Volkswagen Junior Research Group

'Special Geometries in Mathematical Physics'

* * * * * *

The E_8 challenge

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HUMBOLDT-UNIVERSITÄT · ZU BERLIN ·

E_8 in the Media / March 2007. . .

AIM Press release headline: *A calculation the size of Manhattan* + picture (answer is a matrix – compare it to an area)

• articles in: The New York Times, Times (London), Scientific American, Nature, Le Monde, Spiegel, Berliner Zeitung. . .

• TV spots on CNN, NBC, BBC. . .

• Coverage in the following languages: Chinese – Dutch – Finnish – French – German – Greek – Hebrew – Hungarian – Italian – Portugese – Vietnamese

• Jerry McNerney (D-California) delivered a statement to Congress about the result

In this talk:

• What is E_8?

• Why is it interesting?

• What was the computation and why is it important?

Classical Lie groups

Appear in families associated with certain types of *geometry:*

Family A: (Pseudo-)Hermitian geometry

- $\mathrm{SL}(n,\mathbb{R}) := \{A \in \mathrm{GL}(n,\mathbb{R}) : \ \det A = 1\}$ (non compact)

h: a Hermitian product, for example $h(x,y) = x^t\bar{y}$:

- $\mathrm{SU}(n) := \{A \in \mathrm{GL}(n,\mathbb{C}) : \ h(x,y) = h(Ax,Ay)\ \forall x,y \in \mathbb{C}^n\}$ (compact)

– both are real forms of their complexification $\mathrm{SL}(n,\mathbb{C})$ –

Family B and D: (Pseudo-)Riemannian geometry

g : a scalar product of signature (p,q), $p+q=n=\begin{cases}\text{odd: family B}\\ \text{even: family D}\end{cases}$

- $\mathrm{SO}(p,q) = \{A \in \mathrm{SL}(n,\mathbb{R}) : \ g(x,y) = g(Ax,Ay)\ \forall x,y \in \mathbb{R}^n\}$

– all of them are real forms of $\mathrm{SO}(n,\mathbb{C})$ –

Family C: Symplectic geometry

$\Omega \in \Lambda^2(\mathbb{C}^{2n})$: a generic 2-form (i. e. with dense $\mathrm{GL}(2n, \mathbb{C})$-orbit in $\Lambda^2(\mathbb{C}^{2n})$)

- $\mathrm{Sp}(n, \mathbb{C}) := \{A \in \mathrm{GL}(n, \mathbb{C}) : \ \Omega(Ax, Ay) = \Omega(x, y)\}$

has again compact and non compact real forms

Linearisation of a Lie group

For any Lie group G: $\mathfrak{g} := T_e G$ is a vector space with a natural skew-symmetric bilinear product $[,]$ satisfying the

Jacobi identity: $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$ for all $X, Y, Z \in \mathfrak{g}$

and called the *Lie algebra* of G.

N.B. For the Lie algebra of a matrix group, $[,]$ is just the commutator of matrices: $[X, Y] = X \cdot Y - Y \cdot X$ for all $X, Y \in \mathfrak{g} \subset \mathfrak{gl}(n, \mathbb{C}) = \mathrm{End}(\mathbb{C}^n)$

– as a vector space, $\mathfrak{g}$ is a much more tractable object than G ! –

Dfn. A Lie algebra $\mathfrak{g}$ is called *simple* if its only ideals $\mathfrak{m}$ ($\Leftrightarrow [\mathfrak{m}, \mathfrak{g}] \subset \mathfrak{m}$) are 0 and $\mathfrak{g}$.

All classical complex Lie algebras ($\neq \mathfrak{so}(4, \mathbb{C})$) are simple.

Thm (W. Killing, 1889). The only simple complex Lie algebras are $\mathfrak{so}(n, \mathbb{C})$, $\mathfrak{sp}(n, \mathbb{C})$, $\mathfrak{sl}(n, \mathbb{C})$ as well as five exceptional Lie algebras,

$$\mathfrak{g}_2 \ := \mathfrak{g}_2^{14},\ \mathfrak{f}_4^{52},\ \mathfrak{e}_6^{78},\ \mathfrak{e}_7^{133}, \mathfrak{e}_8^{248}.$$

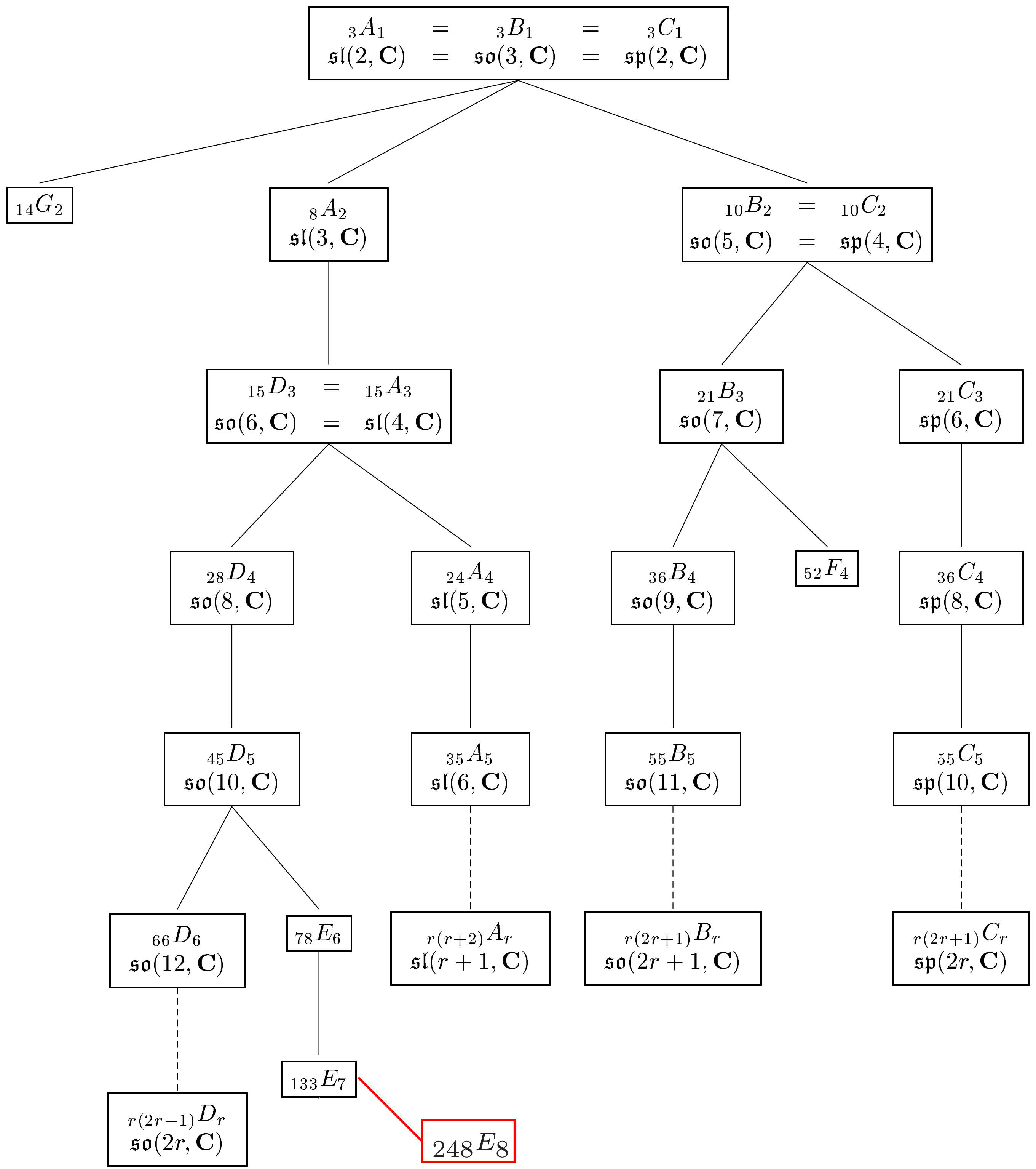
(upper index: dimension, lower index: rank)

Notation:

- E_8, $\mathfrak{e}_8$: complex Lie group, Lie algebra [exa.: $\mathrm{SO}(p+q, \mathbb{C})$]

It has 3 real forms:

- E_8^c, $\mathfrak{e}_8^c$: *compact* real form of E_8, $\mathfrak{e}_8$ [exa.: $\mathrm{SO}(p+q)$]
- E_8^*, $\mathfrak{e}_8^*$: *non compact* split real form of E_8, $\mathfrak{e}_8$ [exa.: $\mathrm{SO}(p,p)$, i. e. $p = q$]
- E_8^r, $\mathfrak{e}_8^r$: *non compact* non split real form of E_8, $\mathfrak{e}_8$ [exa.: all other $\mathrm{SO}(p,q)$]


$_3A_1 = {}_3B_1 = {}_3C_1$
$\mathfrak{sl}(2,\mathbf{C}) = \mathfrak{so}(3,\mathbf{C}) = \mathfrak{sp}(2,\mathbf{C})$
$_{14}G_2$
$_8A_2$
$\mathfrak{sl}(3,\mathbf{C})$
$_{10}B_2 = {}_{10}C_2$
$\mathfrak{so}(5,\mathbf{C}) = \mathfrak{sp}(4,\mathbf{C})$
$_{15}D_3 = {}_{15}A_3$
$\mathfrak{so}(6,\mathbf{C}) = \mathfrak{sl}(4,\mathbf{C})$
$_{21}B_3$
$\mathfrak{so}(7,\mathbf{C})$
$_{21}C_3$
$\mathfrak{sp}(6,\mathbf{C})$
$_{28}D_4$
$\mathfrak{so}(8,\mathbf{C})$
$_{24}A_4$
$\mathfrak{sl}(5,\mathbf{C})$
$_{36}B_4$
$\mathfrak{so}(9,\mathbf{C})$
$_{52}F_4$
$_{36}C_4$
$\mathfrak{sp}(8,\mathbf{C})$
$_{45}D_5$
$\mathfrak{so}(10,\mathbf{C})$
$_{35}A_5$
$\mathfrak{sl}(6,\mathbf{C})$
$_{55}B_5$
$\mathfrak{so}(11,\mathbf{C})$
$_{55}C_5$
$\mathfrak{sp}(10,\mathbf{C})$
$_{66}D_6$
$\mathfrak{so}(12,\mathbf{C})$
$_{78}E_6$
$_{r(r+2)}A_r$
$\mathfrak{sl}(r+1,\mathbf{C})$
$_{r(2r+1)}B_r$
$\mathfrak{so}(2r+1,\mathbf{C})$
$_{r(2r+1)}C_r$
$\mathfrak{sp}(2r,\mathbf{C})$
$_{133}E_7$
$_{r(2r-1)}D_r$
$\mathfrak{so}(2r,\mathbf{C})$
$_{248}E_8$

Root Geometry

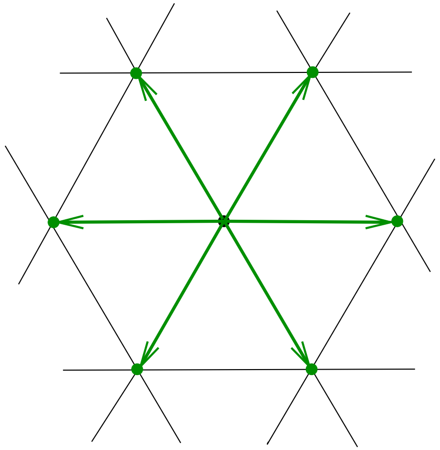
Idea of classification: Choose a maximal abelian subalgebra $\mathfrak{h}$ ('Cartan subalgebra') and find a basis of $\mathfrak{g}$ on which it acts diagonally:

$$\mathfrak{g} = \mathfrak{h} \bigoplus_{\alpha \in \mathfrak{h}^*} \mathfrak{g}_\alpha, \quad [H, X] = \alpha(H) X \quad \forall H \in \mathfrak{h},\, X \in \mathfrak{g}_\alpha.$$

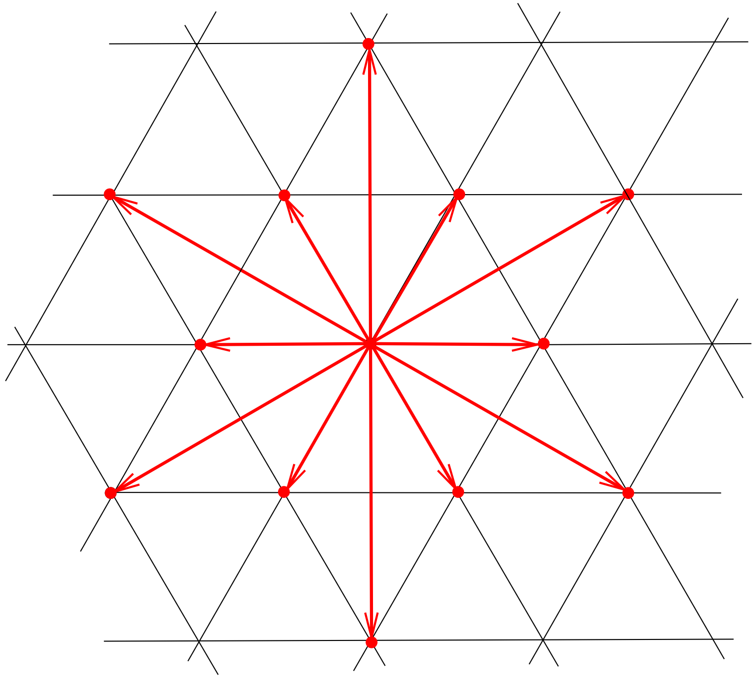
- $0 \neq \alpha \in \mathfrak{h}^*$: 'roots'; all roots together $\subset \mathfrak{h}^*$ form the 'root diagram' and span the 'root lattice'
- $\mathfrak{g}_\alpha$: 'root spaces'; they are all 1-dimensional
- $\dim \mathfrak{h}$: 'rank of $\mathfrak{g}$'
- $\mathfrak{h}$ is the zero eigenspace under its own action; by dfn, 0 is not a root
- multiplication: $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \begin{cases} \mathfrak{g}_{\alpha+\beta} & \text{if } \alpha+\beta \text{ is a root} \\ 0 & \text{otherwise} \end{cases}$

KEY FACT: geometry of root diagram encodes almost everything you (may) want to know about $\mathfrak{g}$

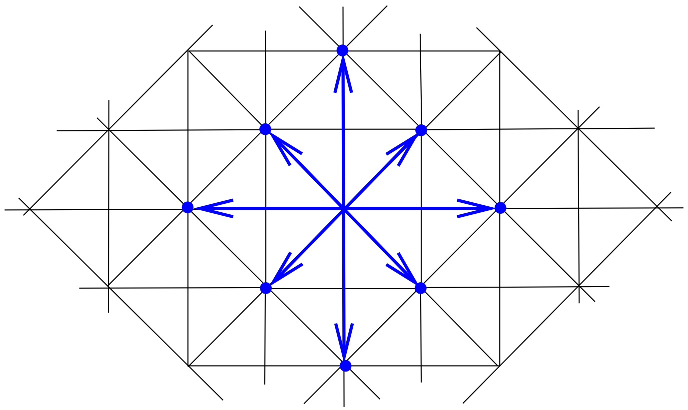
Root diagrams of rank 2 ($= \dim \mathfrak{h}$)



$A_2 = \mathfrak{sl}(3, \mathbb{C})$
(hexagonal lattice)



exceptional G_2
(hexagonal lattice)



$B_2 = C_2 = \mathfrak{sp}(4, \mathbb{C}) = \mathfrak{so}(5, \mathbb{C})$
(quadratic lattice)

Root diagram of E_8

$e_1, \ldots, e_8$: standard basis of $\mathbb{C}^8 = \mathfrak{h}^*(E_8)$.

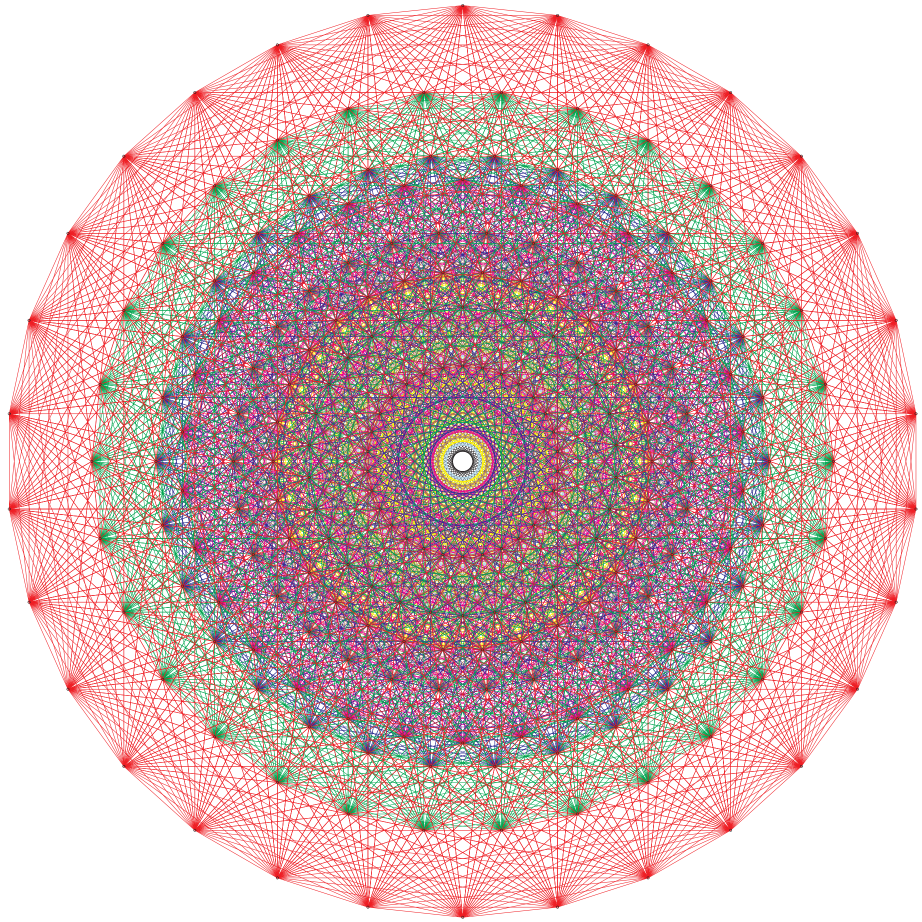
E_8 roots:

- $\pm e_i \pm e_j$: makes 112 roots
  (= roots of $\mathfrak{so}(16, \mathbb{C})$)
- $\frac{1}{2}(\pm e_1 \pm e_2 \pm \ldots \pm e_8)$ with an even number of $-$'s, yielding 128 roots

$\ldots 8+112+128 = 248 = \dim E_8$!

- All roots have same length

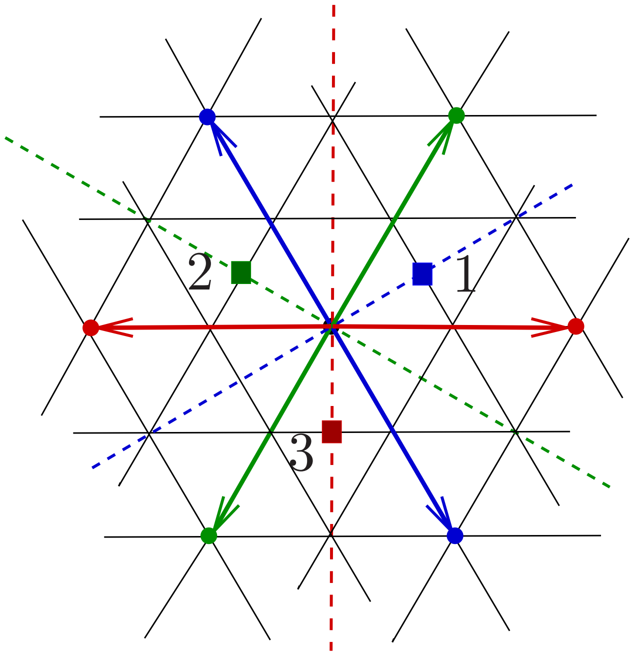
Picture:
2-dimensional projection of E_8 root diagram, where each root is connected to its nearest neighbours by lines (corners: 8 inscribed 30-gons)

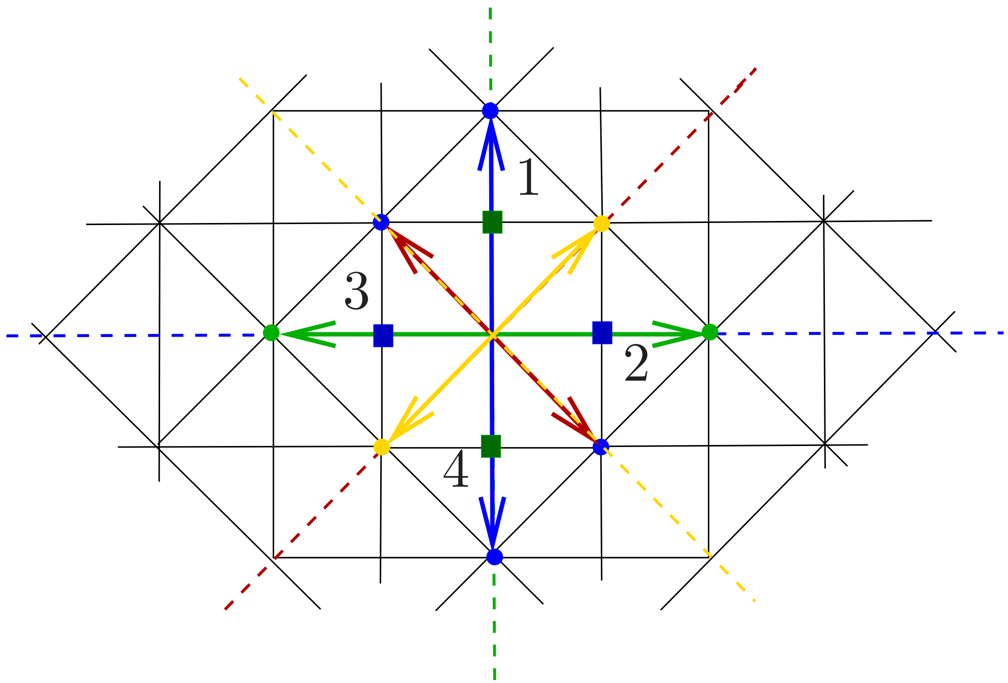


… tells us: E_8 is very symmetric, highly non-trivial, and extremely 'crammed'

Weyl group I

W is the group generated by reflections at hyperplanes $V_\alpha \subset \mathfrak{h}^*$ orthogonal to the roots:

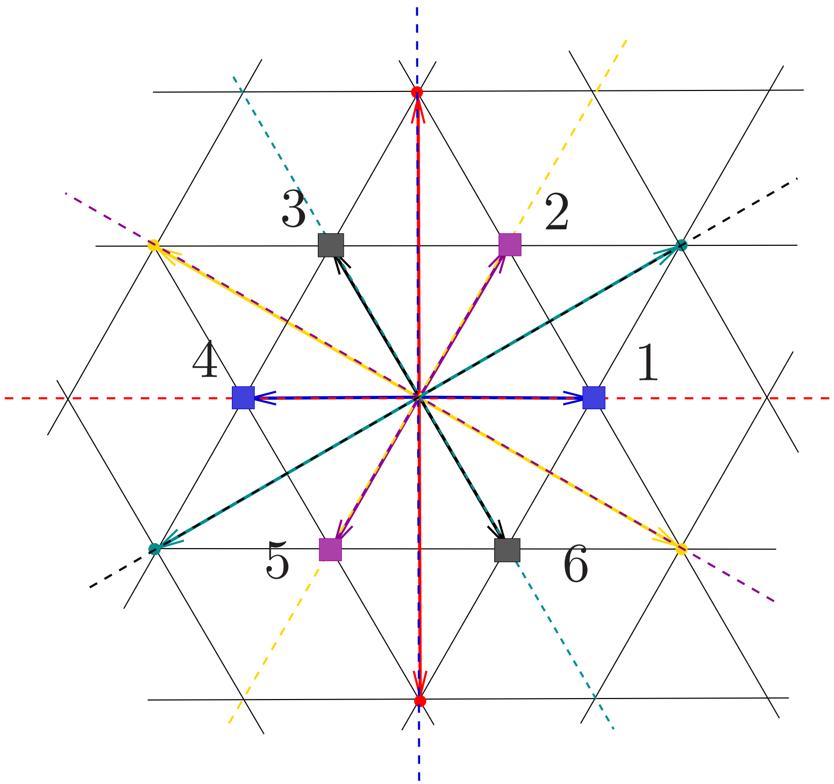




$W(A_2) = \langle (12), (13), (23) \rangle$
$= S_3$, order 6

$W(BC_2) = \langle (23), (14), (12)(34), (13)(24) \rangle$
$= (\mathbb{Z}_2)^2 \rtimes S_2$, order 8

Weyl group II



$W(G_2) = \langle r_{\pi/3}, s \rangle = D_6$:
dihedral group of order 12

More generally:

- $W(A_n) = S_{n+1}$ of order $(n+1)!$
- $W(BC_n) = (\mathbb{Z}_2)^n \rtimes S_n$ of order $2^n n!$
- $W(D_n) = (\mathbb{Z}_2)^{n-1} \rtimes S_n$ of order $2^{n-1} n!$

In particular:

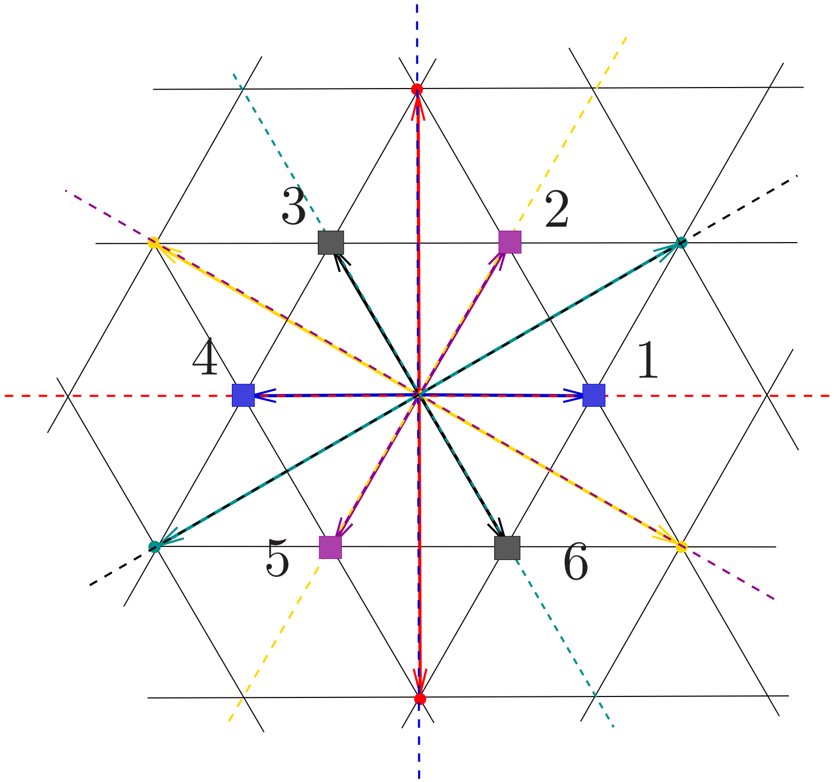
$|W(A_8)| = 9! = 362\,880$

$|W(BC_8)| = 2^8 8! = 10\,321\,920$

$|W(D_8)| = 2^7 8! = 5\,160\,960$

. . . and what about E_8?

Weyl group II



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$|W(E_8)| = 2^{14} 3^5 5^2 7 = 696\,729\,600$
and it is a group of high complexity!

This has dramatic consequences for all computational questions

- that need an explicit realisation of W
- whose complexity grows like a polynomial in $|W|$

Example:

Any representation V of G is determined by a 'highest weight' λ in the lattice

H: subgroup of G with Lie algebra $\mathfrak{h}$

e^{α}: the function on H induced by $\alpha \in \mathfrak{h}^* \cong \mathfrak{h}$

$\chi(V)$: character of G-repr. on V, viewed as function on H, $\dim V = \chi(V)(e)$

ϱ: a certain *fixed* element in $\mathfrak{h}^*$

$\mathrm{sgn}(s) = \pm 1$ (even/odd number of reflections)

Thm (H. Weyl, 1925) $$\chi(V) = \frac{\sum_{s \in W} \mathrm{sgn}(s) e^{s(\lambda+\varrho)}}{\sum_{s \in W} \mathrm{sgn}(s) e^{s\varrho}}$$

What makes E_8 interesting?

E_8 appears in connection with

- sphere packing problems $(*)$
- the 'Monster', the largest of the (finite) sporadic groups
- superstring theory $(*)$
- quasicrystals with 5-fold symmetry

. . . and than there are wild speculations about E_8 as explanation for everything, ranging from Fermat's Theorem to elementary particles

Sphere packings

In n-dimensional Euclidean space, consider the following questions:

Sphere Packing Problem (SPP): Given a huge number of equal spheres, what is the densest way to pack them together? ($\sim$ global problem)

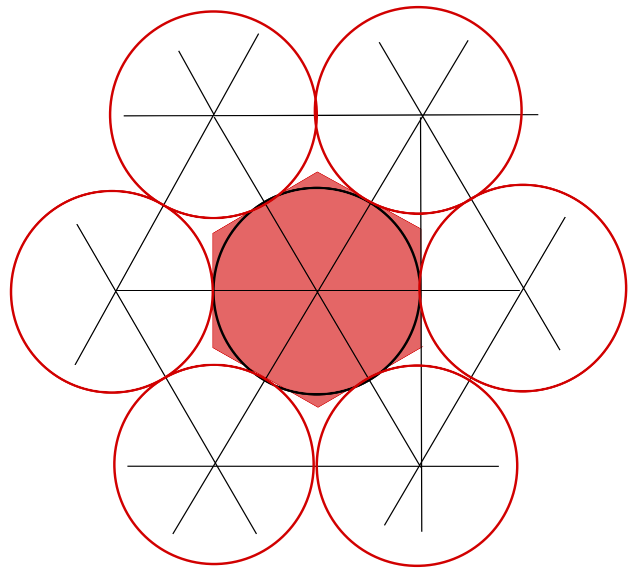
Kissing Number Problem (KNP): How many spheres can be arranged so that they all touch one central sphere of the same size? ($\sim$ local problem)

Step I: represent spheres by their centers; these will sometimes form a *lattice*.

For $n = 2$, the answer to both problems is given mainly by the hexagonal lattice:

density $= \frac{\text{circle area}}{\text{circumscr. hexagon a.}} = \frac{\pi}{\sqrt{12}} = 0,9069\ldots$

kissing number $= 6$



The case $n = 3$ – a still open problem

The classical root systems A_3 and D_3 generate the same lattice – the fcc lattice ('face-centered cubic')

density $= \frac{\pi}{\sqrt{18}} = 0{,}7405\ldots$, kissing number $= 12$

Thm (Gauss, 1831). The fcc lattice is the densest lattice packing for $n = 3$.

But. . .

- nonlattice packings are known that are as dense as the fcc lattice ('hcp packing', still periodical)
- local partial packings of higher density are known

Thm (Bender, 1874). In 3 dimensions, the highest possible kissing number is 12.

But there are infinitely many possible arrangements

Highe values of n

Thm (Korkine-Zolotarev, 1872/77)

The D_4 and D_5 lattices are the densest lattice packings in 4 and 5 dimensions.

Furthermore, they described E_6, E_7, E_8 and conjectured that they are also optimal among lattices!

Thm (Blichfeldt, 1935)

The E_6, E_7, E_8 lattices are the densest lattice packings in 6,7,8 dimensions.

These are the best known packings in these dimensions.

For the KNP, only two case (besides $n = 2, 3$) are settled:

Thm (Odlyzko-Sloane, 1979).

a) The highest kissing number in $n = 8$ is 240 and realized only by the E_8 lattice;

b) The highest kissing number in $n = 24$ is 196 560 and realized only by the Leech lattice.

E_8 and supersymmetric theories

Objective: Unification of standard model of elementary particles and general gravity

Since 1980ies: Construction of field theories with *local supersymmetries*, i. e. transformations that exchange fermions and bosons.

Models with 3-dimensional space-time

- are instructive toy models for higher-dimensional physical theories
- appear in dimensional reductions of lowe and higher dimensional theories

N: $\#$ of supersymmetries – increasing N means increasing the geometric constraints on the 'target manifold' M!

Study

- commutator relations of extended supersymmetry algebra
- its possible 'supermultiplets' $=$ representations and
- compatibility conditions with Langragian

Supersymmetric theories II

N: # of supersymmetries, d_N: # of bosonic states, k: # of supermultiplets

$$\dim M = k{\cdot}d_N$$

N	1	2	3	4	5	6	7	8	9	10	12	16
d_N	1	2	4	4	8	8	8	8	16	32	64	128

a) Compute isotropy group of supersymmetry algebra: $\mathrm{SO}(N) \times H$

Want: $\mathrm{Hol}(M) \subset \mathrm{SO}(N) \times H$ and acts irreducibly on TM

$N = 1$: any Riemannian manifold as 'target space' M

$N = 2$: Kähler manifold ($\dim M/2 \in \mathbb{N}$)

$N = 3, 4$: 3 almost complex structures (quaternionic or product of two quaternionic spaces; $\dim M/4 \in \mathbb{N}$)

$N \geq 5$: Einstein space, $\mathrm{Scal} < 0$, and $\mathrm{SO}(N) \times H$ has no transitive sphere action! Berger's theorem $\Rightarrow$ M is a non-compact symmetric space

For $N \geq 9$, $k = 1$ (the target space is unique) and

For $N = 16$: $M = E_8^*/\mathrm{SO}(16)$

($N = 9, 10, 12$: F_4, E_6, E_7-spaces) [Marcus-Schwarz '83; de Wit-Nicolai-Tollstén '93]

E_8 and computations

In the 1980ies, the character of Lie algebra computations changed drastically:

- Fast recursion algorithms were derived, making (some) sums over Weyl groups unnecessary [Typical idea: introduce partial orderings on weights]
- Suitable software then implemented these algorithms
- Typically, E_8 was used as a test case

In the beginning, the results were published as long lists of tables in journals, then books – see for example

McKay, W.G., Patera, J. *Tables of dimensions, indices, and branching rules for representations of simple Lie algebras*, Marcel Dekker, 1981.

Bremner, M.R., Moody, R.V., Patera, J., *Tables of dominant weight multiplicities for representations of simple Lie algebras*, Marcel Dekker, 1985.

McKay, W.G., Patera, J., Rand, D.W., *Tables of representations of simple Lie algebras. Volume I: Exceptional simple Lie algebras*, Montréal/Centre de Recherches Mathématiques, 1990.

Since July 1996, **LiE** is publically available for free (Centre for Mathematics and Computer Science/Amsterdam).

With **LiE**, problems that were unsolvable became accessible for any graduate student!

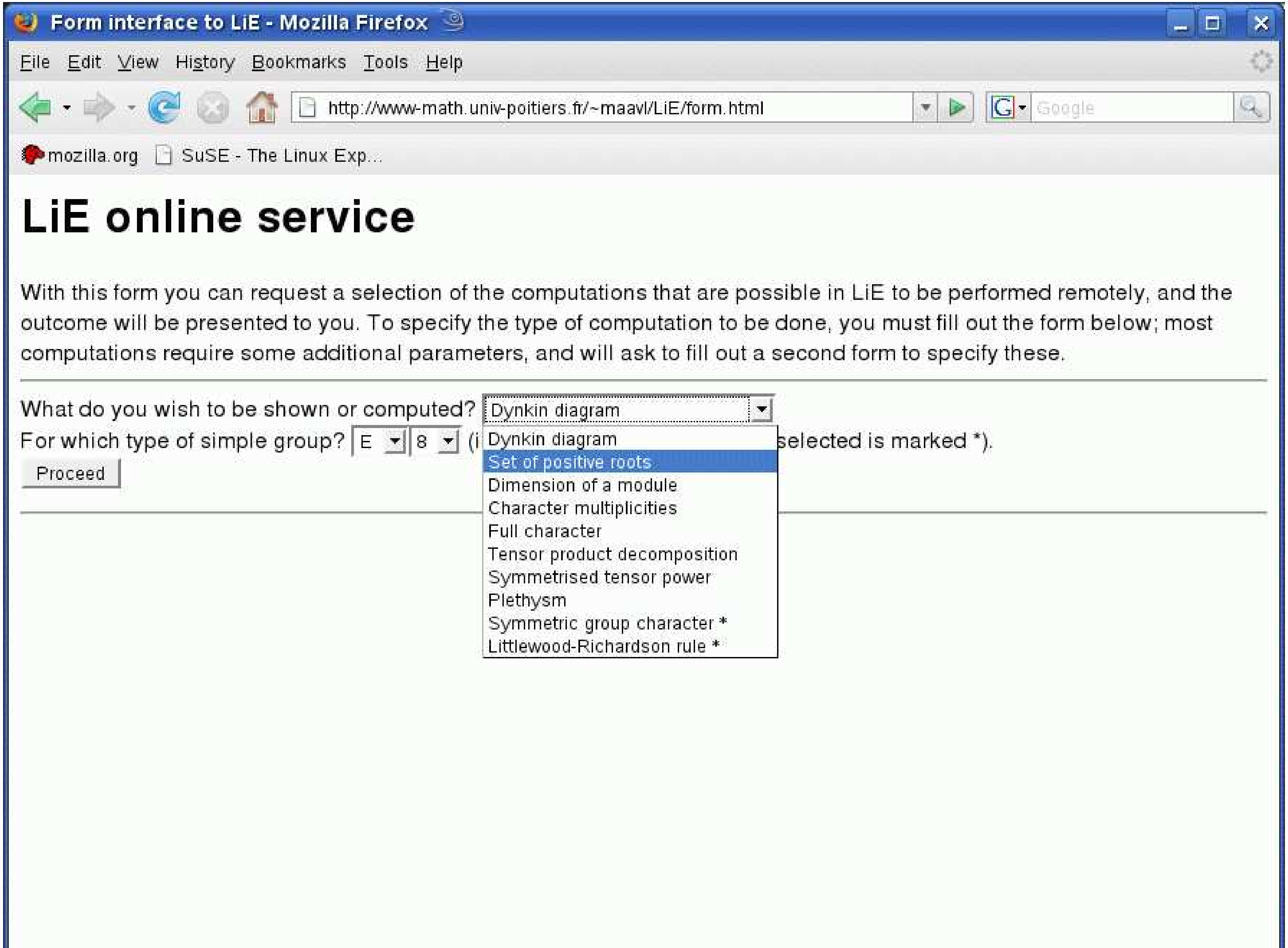
LiE was used to answer many problems of representation theory, like

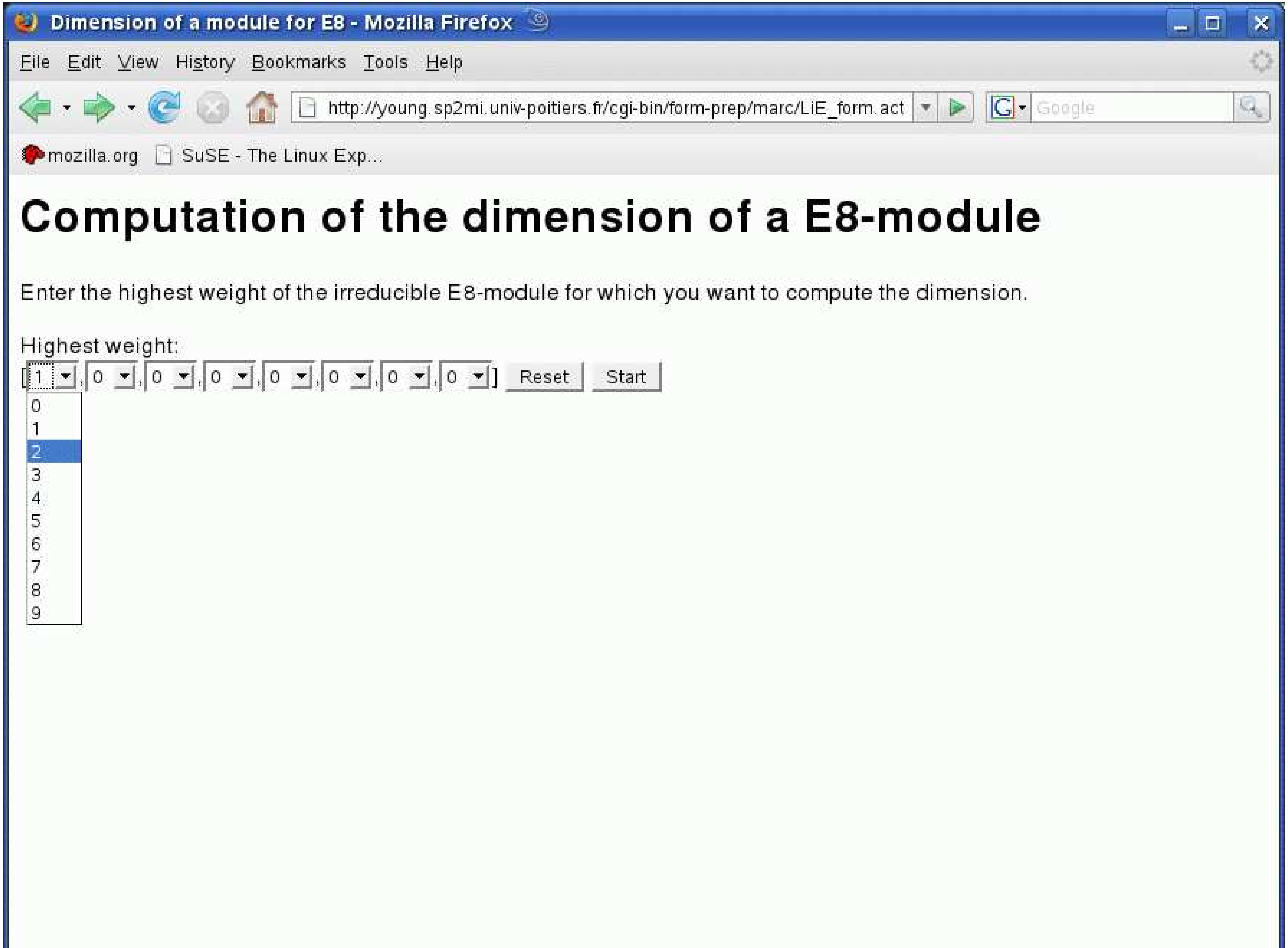
• **big problem:** Kostant's conjecture on subgroups of exceptional Lie groups (relates the Coxeter number to finite simple groups in simple complex Liwe groups)

• **tiny erxercise:** Adam's conjecture on antisymmetric tensor powers of fundamental representations for E_8

From the beginning, it was one of **LiE**'s objectives to provide implementations for computing **Kazhdan-Lusztig polynomials**.

— LiE offline demo: —


Form interface to LiE - Mozilla Firefox
File Edit View History Bookmarks Tools Help
http://www-math.univ-poitiers.fr/~maavl/LiE/form.html
Google
mozilla.org
SuSE - The Linux Exp...
LiE online service
With this form you can request a selection of the computations that are possible in LiE to be performed remotely, and the outcome will be presented to you. To specify the type of computation to be done, you must fill out the form below; most computations require some additional parameters, and will ask to fill out a second form to specify these.
What do you wish to be shown or computed?
Dynkin diagram
For which type of simple group?
E
8
selected is marked *).
Proceed
Dynkin diagram
Set of positive roots
Dimension of a module
Character multiplicities
Full character
Tensor product decomposition
Symmetrised tensor power
Plethysm
Symmetric group character *
Littlewood-Richardson rule *


Dimension of a module for E8 - Mozilla Firefox
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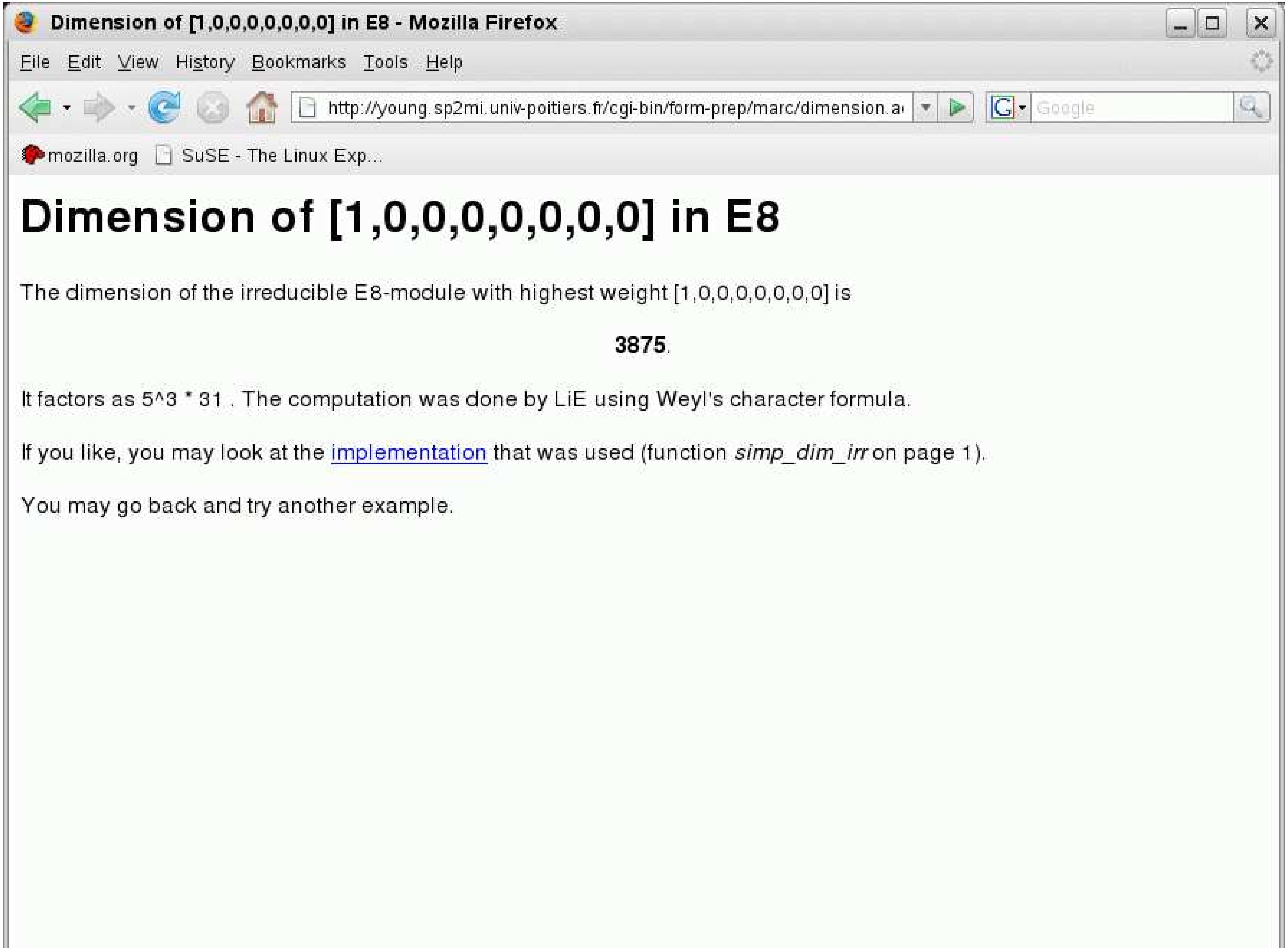
http://young.sp2mi.univ-poitiers.fr/cgi-bin/form-prep/marc/LiE_form.act
Google
mozilla.org
SuSE - The Linux Exp...
Computation of the dimension of a E8-module
Enter the highest weight of the irreducible E8-module for which you want to compute the dimension.
Highest weight:
[1, 0, 0, 0, 0, 0, 0, 0]
Reset
Start
0
1
2
3
4
5
6
7
8
9


Dimension of [1,0,0,0,0,0,0,0] in E8 - Mozilla Firefox

File Edit View History Bookmarks Tools Help

http://young.sp2mi.univ-poitiers.fr/cgi-bin/form-prep/marc/dimension.a

mozilla.org SuSE - The Linux Exp...

Dimension of [1,0,0,0,0,0,0,0] in E8

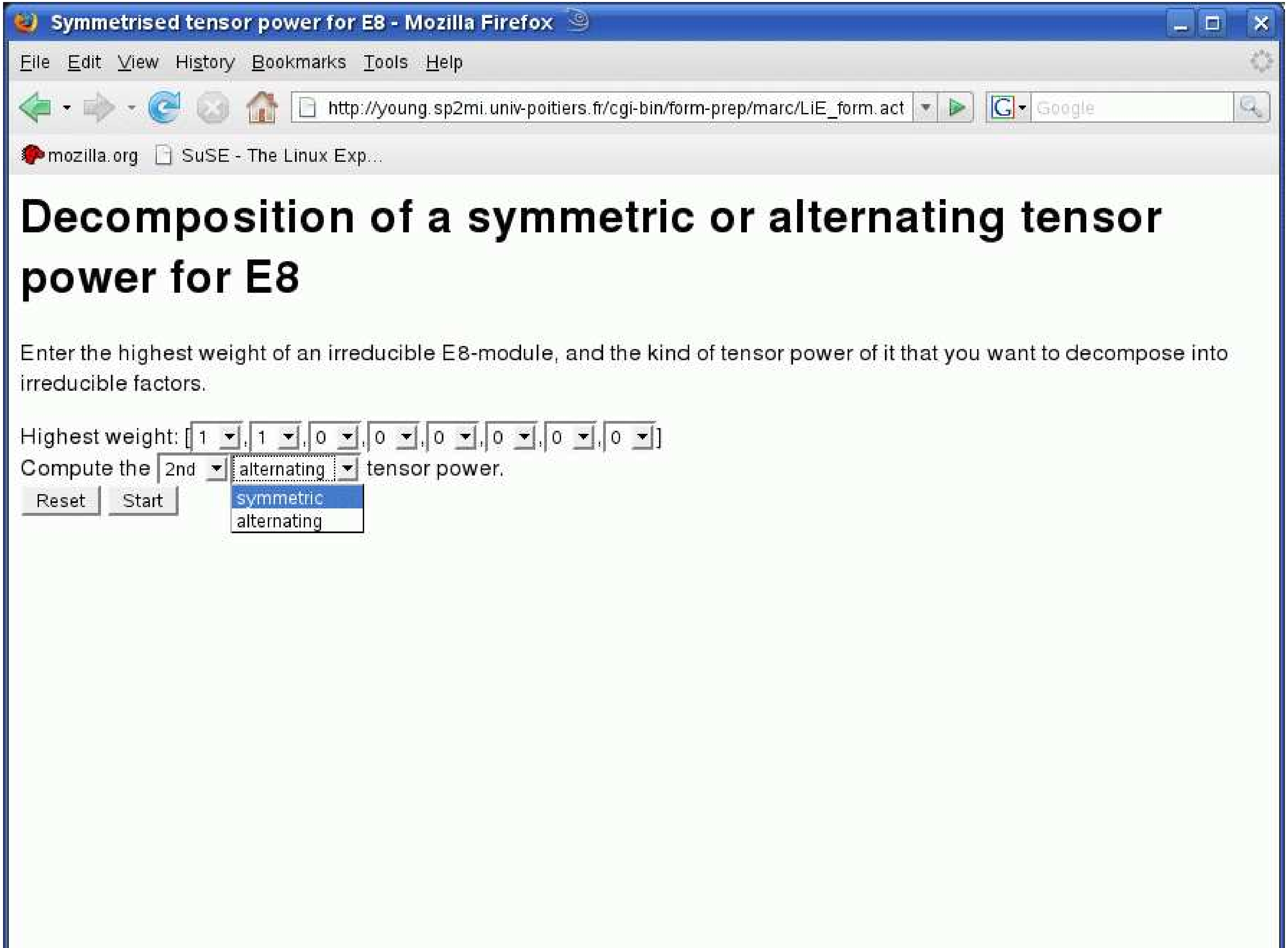
The dimension of the irreducible E8-module with highest weight [1,0,0,0,0,0,0,0] is

3875.

It factors as 5^3 * 31 . The computation was done by LiE using Weyl's character formula.

If you like, you may look at the implementation that was used (function *simp_dim_irr* on page 1).

You may go back and try another example.


Symmetrised tensor power for E8 - Mozilla Firefox
File Edit View History Bookmarks Tools Help
http://young.sp2mi.univ-poitiers.fr/cgi-bin/form-prep/marc/LiE_form.act
Google
mozilla.org
SuSE - The Linux Exp...
Decomposition of a symmetric or alternating tensor power for E8
Enter the highest weight of an irreducible E8-module, and the kind of tensor power of it that you want to decompose into irreducible factors.
Highest weight: [1, 1, 0, 0, 0, 0, 0, 0]
Compute the 2nd alternating tensor power.
symmetric
alternating
Reset
Start

2nd alternating power of [1,1,0,0,0,0,0,0] in E8 - Mozilla Firefox

File Edit View History Bookmarks Tools Help

http://young.sp2mi.univ-poitiers.fr/cgi-bin/form-prep/marc/sym-alt.act?

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Decomposition of the 2nd alternating tensor power of [1,1,0,0,0,0,0,0] in E8

Below you find the decomposition of the 2nd alternating tensor power of the irreducible E8-module with highest weight [1,1,0,0,0,0,0,0] into its irreducible factors, as computed by LiE. Each term represents a different highest weight of an irreducible module occurring in the decomposition, prefixed by its multiplicity of occurrence. The 2nd alternating tensor power of [1,1,0,0,0,0,0,0] has dimension 45509929120972800.

```
 1X[2,0,0,1,0,0,0,0] + 1X[0,2,1,0,0,0,0,0] + 1X[1,1,0,0,1,0,0,0] +
 1X[1,0,1,0,0,1,0,0] + 2X[0,0,0,1,0,1,0,0] + 2X[2,1,0,0,0,0,1,0] +
 1X[0,1,1,0,0,0,1,0] + 1X[4,0,0,0,0,0,0,1] + 2X[1,0,0,0,1,0,1,0] +
 1X[2,0,1,0,0,0,0,1] + 2X[1,2,0,0,0,0,0,1] + 2X[0,0,2,0,0,0,0,1] +
 2X[1,0,0,1,0,0,0,1] + 1X[0,1,0,0,0,1,1,0] + 3X[0,1,0,0,1,0,0,1] +
 2X[3,1,0,0,0,0,0,0] + 4X[2,0,0,0,0,1,0,1] + 4X[1,1,1,0,0,0,0,0] +
 2X[0,0,1,0,0,0,2,0] + 2X[0,0,1,0,0,1,0,1] + 3X[0,1,0,1,0,0,0,0] +
 4X[2,0,0,0,1,0,0,0] + 6X[0,0,1,0,1,0,0,0] + 4X[1,1,0,0,0,0,1,1] +
 7X[1,1,0,0,0,1,0,0] + 2X[0,0,0,0,0,2,0,1] + 2X[0,0,0,0,1,0,1,1] +
 4X[3,0,0,0,0,0,1,0] + 3X[1,0,1,0,0,0,0,2] + 9X[1,0,1,0,0,0,1,0] +
 4X[0,0,0,0,1,1,0,0] + 5X[0,2,0,0,0,0,1,0] + 5X[0,0,0,1,0,0,0,2] +
 6X[0,0,0,1,0,0,1,0] + 9X[2,1,0,0,0,0,0,1] + 2X[1,0,0,0,0,0,2,1] +
 2X[1,0,0,0,0,1,0,2] +11X[0,1,1,0,0,0,0,1] + 9X[1,0,0,0,0,1,1,0] +
14X[1,0,0,0,1,0,0,1] + 8X[2,0,1,0,0,0,0,0] + 3X[0,1,0,0,0,0,1,2] +
 5X[1,2,0,0,0,0,0,0] + 3X[0,1,0,0,0,0,2,0] + 2X[0,0,2,0,0,0,0,0] +
16X[1,0,0,1,0,0,0,0] +12X[0,1,0,0,0,1,0,1] + 3X[2,0,0,0,0,0,0,3] +
10X[2,0,0,0,0,0,1,1] + 9X[0,1,0,0,1,0,0,0] + 1X[0,0,1,0,0,0,0,3] +
13X[0,0,1,0,0,0,1,1] + 9X[2,0,0,0,0,1,0,0] +19X[0,0,1,0,0,1,0,0] +
13X[1,1,0,0,0,0,0,2] +23X[1,1,0,0,0,0,1,0] + 3X[0,0,0,0,0,1,0,3] +
 2X[0,0,0,0,0,0,3,0] + 7X[0,0,0,0,0,1,1,1] + 8X[0,0,0,0,1,0,0,2] +
 2X[0,0,0,0,0,2,0,0] + 9X[3,0,0,0,0,0,0,1] +18X[0,0,0,0,1,0,1,0] +
```

Hecke Algebras

- W: a Weyl group (more generally: a Coxeter group)
- $S \subset W$: a set of reflections generating W

'Braid relations': For $s, s' \in S$, $m(s,s') := \mathrm{ord}(ss')$

$$(ss')^{m(s,s')} = 1 \Leftrightarrow ss'ss' \ldots = s'ss's \ldots \ (m(s,s')\text{-times})$$

Dfn. Let $A := \mathbb{Z}[v, v^{-1}]$ and consider all formal elements T_w for $w \in W$. Then the *Hecke algebra of* (W,S) is the associative algebra $\mathcal{H} := \bigoplus_{w \in W} A \cdot T_w$ with the relations

a) $T_s T_{s'} T_s \ldots = T_{s'} T_s T_{s'} \ldots$ for $m(ss') < \infty$, ('braid relations')
b) $T_s^2 = (v^{-1} - v)T_s + 1$ for all $s \in S$. ('quadratic relations')

Comments:

- $T_e = 1$ and $T_s^{-1} = T_s + (v - v^{-1})$
- If $w = s_1 \ldots s_n$ is a reduced expression for w, then $T_w = T_{s_1} \ldots T_{s_n}$
- For $v = 1$, this is just the group algebra of W

In particular, the elements T_w form a basis of $\mathcal{H}$.

Involution d: Set $d(v) := v^{-1}$ and $d(T_s) = T^{-1} = T_s + (v - v^{-1})$

$\longrightarrow$ extends to a ring homomorphism $d : \mathcal{H} \to \mathcal{H}$.

Thm (Kazhdan-Lusztig, 1979). $\mathcal{H}$ has a unique basis $\{C_w\}_{w \in W}$ such that

a) $d(C_w) = C_w$,

b) $C_w \in T_w + \bigoplus_{w' \in W} v \cdot \mathbb{Z}[v] T_{w'}$.

These are called the *Kazhdan-Lusztig polynomials of* (W, S).

Example. Take $W = S_2 = \{12 = e, 21\}$, the Weyl group of type A_2.

- $C_{12} = T_{12} = 1$,
- $C_{21} = T_{21} + vT_{12}$

check: $d(T_{21} + v) = T_{21} + (v - v^{-1}) + v^{-1} = T_{21} + v$ (o.k.)

Easy Kazhdan-Lusztig polynomials

Kazhdan-Lusztig polynomials for Weyl group of type A_1:

$C_{12} = T_{12}$

$C_{21} = T_{21} + vT_{12}$

Kazhdan-Lusztig polynomials for Weyl group of type A_2:

$C_{123} = T_{123}$

$C_{132} = T_{132} + vT_{123}$

$C_{213} = T_{213} + vT_{123}$

$C_{231} = T_{231} + vT_{132} + vT_{213} + v^2T_{123}$

$C_{312} = T_{312} + vT_{132} + vT_{213} + v^2T_{123}$

$C_{321} = T_{321} + vT_{231} + vT_{312} + v^2T_{132} + v^2T_{213} + v^3T_{123}$

Kazhdan-Lusztig polynomials for Weyl group of type A_3:

T_{1234} // $T_{1243} + vT_{1234}$ // $T_{1324} + vT_{1234}$ // $T_{2134} + vT_{1234}$

$T_{1342} + vT_{1243} + vT_{1324} + v^2T_{1234}$ // $T_{1423} + vT_{1243} + vT_{1324} + v^2T_{1234}$

$T_{2143} + vT_{1243} + vT_{2134} + v^2T_{1234}$ // $T_{2314} + vT_{1324} + vT_{2134} + v^2T_{1234}$

$T_{3124} + vT_{1324} + vT_{2134} + v^2T_{1234}$

$T_{1432} + vT_{1342} + vT_{1423} + v^2T_{1243} + v^2T_{1324} + v^3T_{1234}$

$T_{3214} + vT_{2314} + vT_{3124} + v^2T_{1324} + v^2T_{2134} + v^3T_{1234}$

$T_{2341} + vT_{1342} + vT_{2143} + vT_{2314} + v^2T_{1243} + v^2T_{1324} + v^2T_{2134} + v^3T_{1234}$

$T_{2413} + vT_{1423} + vT_{2143} + vT_{2314} + v^2T_{1243} + v^2T_{1324} + v^2T_{2134} + v^3T_{1234}$

$T_{3142} + vT_{1342} + vT_{2143} + vT_{3124} + v^2T_{1243} + v^2T_{1324} + v^2T_{2134} + v^3T_{1234}$

- $T_{2431} + vT_{1423} + vT_{2431} + vT_{2413} + v^2T_{1342} + v^2T_{1423} + v^2T_{2143} + v^2T_{2314} + v^3T_{1243} + v^3T_{1324} + v^3T_{2134} + v^4T_{1234}$

- $T_{3241} + vT_{2341} + vT_{3142} + vT_{3214} + v^2T_{1342} + v^2T_{2143} + v^2T_{2314} + v^2T_{3124} + v^3T_{1243} + v^3T_{1324} + v^3T_{2134} + v^4T_{1234}$

⋮ [7 are missing]

- $T_{4231} + vT_{2431} + vT_{3241} + vT_{4132} + vT_{4213} + v^2T_{1432} + v^2T_{2341} + v^2T_{2413} + v^2T_{3142} + v^2T_{3214} + +v^2T_{4123} + v^3T_{1342} + v^3T_{1423} + (v^3 + v)T_{2143} + v^3T_{2314} + v^3T_{3124} + (v^4 + v^2)T_{1243} + v^4T_{1324} + (v^4 + v^2)T_{2134} + (v^5 + v^3)T_{1234}$

The meaning of KL polynomials

combinatorics		analytical representation theory
KL polynomials of Weyl group W of G	KL Conjecture → KL (1979)	characters of highest weight and Verma modules of G

The meaning of KL polynomials

combinatorics

KL polynomials
of Weyl group
W of G

KL Conjecture

KL (1979)

analytical representation theory

characters of highest
weight and Verma
modules of G

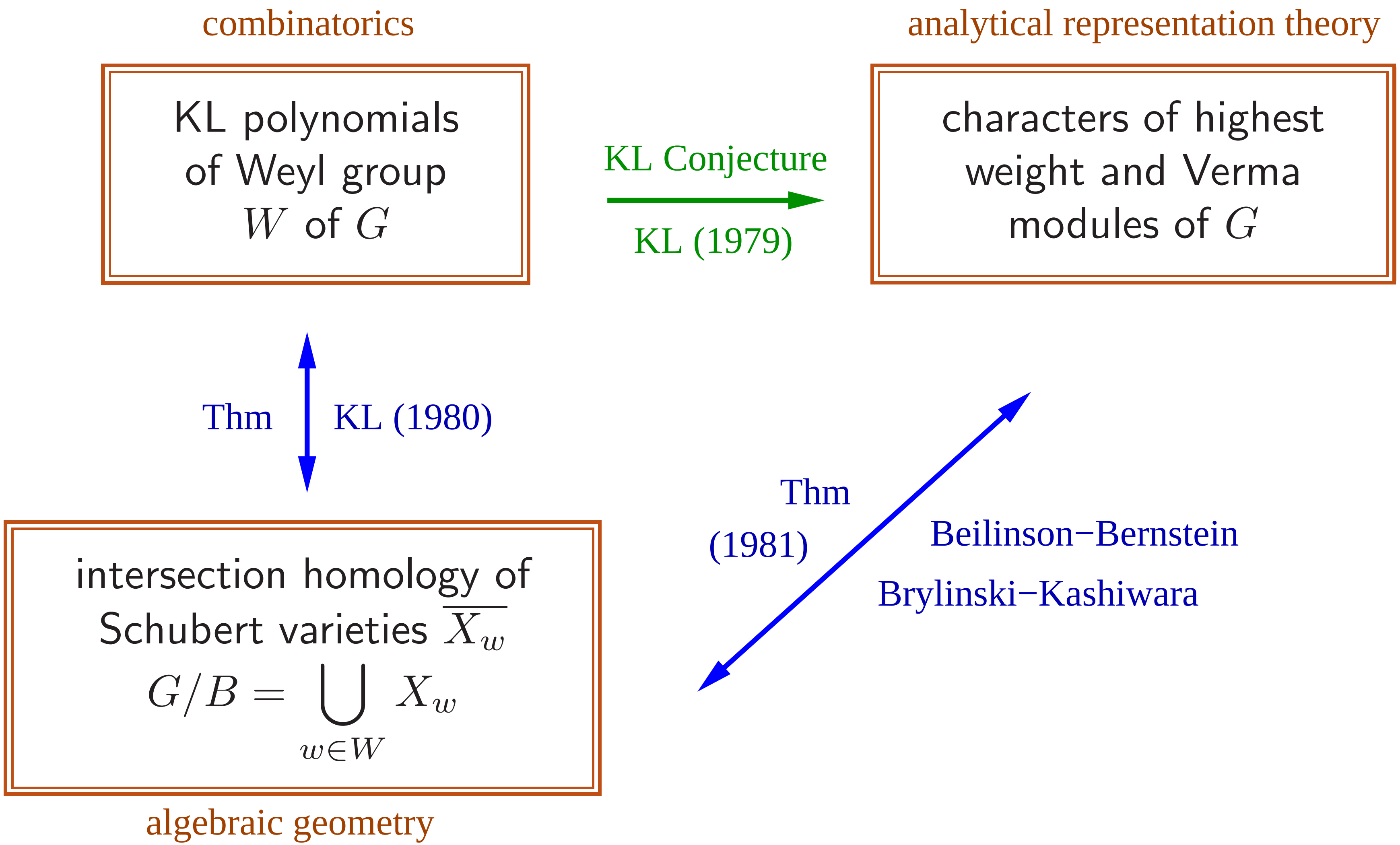
Thm KL (1980)

intersection homology of
Schubert varieties $\overline{X_w}$

$$G/B = \bigcup_{w \in W} X_w$$

algebraic geometry

The meaning of KL polynomials



1983: extension to representations of real simple Lie groups (L-Vogan)

The 'Atlas of Lie groups and representations' Project

Ultimate goal: website with information on complex & real semisimple Lie groups; in particular, their infinite-dimensional unitary representations in code.

2002: Started by J. Adams, now a team of 18 mathematicians (including F. du Cloux, M. van Leeuwen, D. Vogan)

Nov. 2005: KL polynomials for all real forms of F_4, E_6, E_7 and the non-split form E_8^r of E_8: holds in a $73\,410^2$ triangular integer matrix.

For E_8^*: character table holds in a $453\,060^2$ triangular integer matrix (eval. at 1 of KL polynomials).

Trick: compute KL polynomials $\bmod m$ for $m = 253, 255, 256$, then use Chinese Remainder Theorem to reconstruct answer $\bmod 253 \cdot 255 \cdot 256 = 16\,515\,840 \longrightarrow$ saves memory!

. . . . . .

Monday Januar 8, 2007: Result for E_8^* was written to disk (60 GB) by 'sage', a computer at the University of Seattle.

Summary

Mathematical ‘monsters’ like E_8 are in many senses similar to the monsters of your childhood:

- they are frightening, at least at the beginning,
- they are nevertheless exciting & fascinating,
- they do not really exist if you think it over seriously.

Hence, there are two types of monster stories:

- the excellent ones involving great plots and heroic efforts,
- the ‘Loch Ness’ type fairy tales that you should not believe in.

I.A.