Philipps Universität Marburg

Non-integrable geometries, torsion, and holonomy II a): Geometric structures and connections

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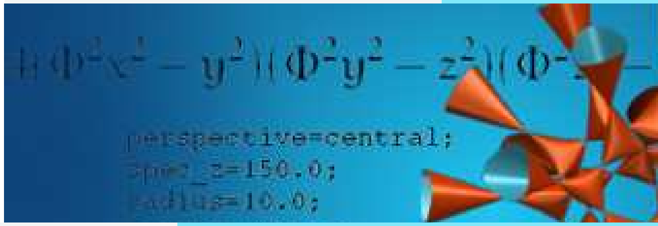


Torino, Carnival Differential Geometry school

General philosophy:

Given a mnfd M^n with G-structure ($G \subset \mathrm{SO}(n)$), replace ∇^g by a *metric connection ∇ with skew torsion that preserves the geometric structure!*

$$\text{torsion:} \quad T(X,Y,Z) \;:=\; g(\nabla_X Y - \nabla_Y X - [X,Y], Z)$$

Special case: require $T \in \Lambda^3(M^n)$ ($\Leftrightarrow$ same geodesics as ∇^g)

$$\Rightarrow \quad g(\nabla_X Y, Z) \;=\; g(\nabla^g_X Y, Z) + \tfrac{1}{2} T(X,Y,Z)$$

1) representation theory yields

- a clear answer *which* G-structures admit such a connection; if existent, it's unique and called the *'characteristic connection'*

- a *classification scheme* for G-structures with characteristic connection: $T_x \in \Lambda^3(T_x M) \stackrel{G}{=} V_1 \oplus \ldots \oplus V_p$

2) Analytic tool: Dirac operator $\not{D}$ of the metric connection with torsion $T/3$: *'characteristic Dirac operator'* (generalizes the Dolbeault operator)

In this lecture:

1) Algebra of 3-forms, and in particular, a ‘Skew Holonomy Theorem’

2) Characteristic connections: Existence, examples, uniqueness

3) An important class of examples: Naturally reductive homogeneous spaces

Algebraic Torsion Forms in $\mathbb{R}^n$

Consider $T \in \Lambda^3(\mathbb{R}^n)$, an algebraic 3-form in $\mathbb{R}^n =: V$, fix a positive def. scalar product $\langle -, - \rangle$ on V.

- T defines a metric connection: $\nabla_X Y := \nabla^g_X Y + \frac{1}{2} T(X, Y, -)$.
- ∇ lifts to a connection on spinor fields $\psi : \mathbb{R}^n \longrightarrow \Delta_n$,

$$\nabla_X \psi := \nabla^g_X \psi + \frac{1}{4}(X \lrcorner T) \cdot \psi$$

Dfn. For T 3-form, define [introduced in AFr, 2004]

- kernel: $\ker T = \{X \in R^n \,|\, X \lrcorner T = 0\}$ (for later)
- Lie algebra generated by its image: $\mathfrak{g}_T := \mathrm{Lie}\langle X \lrcorner T \,|\, X \in \mathbb{R}^n \rangle$

isotropy Lie algebra : $\mathfrak{h}_T := \{A \in \mathfrak{gl}(n, \mathbb{R}) \mid A^* T = 0\}$

$\mathfrak{g}_T$ is *not* related in any obvious way to $\mathfrak{h}_T$!

Examples:

- $n = 3, 4$, $T = e_{123}$: then $e_i \lrcorner T = e_{23}, -e_{13}, e_{12}$, so $\mathfrak{g}_T = \mathfrak{so}(3)$, and $\mathfrak{h}_T = \mathfrak{so}(3)$.

- $n = 5$: $T = \varrho e_{125} + \lambda_{345} \neq 0$, then
  * $\varrho\lambda = 0$: $\mathfrak{g}_T = \mathfrak{so}(3)$, $\mathfrak{h}_T = \mathfrak{so}(3) \oplus \mathfrak{so}(2)$
  * $\varrho\lambda \neq 0$: $\mathfrak{g}_T = \mathfrak{so}(5)$, $\mathfrak{h}_T = \mathfrak{so}(2) \oplus \mathfrak{so}(2)$ (if $\varrho \neq \lambda$), else $\mathfrak{h}_T = \mathfrak{u}(2)$.

- $n = 7$, $= e_{127} + e_{135} - e_{146} - e_{236} - e_{245} + e_{347} + e_{567}$ a 3-form with stabilizer G_2, i. e. $\mathfrak{h}_T = \mathfrak{g}_2$. Moreover, $\mathfrak{so}(7) \stackrel{G_2}{=} \mathfrak{g}_2 \oplus \mathfrak{m}$, where $\mathfrak{m}$ is the space of all inner products $X \lrcorner \mathrm{T}$. The Lie algebra generated by these elements is isomorphic to $\mathfrak{so}(7) = \mathfrak{g}_T$.

- $\mathfrak{g}$ a compact, semisimple Lie algebra acting on itself $\mathfrak{g} \cong \mathbb{R}^n$ by the adjoint rep., β its Killing form, $T(X, Y, Z) := \beta([X, Y], Z)$. Then $\mathfrak{g}_T = \mathfrak{g}$.

Observe: $\mathfrak{g}_T$ does not always act irreducibly on $V = \mathbb{R}^n$.

Thm. The representation $(\mathfrak{g}_T, V)$ is reducible iff there exists a proper subspace $W \subset \mathbb{R}^n$ and two 3-forms $T_1 \in \Lambda^3(W)$ and $T_2 \in \Lambda^3(W^\perp)$ such that $T = T_1 + T_2$. In this case, $\mathfrak{g}_T = \mathfrak{g}_{T_1} \oplus \mathfrak{g}_{T_2}$.

Proof. Consider a $\mathfrak{g}_T$-invariant subspace W, fix bases $e_1, \cdots, e_k$ of W, $e_{k+1}, \cdots, e_n$ of $W^\perp$. Then $\forall X \in \mathbb{R}^n$, $\forall i = 1, \ldots, k$, $\alpha = k+1, \ldots, n$, we obtain $T(X, e_i, e_\alpha) = 0$.

Since T is skew-symmetric, we conclude

$$T(e_i, e_j, e_\alpha) = 0 \quad \text{and} \quad T(e_i, e_\alpha, e_\beta) = 0.$$

□

Next step: In its original version, Berger's holonomy theorem is not suitable for generalization to connections with skew torsion.

Formulate a holonomy theorem in terms of $\mathfrak{g}_T$!

The skew torsion holonomy theorem

Dfn. Let $0 \neq T \in \Lambda^3(V)$, $\mathfrak{g}_T$ as before, $G_T \subset \mathrm{SO}(n)$ its Lie group. Hence, $X \lrcorner T \in \mathfrak{g}_T \subset \mathfrak{so}(V) \cong \Lambda^2(V)\ \forall\ X \in V$. Then (G_T, V, T) is called a *skew-torsion holonomy system (STHS)*. It is said to be

- *irreducible* if G_T acts irreducibly on V,
- *transitive* if G_T acts transitively on the unit sphere of V,
- and *symmetric* if T is G_T-invariant.

Recall: The only transitive sphere actions are:

$\mathrm{SO}(n)$ on $S^{n-1} \subset \mathbb{R}^n$, $\mathrm{SU}(n)$ on $S^{2n-1} \subset \mathbb{C}^n$, $\mathrm{Sp}(n)$ on $S^{4n-1} \subset \mathbb{H}^n$, G_2 on S^6, $\mathrm{Spin}(7)$ on S^7, $\mathrm{Spin}(9)$ on S^{15}. [Montgomery-Samelson, 1943]

Thm (STHT). Let (G_T, V, T) be an irreducible STHS. If it is transitive, $G_T = \mathrm{SO}(n)$. If it is not transitive, it is symmetric, and

- V is a simple Lie algebra of rank ≥ 2 w. r. t. the bracket $[X, Y] = T(X, Y)$, and G_T acts on V by its adjoint representation,
- T is unique up to a scalar multiple.

[transitive: AFr 2004, general: Olmos-Reggiani, 2012; Nagy 2013]

The newer proofs are based on general holonomy theory. The statement about transitive actions is easily verified, for example:

Thm. Let $T \in \Lambda^3(\mathbb{R}^{2n})$ be a 3-form s. t. there exists a 2-form Ω such that

$$\Omega^n \neq 0 \ \text{ and } [\mathfrak{g}_T, \Omega] = 0.$$

Then T is zero, $T = 0$.

Sketch of Proof. Fix an ONB in $\mathbb{R}^{2n}$ s. t. Ω is given by

$$\Omega = A_1\, e_{12} + \ldots + A_k\, e_{2n-1,2n}, \quad A_1 \cdot \ldots \cdot A_k \neq 0.$$

The condition $[\hat{\mathfrak{g}}_{\mathrm{T}}, \Omega] = 0$ is equivalent to $\sum_{j=1}^{2n} \Omega_{\alpha j} \cdot \mathrm{T}_{\beta j \gamma} = \sum_{j=1}^{2n} \mathrm{T}_{\beta \alpha j} \cdot \Omega_{j\gamma}$ for any $1 \leq \alpha, \beta, \gamma \leq 2n$. Using the special form of Ω we obtain the equations $(1 \leq \alpha, \gamma \leq k)$:

$$A_\alpha \cdot \mathrm{T}_{\beta,2\alpha,2\gamma-1} = -A_\gamma \cdot \mathrm{T}_{\beta,2\alpha-1,2\gamma}, \quad A_\alpha \cdot \mathrm{T}_{\beta,2\alpha-1,2\gamma-1} = A_\gamma \cdot \mathrm{T}_{\beta,2\alpha,2\gamma}.$$

This system of algebraic equations implies that $\mathrm{T} = 0$. $\square$

Want to apply this to *existence of characteristic connections!*

The characteristic connection of a geometric structure

Fix $G \subset \mathrm{SO}(n)$, $\Lambda^2(\mathbb{R}^n) \cong \mathfrak{so}(n) = \mathfrak{g} \oplus \mathfrak{m}$, $\mathcal{F}(M^n)$: frame bundle of (M^n, g).

Dfn. A geometric G-structure on M^n is a G-PFB $\mathcal{R}$ which is subbundle of $\mathcal{F}(M^n)$: $\mathcal{R} \subset \mathcal{F}(M^n)$.

Choose a G-adapted local ONF $e_1, \ldots, e_n$ in $\mathcal{R}$ and define *connection 1-forms of* ∇^g*:*

$$\omega_{ij}(X) := g(\nabla^g_X e_i, e_j), \quad g(e_i, e_j) = \delta_{ij} \ \Rightarrow \ \omega_{ij} + \omega_{ji} = 0.$$

Define a skew symmetric matrix Ω with values in $\Lambda^1(\mathbb{R}^n) \cong \mathbb{R}^n$ by $\Omega(X) := \big(\omega_{ij}(X)\big) \in \mathfrak{so}(n) = \mathfrak{g} \oplus \mathfrak{m}$ und set

$$\Gamma := \mathrm{pr}_{\mathfrak{m}}(\Omega).$$

- Γ is a 1-Form on M^n with values in $\mathfrak{m}$, $\Gamma_x \in \mathbb{R}^n \otimes \mathfrak{m}$ $(x \in M^n)$ [“intrinsic torsion”, Swann/Salamon]

Fact: $\Gamma = 0 \Leftrightarrow \nabla^g$ is a G-connection $\Leftrightarrow \mathrm{Hol}(\nabla^g) \subset G$

Via Γ, geometric G-structures $\mathcal{R} \subset \mathcal{F}(M^n)$ correspond to irreducible components of the G-representation $\mathbb{R}^n \otimes \mathfrak{m}$.

Thm. A geometric G-structure $\mathcal{R} \subset \mathcal{F}(M^n)$ admits a metric G-connection with antisymmetric torsion iff Γ lies in the image of Θ,

$$\Theta : \Lambda^3(M^n) \to T^*(M^n) \otimes \mathfrak{m}, \quad \Theta(T) \ := \ \sum_{i=1}^n e_i \otimes \mathrm{pr}_{\mathfrak{m}}(e_i \lrcorner T).$$

[Fr, 2003]

If such a connection exists, it is called the *characteristic connection* ∇^c $\to$ replace the (unadapted) LC connection by ∇^c.

Thm. If $G \not\subset \mathrm{SO}(n)$ acts irreducibly and not by its adjoint rep. on $\mathbb{R}^n \cong T_pM^n$, then $\ker\Theta = \{0\}$, and hence the characteristic connection of a G-structure on a Riemannian manifold (M^n, g) is, if existent, unique.

[A-Fr-Höll, 2013]

Uniqueness of characteristic connections

This is a consequence of the STHT:

Proof. $T \in \ker\Theta$ iff all $X \lrcorner T \in \mathfrak{g} \subset \mathfrak{so}(n)$, that is,

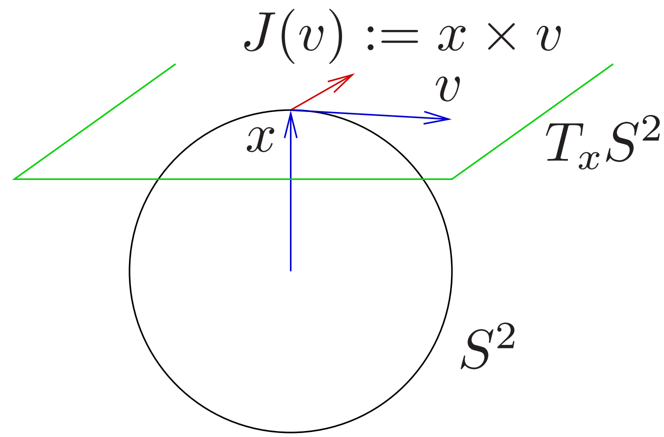
$$\ker\Theta = \{T \in \Lambda^3(\mathbb{R}^n) \,|\, \mathfrak{g}_T \subset \mathfrak{g}\},$$

so $(T, G, \mathbb{R}^n)$ defines an irreducible STHS, which by assumption is non transitive (because $G \not\subset \mathrm{SO}(n)$). By the STHT, it has to be a Lie algebra with the adjoint representation. Since this was excluded as well, it follows that $\ker\Theta = \{0\}$. □

For many G-structures, uniqueness can be proved directly case by case – including a few cases where the G-action is not irreducible.

$\mathrm{U}(n)$ structures in dimension $2n$

for $S^2 \subset \mathbb{R}^3$:



- (S^6, g_{can}): $S^6 \subset \mathbb{R}^7$ has an almost complex structure J ($J^2 = -\mathrm{id}$) inherited from "cross product" on $\mathbb{R}^7$.
- J is not integrable, $\nabla^g J \neq 0$
- **Problem (Hopf):** Does S^6 admit an (integrable) complex structure ?

J is an example of a nearly Kähler structure: $\nabla^g_X J(X) = 0$

More generally: (M^{2n}, g, J) almost Hermitian mnfd: J almost complex structure, g a compatible Riemannian metric.

Fact: structure group $G \subset \mathrm{U}(n) \subset \mathrm{SO}(2n)$, but $\mathrm{Hol}_0(\nabla^g) = \mathrm{SO}(2n)$.

Examples: twistor spaces $(\mathbb{CP}^3, F_{1,2})$ with their nK str., compact complex mnfd with $b_1(M)$ odd ($\nexists$ Kähler metric) . . .

Thm. An almost hermitian manifold (M^{2n}, g, J) admits a characteristic connection ∇ if and only if the Nijenhuis tensor

$$N(X,Y,Z) \;:=\; g\big(N(X,Y),\, Z\big)$$

is skew-symmetric. Its torsion is then

$$T(X,Y,Z) \;= -d\Omega(JX,JY,JZ) + N(X,Y,Z)$$

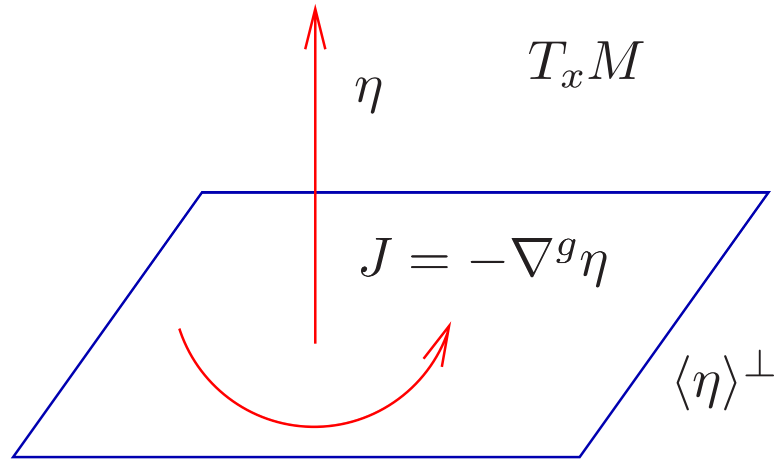
and it satisfies: $\nabla\Omega = 0$, $\mathrm{Hol}(\nabla) \subset \mathrm{U}(n)$. [Fr-Ivanov, 2002]

'Trivial case': If (M^{2n}, g, J) is Kähler ($N = 0$ and $d\Omega = 0$), then $T = 0$, the LC connection ∇^g *is* the characteristic connection.

In particular for $n = 3$: [Gray-Hervella]

- $\mathfrak{so}(6) = \mathfrak{u}(3) \oplus \mathfrak{m}^6$, $\Gamma \in \mathbb{R}^6 \otimes \mathfrak{m}^6\big|_{\mathrm{U}(3)} \cong W_1^2 \oplus W_2^{16} \oplus W_3^{12} \oplus W_4^6$
- N is skew-symmetric $\Leftrightarrow$ Γ has no W_2-part
- $\Gamma \in W_1$: nearly Kähler manifolds $(S^6, S^3 \times S^3, F(1,2), \mathbb{CP}^3)$
- $\Gamma \in W_3 \oplus W_4$: hermitian manifolds $(N = 0)$

Contact structures



• (M^{2n+1}, g, η) contact mnfd, η: 1-form ($\cong$ vector field)

• $\langle\eta\rangle^\perp$ admits an almost complex structure J compatible with g

• Contact condition: $\eta \wedge (d\eta)^n \neq 0 \Rightarrow \nabla^g \eta \neq 0$, i. e. contact structures are never integrable ! (no analogue on Berger's list)

• structure group: $G \subset \mathrm{U}(n) \subset \mathrm{SO}(2n+1)$

Examples: $S^{2n+1} = \frac{\mathrm{SU}(n+1)}{\mathrm{SU}(n)}$, $V_{4,2} = \frac{\mathrm{SO}(4)}{\mathrm{SO}(2)}$, $M^{11} = \frac{G_2}{\mathrm{Sp}(1)}$, $M^{31} = \frac{F_4}{\mathrm{Sp}(3)}$

Thm. An almost metric contact manifold (M^{2n+1}, g, η) admits a connection ∇ with skew-symmetric torsion and preserving the structure if and only if ξ is a Killing vector field and the tensor $N(X, Y, Z) := g(N(X, Y), Z)$ is totally skew-symmetric. In this case, the connection is unique, and its torsion form is given by the formula

$$T = \eta \wedge d\eta + d^\phi F + N - \eta \wedge \xi \lrcorner N.$$

[Fr-Ivanov, 2002]

A large class of almost metric contact manifolds thus admits a char. connection ∇, and for these: $\mathrm{Hol}_0(\nabla) \subset \mathrm{U}(n) \subset \mathrm{SO}(2n+1)$.

A special class: Sasaki manifolds: Riemannian manifolds (M^{2n+1}, g) equipped with a contact form η, its dual vector field ξ and an endomorphism $\varphi : TM^7 \to TM^7$ s. t.:

- $\eta \wedge (d\eta)^n \neq 0, \quad \eta(\xi) \;=\; 1, \quad g(\xi,\xi) \;=\; 1$
- $g(\varphi X, \varphi Y) = g(X,Y)$ and $\varphi^2 = -\mathrm{Id}$ on $\langle\eta\rangle^\perp$,
- $\nabla^g_X \xi = -\varphi X,\ (\nabla^g_X \varphi)(Y) = g(X,Y)\cdot\xi - \eta(Y)\cdot X.$

For Sasaki manifolds, the formula is particularly simple,

$$g(\nabla^c_X Y, Z) \;=\; g(\nabla^g_X Y, Z) + \tfrac{1}{2}\eta \wedge d\eta(X,Y,Z),$$

and $\nabla T = 0$ holds. [Kowalski-Wegrzynowski, 1987]

G_2 structures in dimension 7

Fix $G_2 \subset \mathrm{SO}(7)$, $\mathfrak{so}(7) = \mathfrak{g}_2 \oplus \mathfrak{m}^7 \cong \mathfrak{g}_2 \oplus \mathbb{R}^7$.

Intrinsic torsion Γ lies in $\mathbb{R}^7 \otimes \mathfrak{m}^7 \cong \mathbb{R}^1 \oplus \mathfrak{g}_2 \oplus \mathrm{S}_0(\mathbb{R}^7) \oplus \mathbb{R}^7 =: \bigoplus_{i=1}^4 \mathcal{X}_i$

$\Rightarrow$ four classes of geometric G_2 structures [Fernandez-Gray, '82]

- Decomposition of 3-forms: $\Lambda^3(\mathbb{R}^7) = \mathbb{R}^1 \oplus \mathrm{S}_0(\mathbb{R}^7) \oplus \mathbb{R}^7$.

G_2 is the isotropy group of a generic element of $\omega \in \Lambda^3(\mathbb{R}^7)$:

$$G_2 \;=\; \{A \in \mathrm{SO}(7) \mid A \cdot \omega = \omega\}.$$

Thm. A 7-dimensional Riemannian mfd (M^7, g, ω) with a fixed G_2 structure $\omega \in \Lambda^3(M^7)$ admits a characteristic connection ∇

$\Leftrightarrow$ the $\mathfrak{g}_2$ component of Γ vanishes

$\Leftrightarrow$ There exists a VF β with $\delta\omega = -\beta \lrcorner \omega$

The torsion of ∇ is then $T = -* d\omega - \frac{1}{6}(d\omega, *\omega)\omega + *(\beta \wedge \omega)$, and ∇ admits (at least) one parallel spinor. [Fr-Ivanov, 2002]

name	class	characterization
parallel G_2-manifold	$\{0\}$	a) $\nabla^g \omega = 0$ b) $\exists$ a ∇^g-parallel spinor
nearly parallel G_2-manifold	$\mathcal{X}_1$	a) $d\omega = \lambda * \omega$ for some $\lambda \in \mathbb{R}$ b) $\exists$ real Killing spinor
almost parallel or closed (or calibrated symplectic) G_2-m.	$\mathcal{X}_2$	$d\omega = 0$
balanced G_2-manifold	$\mathcal{X}_3$	$\delta\omega = 0$ and $d\omega \wedge \omega = 0$
locally conformally parallel G_2-m.	$\mathcal{X}_4$	$d\omega = \frac{3}{4}\theta \wedge \omega$ and $d * \omega = \theta \wedge *\omega$ for some θ
cocalibrated (or semi-parallel or cosymplectic) G_2-manifold	$\mathcal{X}_1 \oplus \mathcal{X}_3$	$\delta\omega = 0$
locally conformally (almost) parallel G_2-manifold	$\mathcal{X}_2 \oplus \mathcal{X}_4$	$d\omega = \frac{3}{4}\theta \wedge \omega$
G_2T-manifold	$\mathcal{X}_1 \oplus \mathcal{X}_3 \oplus \mathcal{X}_4$	a) $d * \omega = \theta \wedge *\omega$ for some θ b) $\exists$ char. connection ∇^c

Easiest examples:

- $S^7 = \frac{\mathrm{Spin}(7)}{G_2}$, $M_{k,l}^{AW} = \frac{\mathrm{SU}(3)}{\mathrm{U}(1)_{k,l}}$, $V_{5,2} = \frac{\mathrm{SO}(5)}{\mathrm{SO}(3)}$, ...
- Explicit constructions of G_2 structures:

[Friedrich-Kath, Fernandez-Gray, Fernandez-Ugarte, Aloff-Wallach, Boyer-Galicki. . .]

- Every orientable hypersurface in $\mathbb{R}^8$ carries a cocalibrated G_2-structure
- S^1-PFB over 6-dim. Kähler manifolds, nearly Kähler manifolds. . .

Example

[Fernandez-Ugarte, '98]

N^6: 3-dimensional complex solvable group, $M^7 := N^6 \times \mathbb{R}^1$. There exists a left invariant metric and a left invariant G_2-structure on M^7 such that the structural equations are:

$$de_3 = e_{13} - e_{24}, \quad de_4 = e_{23} + e_{14}, \quad de_5 = -e_{15} + e_{26}, \quad de_6 = -e_{25} - e_{16},$$

all other $de_i = 0$.

M^7 has a G_2-invariant characteristic connection ∇^c and

- $T = 2 \cdot e_{256} - 2 \cdot e_{234}, \quad \delta(T) = 0.$
- $\mathrm{Scal}^c = -16.$
- There are two ∇^c-parallel spinors, and both satisfy $T^c \cdot \Psi = 0$.

An interesting subclass of G_2-mnfds: 7-dim. 3-Sasaki mnfds

M^7: 3-Sasaki mnfd, corresponds to $\mathrm{SU}(2) \subset G_2 \subset \mathrm{SO}(7)$.

- 3 orth. Sasaki structures $\eta_i \in T^*M^7$, $[\eta_1, \eta_2] = 2\,\eta_3$, $[\eta_2, \eta_3] = 2\,\eta_1$, $[\eta_3, \eta_1] = 2\,\eta_2$ and $\varphi_3 \circ \varphi_2 = -\varphi_1$ etc. on $\langle \eta_2, \eta_3 \rangle^\perp$
- Known: A 3-Sasaki mnfd is always Einstein and has 3 Riemannian Killing spinors, define $T^v := \langle \xi_1, \xi_2 \xi_3 \rangle$, $T^h = (T^v)^\perp$
- each Sasaki structures η_i induces a characteristic connection ∇^i, but $\nabla^1 \neq \nabla^2 \neq \nabla^3$?!? $\Rightarrow$ Ansatz: $\quad T = \sum_{i,j=1}^{3} \alpha_{ij}\, \eta_i \wedge d\eta_j + \gamma\, \eta_1 \wedge \eta_2 \wedge \eta_3$

Thm. Every 7-dimensional 3-Sasaki mnfd admits a $\mathbb{P}^2$-family of metric connections with skew torsion and parallel spinors. Its holonomy is G_2.

[A-Fr, 2003]

Thm. There exists a cocalibrated G_2-structure with char. connection ∇^c with parallel spinor ψ on M^7 with the properties:

- ∇^c preserves T^v and T^h, and $\nabla^c T = 0$
- $\xi_i \cdot \psi$ are the 3 Riemannian Killing spinors on M^7 [A-Fr, 2010]

Example: Naturally reductive spaces

• Homogeneous *non symmetric* spaces provide a rich source for manifolds with characteristic connection

Let $M = G/H$ be reductive, i. e. $\exists\, \mathfrak{m} \subset \mathfrak{g}$ s. t. $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ and $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$; isotropy repr. $\mathrm{Ad} : H \to \mathrm{SO}(\mathfrak{m})$. $\langle\, ,\, \rangle$ a pos. def. scalar product on $\mathfrak{m}$.

The PFB $G \to G/H$ induces a distinguished connection on G/H, the so-called *canonical connection* ∇^1. Its torsion is

$$T^1(X, Y, Z) \;=\; -\langle [X, Y]_{\mathfrak{m}}, Z\rangle \qquad (= 0 \text{ for } M \text{ symmetric})$$

Dfn. The metric $\langle\, ,\, \rangle$ is called *naturally reductive* if T^1 defines a 3-form,

$$\langle [X, Y]_{\mathfrak{m}}, Z\rangle + \langle Y, [X, Z]_{\mathfrak{m}}\rangle \;=\; 0 \text{ for all } X, Y, Z \in \mathfrak{m}\,.$$

They generalize symmetric spaces: $\nabla^1 T^1 = 0, \nabla^1 \mathcal{R}^1 = 0$.

Naturally reductive spaces – basic facts

Thm. A Riemannian manifold equipped with a [regular] homogeneous structure, i. e. a metric connection ∇ with torsion T and curvature $\mathcal{R}$ such that $\nabla\mathcal{R} = 0$ and $\nabla T = 0$, is locally isometric to a homogeneous space. [Ambrose-Singer, 1958, Tricerri 1993]

Hence: Naturally reductive spaces have a metric connection ∇ with skew torsion such that $\nabla T = \nabla\mathcal{R} = 0$

N.B. Well-known: Some mnfds carry several nat.red.structures, for exa.

$$S^{2n+1} = \mathrm{SO}(2n+2)/\mathrm{SO}(2n+1) = \mathrm{SU}(n+1)/\mathrm{SU}(n),$$

$$S^6 \;=\; G_2/\mathrm{SU}(3),\;\; S^7 \;=\; \mathrm{Spin}(7)/G_2,\;\; S^{15} = \mathrm{Spin}(9)/\mathrm{Spin}(7).$$

But, another consequence of the STHT:

Thm. If (M, g) is not loc. isometric to a sphere or a Lie group, then its admits at most one naturally reductive homogeneous structure.

[Olmos-Reggiani, 2012]

Classical construction of naturally reductive spaces

General construction:
Consider $M = G/H$ with restriction of the Killing form to $\mathfrak{m}$:

$$\beta(X,Y) := -\mathrm{tr}(X^tY),\ \langle X,Y\rangle = \beta(X,Y) \text{ for } X,Y \in \mathfrak{m}.$$

Suppose that $\mathfrak{m}$ is an orthogonal sum $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2$ such that

$$[\mathfrak{h}, \mathfrak{m}_2] = 0,\ [\mathfrak{m}_2, \mathfrak{m}_2] \subset \mathfrak{m}_2\,.$$

Then the new metric, depending on a parameter $s > 0$

$$\tilde{\beta}_s = \beta\big|_{\mathfrak{m}_1} \oplus s \cdot \beta\big|_{\mathfrak{m}_2}$$

is naturally reductive for $s \neq 1$ w. r. t. the realisation as

$$M = (G \times M_2)/(H \times M_2) =: \overline{G}/\overline{H}\,.$$

[Chavel, 1969; Ziller / D'Atri, 1979]

This description gets quickly rather tedious – thus, we shall usually describe nat. reductive spaces through their connections with parallel torsion and curvature.

Example: Lie groups

Let $M = G$ be a connected Lie group, $\mathfrak{g} = T_e G$.

the metric g on G is biinvariant

$\Leftrightarrow L_a,\ R_a$ are isometries $\forall a \in G$

$\Rightarrow \mathrm{ad} V \in \mathfrak{so}(\mathfrak{g})$, i. e. $g(\mathrm{ad}(V)X, Y) + g(X, \mathrm{ad}(V)Y) = 0 \qquad (*)$

Easy: $\nabla^g_X Y = \frac{1}{2}[X, Y]$.

Ansatz: T proportional to $[,]$, i. e. $\nabla_X Y = \lambda[X, Y]$

- torsion: $T^\nabla(X, Y) = (2\lambda - 1)[X, Y]$, hence $T \in \Lambda^3(G) \Leftrightarrow (*)$
- curvature:

$$\mathcal{R}^\nabla(X, Y)Z = \lambda(1-\lambda)[Z, [X, Y]] = \begin{cases} \frac{1}{4}[Z, [X, Y]] & \text{for LC conn.} (\lambda = \frac{1}{2}) \\ 0 & \text{for } \lambda = 0, 1 \end{cases}$$

[±-connection, Cartan-Schouten, 1926]

Interpretation in the framework of homogeneous spaces

Take $\tilde{G} = G \times G$ with involution $\theta(a,b) = (b,a)$.

- $K := \tilde{G}^\theta = \{(a,a) \in \tilde{G}\} = \Delta G$ with Lie alg. $\mathfrak{k} = \{(X,X)|X \in \mathfrak{g}\} \subset \tilde{\mathfrak{g}}$

To make $\tilde{G}/\Delta G$ symmetric, one usually chooses as complement of $\mathfrak{k}$ in $\mathfrak{g}$

$$\mathfrak{m}_{\rm sym} := \{(X,-X)|X \in \mathfrak{g}\},$$

for it satisfies $[\mathfrak{m}_{\rm sym}, \mathfrak{m}_{\rm sym}] \subset \mathfrak{k}$. But every space

$$\mathfrak{m}_t := \{X_t := \big(tX,(t-1)X\big)|X \in \mathfrak{g}\}, \quad t \in \mathbb{R},$$

also defines a reductive complement, $[\mathfrak{k}, \mathfrak{m}_t] \subset \mathfrak{m}_t$.

Fact: Every reductive homogeneous space has a canonical connection ∇^c induced from the PFB $\tilde{G} \to \tilde{G}/\Delta G$ (the ∇^c-parallel tensors are exactly the $\tilde{G}$-invariant ones), $[\,,\,] = [\,,\,]_{\mathfrak{k}} + [\,,\,]_{\mathfrak{m}}$

$$\nabla^c T^c = 0, \quad T^c(X,Y) = -[X,Y]_{\mathfrak{m}},$$
$$\nabla^c \mathcal{R}^c = 0, \quad \mathcal{R}^c(X,Y) = -[[X,Y]_{\mathfrak{k}}, Z].$$

This turns G into a naturally reductive space.

One checks for $X_t = \big(tX, (t-1)X\big),\ Y_t = \big(tY, (t-1)Y\big) \in \mathfrak{m}_t$

$$[X_t, Y_t] = (t^2[X,Y], (t-1)^2[X,Y]) : \text{ hence } " \in \mathfrak{m}_t'' \ \Leftrightarrow\ t = 0,1$$

i. e. $\mathcal{R}^c = 0$ for $t = 0,1$ – these are again the $\pm$-connections of Cartan-Schouten.

In particular, $\nabla T = 0$ for these connections on Lie groups.

Connections on homogeneous spaces – Wang's Theorem

Wang's Thm. [see Kobayashi-Nomizu]

Let $M^n = G/H$ be reductive, $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$. Then there is a bijection between $\mathrm{GL}(n)$-invariant connections ∇ on M^n and maps $\Lambda : \mathfrak{m} \to \mathfrak{gl}(n)$ satisfying

$$\Lambda(\mathrm{Ad}h)X = \mathrm{Ad}(h)\Lambda(X)\mathrm{Ad}(h)^{-1} \text{ for all } h \in H \text{ and } X \in \mathfrak{m}.$$

[Idea: Λ is the evaluation of the connection form at eH]

Comments:

- If $\Lambda : \mathfrak{m} \to \mathfrak{so}(n)$, the corresponding connection is metric
- $\Lambda = 0$ is a solution, corresponds to the canonical connection
- Torsion: $T(X,Y) = \Lambda(X)Y - \Lambda(Y)X - [X,Y]_{\mathfrak{m}}$
- Curvature: $\mathcal{R}(X,Y)Z = \Lambda(X)\Lambda(Y)Z - \Lambda(Y)\Lambda(X)Z - \mathrm{Ad}([X,Y]_{\mathfrak{h}})Z$

Example: The Berger sphere $M^5 = \mathrm{SU}(3)/\mathrm{SU}(2)$

$$\mathfrak{su}(3) \subset \mathcal{M}_3(\mathbb{C}), \mathfrak{su}(2) \cong \left\{ \begin{bmatrix} 0 & 0 \\ 0 & B \end{bmatrix} : B \in \mathfrak{su}(2) \right\}, \mathfrak{m}_0 := \left\{ \begin{bmatrix} 0 & -\bar{v}^t \\ v & 0 \end{bmatrix} : v \in \mathbb{C}^2 \right\}$$

Hence, we get a reductive decomposition

$$\mathfrak{su}(3) = \mathfrak{su}(2) \oplus \mathfrak{m}, \ \ \mathfrak{m} = \mathfrak{m}_0 \oplus \langle \eta \rangle \ \ \text{with } \eta = \frac{1}{\sqrt{3}} \mathrm{diag}(-2i, i, i)$$

Basis of $\mathfrak{m}_0$: $e_1, \ldots e_4$ corresponding to $v = (1,0),\ (i,0),\ (0,1),\ (0,i)$. Deform the Killing form $\beta(X,Y) = -\mathrm{tr}(XY)/2$ of $\mathfrak{su}(3)$ on $\mathfrak{m}$ to the family of metrics

$$g_\gamma \ := \ \beta\big|_{\mathfrak{m}_0} \oplus \frac{1}{\gamma}\beta\big|_{\langle \eta \rangle}, \quad \gamma > 0.$$

- $\tilde{\eta} = \eta/\sqrt{\gamma} =: e_5$ and $\varphi := e_{12} + e_{34}$ defines an α-Sasakian on M^5, its characteristic connection is described by $\Lambda : \ \mathfrak{m} \to \mathfrak{so}(\mathfrak{m})$

$$\Lambda(e_i) = 0 \text{ for } i = 1, \ldots, 4, \ \Lambda(e_5) = (\sqrt{3/\gamma} - \sqrt{3\gamma})(E_{12} + E_{34}).$$

- Torsion $T = \tilde{\eta} \wedge d\tilde{\eta} = \sqrt{3/\gamma}(e_{12} + +e_{34}) \wedge e_5$

Link to Dirac operators

Without torsion:

- Classical Schrödinger-Lichnerowicz formula on Riemannian spin mnfds

- Parthasarathy formula on symmetric spaces: $(D^g)^2 = \Omega + \frac{1}{8}\mathrm{Scal}^g$, where Ω : Casimir operator

With torsion: Assume (M^n, g) is mnfd with G-structure and characteristic connection ∇ with torsion T

$\not{D}$: Dirac operator of connection with torsion $T/3$

- **Generalized SL formula:** [A-Fr, 2003]

$$\not{D}^2 = \Delta_T + \frac{1}{4}\mathrm{Scal}^g + \frac{1}{8}||T||^2 - \frac{1}{4}T^2$$

[$1/3$ rescaling: Slebarski (1987), Bismut (1989), Kostant, Goette (1999), IA (2002)]

- Similarly, $\not{D}^2 = \Omega + \mathrm{const}$ on naturally reductive homogeneous spaces

[Kostant 1999, A 2002]

Almost hermitian manifolds and their Dirac operators

SL:

$$(D^g)^2 = \Delta + \tfrac{1}{4}\mathrm{Scal}^g$$

B/K/A−F/S:

$$\not{D}^2 = \Delta_T + \tfrac{1}{4}\mathrm{Scal}^g + \tfrac{1}{8}||T||^2 - \tfrac{1}{4}T^2$$

non homog.

Kähler mnfds

$D^g =$ Dolbeault op.

almost Herm. mnfds
(nearly/almost/quasi/semi K,
Hermitian, loc.conf.K etc.)

$D^g \neq \not{D} =$ Dolbeault op.

Hermitian symm. spaces

almost Herm. nat. red.
homogeneous spaces

almost Hermitian mnfds

homog.

Parthasarathy:

$$(D^g)^2 = \Omega + \tfrac{1}{8}\mathrm{Scal}^g$$

Kostant:

$$\not{D}^2 = \Omega + \mathrm{const}$$

symmetric

naturally reductive

$T = 0$ (integrable)

$T \neq 0$ (non integrable)

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