété, les lignes droites sont les *loxodromies*, qui font un angle constant avec les méridiennes. Les seules lignes droites qui réalisent les plus courts chemins sont celles qui sont normales en chaque point à la torsion : *ce sont les méridiennes.* loc. cit.

arbitrary vector field in great detail and show that the behaviour of geodesics is, already in this low dimension, dominated by a highly non-trivial system of ordinary differential equations. In particular, we give an example of a vector field which does admit a second invariant of motion, although it is not a gradient and not geodesically complete (that is, the Hopf-Rinow theorem fails).

The last article in the present collection is the joint work with Roe Goodman on K-invariant vector fields on symmetric spaces G/K [**AG03**]. It can be read independently of the other articles and is (together with [**AT04**]) a purely mathematical paper. Its link with the other articles stems from the fact that it also deals with the properties of invariant differential operators on homogeneous manifolds. While G-invariant operators on symmetric spaces have been intensively studied (see, for example, the books by Helgason [**Hel78**], [**Hel84**], [**Hel94**] for operators on functions, and [**Par72**], [**Wol74**], [**AS77**] for Dirac operators on spinors), nothing was known before on K-invariant operators. Hence, it was necessary to start with the simplest situation, namely, the infinite-dimensional Lie algebra of K-invariant vector fields on a reductive symmetric space G/K. It turned out to be useful to work in the algebraic category, i. e. G is a complex connected reductive linear algebraic group and K is the fixed points of an involutory automorphism θ of G (thus G/K is the complexification of a Riemannian symmetric space). The main case that we are studying— $\mathrm{SL}(2,\mathbb{C})$ with the conjugation action—is of interest in the context of connections with skew-symmetric torsion, too. Indeed, a recent result by Alexandrov, Friedrich and Schoemann ([**AFS04**]) shows that $\mathrm{SL}(2,\mathbb{C})$ is, besides twistor spaces, the only hermitian 6-manifold with parallel characteristic torsion and non-abelian isotropy group of the NS-3-form.
There is a canonical G-module isomorphism between the space $\mathfrak{X}(G/K)$ of regular algebraic vector fields on G/K and the algebraically induced representation $\mathrm{Ind}_K^G(\sigma)$, where σ is the isotropy representation of K. In particular, the space $\mathfrak{X}(G/K)^K$ of K-invariant vector fields on G/K corresponds to the K-fixed vectors in the induced representation. When G is simple and simply connected, Richardson's results [**Ric82**] imply that $\mathfrak{X}(G/K)$ is a free module over the algebra $\mathcal{J}$ of K-biinvariant functions on G. In Theorem [**AG03**, Thm 2.2] we obtain an explicit set of free generators for a localization $\mathfrak{X}(G/K)_\psi^K$, for some $\psi \in \mathcal{J}$.
We next study $\mathfrak{X}(G/K)^K$ as a Lie algebra and obtain a formula for the commutator of K-invariant vector fields in terms of the associated K-covariant mappings. The Cartan embedding $G/K \longrightarrow P \subset G$ given by $gK \mapsto g\theta(g)^{-1}$ is a fundamental tool in the study of symmetric spaces, and it is natural to use it to study $\mathfrak{X}(G/K)^K$. Invariant vector fields on G/K whose horizontal lifts to G are tangent to P are called *flat* (in fact, the Cartan embedding induces *a priori* two different notions of flatness, which we show to be equivalent). We obtain a commutator formula with no curvature term for the action on P of these vector fields. For G simple and simply connected, we prove ([**AG03**, Thm 3.1]) that every element of $\mathfrak{X}(G/K)^K$ is flat if and only if K is semisimple (i.e. G/K is not the complexification of a hermitian symmetric space).
In Section 4 we study the conjugation action of a semisimple group G on itself. This is an example of the Cartan embedding of a symmetric space for the group $G \times G$ and involution $\theta(g,h) = (h,g)$. In this case, $\mathcal{J}$ is just the algebra of regular class functions. Assuming G is simply-connected, no localization is needed anymore:

Theorem 3.11 ([**AG03**, Thm 4.1])**.** *Assume G is simply connected and $\mathfrak{g}$ is semisimple of rank r. Let $\varphi_1, \dots, \varphi_r$ be the characters of the fundamental representations of G. Then the vector fields $X_1, \dots, X_r$ on G corresponding to $\mathrm{grad}\varphi_1, \dots, \mathrm{grad}\varphi_r$ are a $\mathcal{J}$-module basis for $\mathfrak{X}(G)^{\mathrm{Ad}\,G}$. Furthermore, all conjugation-invariant vector fields are flat.*

In the special case of $\mathrm{SL}(n,\mathbb{C})$, we calculate the commutators of an explicit basis of conjugation-invariant vector fields. When $G = \mathrm{SL}(2,\mathbb{C})$, we construct a $\mathbb{C}$-basis for $\mathfrak{X}_2 = \mathfrak{X}(G)^{\mathrm{Ad}\,G}$ and compute the commutators and the action on invariants of this basis. We show that $\mathfrak{X}_2$ is isomorphic to a subalgebra of the Witt algebra ([**AG03**, Thm 4.5]) and we find the highest weight vectors inside $\mathbb{C}[\mathrm{SL}(2,\mathbb{C})]$.
Finally, we establish a separation of variables theorem for $\mathrm{SL}(2,\mathbb{C})$. More precisely, using the preceding results, we construct explicitly a conjugation-invariant differential operator on $\mathrm{SL}(2,\mathbb{C})$ such

that its kernel H realizes the isomorphism

$$\mathbb{C}[\mathrm{SL}(2,\mathbb{C})] \;\cong\; \mathbb{C}[\mathrm{SL}(2,\mathbb{C})]^{\mathrm{Ad}\,\mathrm{SL}(2,\mathbb{C})} \otimes H\,.$$

This result ([**AG03**, Thm 5.3]) is the global version of the separation of variables in the isotropy representation going back to Kostant and Kostant-Rallis ([**Kos63**], [**KR71**]). However, our proof requires extensive representation-theoretic calculations and does not seem to extend to arbitrary conjugation actions or symmetric spaces in any obvious way.

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