Non-integrable geometries, torsion, and holonomy
II a): Geometric structures and connections

Prof. Dr. habil. Ilka Agricola
Philipps-Universität Marburg

Torino, Carnival Differential Geometry school
General philosophy:

Given a mnfd $M^n$ with $G$-structure ($G \subset \text{SO}(n)$), replace $\nabla^g$ by a metric connection $\nabla$ with skew torsion that preserves the geometric structure!

\[
\text{torsion: } T(X, Y, Z) := g(\nabla_X Y - \nabla_Y X - [X, Y], Z)
\]

Special case: require $T \in \Lambda^3(M^n)$ ($\iff$ same geodesics as $\nabla^g$)

\[
\Rightarrow \quad g(\nabla_X Y, Z) = g(\nabla^g_X Y, Z) + \frac{1}{2} T(X, Y, Z)
\]

1) representation theory yields

- a clear answer \textit{which} $G$-structures admit such a connection; if existent, it’s unique and called the ‘\textit{characteristic connection}’

- a \textit{classification scheme} for $G$-structures with characteristic connection:
  \[
  T_x \in \Lambda^3(T_x M) \overset{G}{\cong} V_1 \oplus \ldots \oplus V_p
  \]

2) Analytic tool: Dirac operator $\mathcal{D}$ of the metric connection with torsion $T/3$: ‘\textit{characteristic Dirac operator}’ (generalizes the Dolbeault operator)
In this lecture:

1) Algebra of 3-forms, and in particular, a ‘Skew Holonomy Theorem’

2) Characteristic connections: Existence, examples, uniqueness

3) An important class of examples: Naturally reductive homogeneous spaces
Algebraic Torsion Forms in $\mathbb{R}^n$

Consider $T \in \Lambda^3(\mathbb{R}^n)$, an algebraic 3-form in $\mathbb{R}^n =: V$, fix a positive def. scalar product $\langle -, - \rangle$ on $V$.

- $T$ defines a metric connection: $\nabla_X Y := \nabla^g_X Y + \frac{1}{2} T(X, Y, -)$.

- $\nabla$ lifts to a connection on spinor fields $\psi : \mathbb{R}^n \longrightarrow \Delta_n$,

$$\nabla_X \psi := \nabla^g_X \psi + \frac{1}{4} (X \perp T) \cdot \psi$$

**Dfn.** For $T$ 3-form, define [introduced in AFr, 2004]

- kernel: $\ker T = \{X \in \mathbb{R}^n \mid X \perp T = 0\}$ (for later)
- Lie algebra generated by its image: $\mathfrak{g}_T := \text{Lie} \langle X \perp T \mid X \in \mathbb{R}^n \rangle$

isotropy Lie algebra : $\mathfrak{h}_T := \{A \in \mathfrak{gl}(n, \mathbb{R}) \mid A^* T = 0\}$

$\mathfrak{g}_T$ is not related in any obvious way to $\mathfrak{h}_T$!
Examples:

• \( n = 3, 4, \ T = e_{123} \): then \( e_i \bot T = e_{23}, -e_{13}, e_{12} \), so \( \mathfrak{g}_T = \mathfrak{so}(3) \), and \( \mathfrak{h}_T = \mathfrak{so}(3) \).

• \( n = 5 \): \( T = \varrho e_{125} + \lambda_{345} \neq 0 \), then
  * \( \varrho \lambda = 0 \): \( \mathfrak{g}_T = \mathfrak{so}(3), \ \mathfrak{h}_T = \mathfrak{so}(3) \oplus \mathfrak{so}(2) \)
  * \( \varrho \lambda \neq 0 \): \( \mathfrak{g}_T = \mathfrak{so}(5), \ \mathfrak{h}_T = \mathfrak{so}(2) \oplus \mathfrak{so}(2) \) (if \( \varrho \neq \lambda \)), else \( \mathfrak{h}_T = \mathfrak{u}(2) \).

• \( n = 7 \), \( = e_{127} + e_{135} - e_{146} - e_{236} - e_{245} + e_{347} + e_{567} \) a 3-form with stabilizer \( G_2 \), i.e. \( \mathfrak{h}_T = \mathfrak{g}_2 \). Moreover, \( \mathfrak{so}(7) \cong \mathfrak{g}_2 \oplus \mathfrak{m} \), where \( \mathfrak{m} \) is the space of all inner products \( X \bot T \). The Lie algebra generated by these elements is isomorphic to \( \mathfrak{so}(7) = \mathfrak{g}_T \).

• \( \mathfrak{g} \) a compact, semisimple Lie algebra acting on itself \( \mathfrak{g} \cong \mathbb{R}^n \) by the adjoint rep., \( \beta \) its Killing form, \( T(X, Y, Z) := \beta([X, Y], Z) \). Then \( \mathfrak{g}_T = \mathfrak{g} \).
Observe: \( g_T \) does not always act irreducibly on \( V = \mathbb{R}^n \).

**Thm.** The representation \((g_T, V)\) is reducible iff there exists a proper subspace \( W \subset \mathbb{R}^n \) and two 3-forms \( T_1 \in \Lambda^3(W) \) and \( T_2 \in \Lambda^3(W^\perp) \) such that \( T = T_1 + T_2 \). In this case, \( g_T = g_{T_1} \oplus g_{T_2} \).

**Proof.** Consider a \( g_T \)-invariant subspace \( W \), fix bases \( e_1, \ldots, e_k \) of \( W \), \( e_{k+1}, \ldots, e_n \) of \( W^\perp \). Then \( \forall X \in \mathbb{R}^n, \forall i = 1, \ldots, k, \alpha = k + 1, \ldots, n, \) we obtain \( T(X, e_i, e_\alpha) = 0 \).

Since \( T \) is skew-symmetric, we conclude

\[
T(e_i, e_j, e_\alpha) = 0 \quad \text{and} \quad T(e_i, e_\alpha, e_\beta) = 0.
\]

□

**Next step:** In its original version, Berger’s holonomy theorem is not suitable for generalization to connections with skew torsion.

Formulate a holonomy theorem in terms of \( g_T \)!
The skew torsion holonomy theorem

**Dfn.** Let $0 \neq T \in \Lambda^3(V)$, $\mathfrak{g}_T$ as before, $G_T \subset SO(n)$ its Lie group. Hence, $X \perp T \in \mathfrak{g}_T \subset \mathfrak{so}(V) \cong \Lambda^2(V) \ \forall \ X \in V$. Then $(G_T, V, T)$ is called a *skew-torsion holonomy system (STHS)*. It is said to be

- **irreducible** if $G_T$ acts irreducibly on $V$,
- **transitive** if $G_T$ acts transitively on the unit sphere of $V$,
- and **symmetric** if $T$ is $G_T$-invariant.

**Recall:** The only transitive sphere actions are:

- $SO(n)$ on $S^{n-1} \subset \mathbb{R}^n$,
- $SU(n)$ on $S^{2n-1} \subset \mathbb{C}^n$,
- $Sp(n)$ on $S^{4n-1} \subset \mathbb{H}^n$,
- $G_2$ on $S^6$,
- $Spin(7)$ on $S^7$,
- $Spin(9)$ on $S^{15}$.  

* [Montgomery-Samelson, 1943]

**Thm (STHT).** Let $(G_T, V, T)$ be an irreducible STHS. If it is transitive, $G_T = SO(n)$. If it is not transitive, it is symmetric, and

- $V$ is a simple Lie algebra of rank $\geq 2$ w. r. t. the bracket $[X, Y] = T(X, Y)$, and $G_T$ acts on $V$ by its adjoint representation,
- $T$ is unique up to a scalar multiple.

* [transitive: AFr 2004, general: Olmos-Reggiani, 2012; Nagy 2013]
The newer proofs are based on general holonomy theory. The statement about transitive actions is easily verified, for example:

**Thm.** Let \( T \in \Lambda^3(\mathbb{R}^{2n}) \) be a 3-form s.t. there exists a 2-form \( \Omega \) such that

\[
\Omega^n \neq 0 \quad \text{and} \quad [g_T, \Omega] = 0.
\]

Then \( T \) is zero, \( T = 0 \).

**Sketch of Proof.** Fix an ONB in \( \mathbb{R}^{2n} \) s.t. \( \Omega \) is given by

\[
\Omega = A_1 e_{12} + \ldots + A_k e_{2n-1,2n}, \quad A_1 \cdot \ldots \cdot A_k \neq 0.
\]

The condition \( [\hat{g}_T, \Omega] = 0 \) is equivalent to \( \sum_{j=1}^{2n} \Omega_{\alpha j} \cdot T_{\beta j \gamma} = \sum_{j=1}^{2n} T_{\beta \alpha j} \cdot \Omega_{j \gamma} \) for any \( 1 \leq \alpha, \beta, \gamma \leq 2n \). Using the special form of \( \Omega \) we obtain the equations \((1 \leq \alpha, \gamma \leq k)\):

\[
A_\alpha \cdot T_{\beta,2\alpha,2\gamma-1} = -A_\gamma \cdot T_{\beta,2\alpha-1,2\gamma}, \quad A_\alpha \cdot T_{\beta,2\alpha-1,2\gamma-1} = A_\gamma \cdot T_{\beta,2\alpha,2\gamma}.
\]

This system of algebraic equations implies that \( T = 0 \). \( \square \)

Want to apply this to *existence of characteristic connections!*
The characteristic connection of a geometric structure

Fix $G \subset \text{SO}(n)$, $\Lambda^2(\mathbb{R}^n) \cong \mathfrak{so}(n) = \mathfrak{g} \oplus \mathfrak{m}$, $\mathcal{F}(M^n)$: frame bundle of $(M^n, g)$.

**Dfn.** A geometric $G$-structure on $M^n$ is a $G$-PFB $\mathcal{R}$ which is subbundle of $\mathcal{F}(M^n)$: $\mathcal{R} \subset \mathcal{F}(M^n)$.

Choose a $G$-adapted local ONF $e_1, \ldots, e_n$ in $\mathcal{R}$ and define **connection 1-forms of $\nabla^g$**:

$$
\omega_{ij}(X) := g(\nabla^g_X e_i, e_j), \quad g(e_i, e_j) = \delta_{ij} \implies \omega_{ij} + \omega_{ji} = 0.
$$

Define a skew symmetric matrix $\Omega$ with values in $\Lambda^1(\mathbb{R}^n) \cong \mathbb{R}^n$ by $\Omega(X) := (\omega_{ij}(X)) \in \mathfrak{so}(n) = \mathfrak{g} \oplus \mathfrak{m}$ und set

$$
\Gamma := \text{pr}_\mathfrak{m}(\Omega).
$$

- $\Gamma$ is a 1-Form on $M^n$ with values in $\mathfrak{m}$, $\Gamma_x \in \mathbb{R}^n \otimes \mathfrak{m}$ ($x \in M^n$) ["intrinsic torsion", Swann/Salamon]
Fact: $\Gamma = 0 \Leftrightarrow \nabla^g$ is a $G$-connection $\Leftrightarrow \text{Hol}(\nabla^g) \subset G$

Via $\Gamma$, geometric $G$-structures $\mathcal{R} \subset \mathcal{F}(M^n)$ correspond to irreducible components of the $G$-representation $\mathbb{R}^n \otimes m$.

Thm. A geometric $G$-structure $\mathcal{R} \subset \mathcal{F}(M^n)$ admits a metric $G$-connection with antisymmetric torsion iff $\Gamma$ lies in the image of $\Theta$,

$$\Theta : \Lambda^3(M^n) \to T^*(M^n) \otimes m, \quad \Theta(T) := \sum_{i=1}^n e_i \otimes \text{pr}_m(e_i \triangleleft T).$$

[Fr, 2003]

If such a connection exists, it is called the characteristic connection $\nabla^c$ to replace the (unadapted) LC connection by $\nabla^c$.

Thm. If $G \not\subset \text{SO}(n)$ acts irreducibly and not by its adjoint rep. on $\mathbb{R}^n \cong T_pM^n$, then $\ker \Theta = \{0\}$, and hence the characteristic connection of a $G$-structure on a Riemannian manifold $(M^n, g)$ is, if existent, unique.

[A-Fr-Höll, 2013]
Uniqueness of characteristic connections

This is a consequence of the STHT:

**Proof.** $T \in \ker \Theta$ iff all $X \perp T \in \mathfrak{g} \subset \mathfrak{so}(n)$, that is,

$$\ker \Theta = \{ T \in \Lambda^3(\mathbb{R}^n) | \mathfrak{g}_T \subset \mathfrak{g} \},$$

so $(T, G, \mathbb{R}^n)$ defines an irreducible STHS, which by assumption is non transitive (because $G \not\subset \text{SO}(n)$). By the STHT, it has to be a Lie algebra with the adjoint representation. Since this was excluded as well, it follows that $\ker \Theta = \{0\}$. □

For many $G$-structures, uniqueness can be proved directly case by case – including a few cases where the $G$-action is not irreducible.
**U(n) structures in dimension 2n**

- $(S^6, g_{can})$: $S^6 \subset \mathbb{R}^7$ has an almost complex structure $J$ ($J^2 = -\text{id}$) inherited from "cross product" on $\mathbb{R}^7$.
- $J$ is not integrable, $\nabla^g J \neq 0$
- **Problem (Hopf):** Does $S^6$ admit an (integrable) complex structure?

$J$ is an example of a nearly Kähler structure: $\nabla^g_X J(X) = 0$

**More generally:** $(M^{2n}, g, J)$ almost Hermitian mnfd: $J$ almost complex structure, $g$ a compatible Riemannian metric.

**Fact:** structure group $G \subset U(n) \subset SO(2n)$, but $\text{Hol}_0(\nabla^g) = SO(2n)$.

**Examples:** twistor spaces $(\mathbb{CP}^3, F_{1,2})$ with their nK str., compact complex mnfd with $b_1(M)$ odd (does not admit a Kähler metric) . . .
**Thm.** An almost hermitian manifold \((M^{2n}, g, J)\) admits a characteristic connection \(\nabla\) if and only if the Nijenhuis tensor
\[
N(X, Y, Z) := g(N(X, Y), Z)
\]
is skew-symmetric. Its torsion is then
\[
T(X, Y, Z) = -d\Omega(JX, JY, JZ) + N(X, Y, Z)
\]
and it satisfies: \(\nabla\Omega = 0\), \(\text{Hol}(\nabla) \subset U(n)\). [Fr-Ivanov, 2002]

**‘Trivial case’:** If \((M^{2n}, g, J)\) is Kähler \((N = 0\) and \(d\Omega = 0)\), then \(T = 0\), the LC connection \(\nabla^g\) is the characteristic connection.

In particular for \(n = 3\): [Gray-Hervella]

- \(\mathfrak{so}(6) = \mathfrak{u}(3) \oplus \mathfrak{m}^6, \Gamma \in \mathbb{R}^6 \otimes \mathfrak{m}^6|_{U(3)} \cong W_1^2 \oplus W_2^{16} \oplus W_3^{12} \oplus W_4^6\)
- \(N\) is skew-symmetric \(\Leftrightarrow\) \(\Gamma\) has no \(W_2\)-part
- \(\Gamma \in W_1\): nearly Kähler manifolds \((S^6, S^3 \times S^3, F(1, 2), \mathbb{C}P^3)\)
- \(\Gamma \in W_3 \oplus W_4\): hermitian manifolds \((N = 0)\)
Contact structures

- \((M^{2n+1}, g, \eta)\) contact mnfd, \(\eta\): 1-form \((\cong\) vector field) 
- \(\langle \eta \rangle \perp\) admits an almost complex structure \(J\) compatible with \(g\)

- Contact condition: \(\eta \wedge (d\eta)^n \neq 0 \Rightarrow \nabla^g \eta \neq 0\), i.e. contact structures are never integrable! (no analogue on Berger’s list)

- Structure group: \(G \subset U(n) \subset SO(2n+1)\)

Examples: \(S^{2n+1} = \frac{SU(n+1)}{SU(n)}\), \(V_{4,2} = \frac{SO(4)}{SO(2)}\), \(M^{11} = \frac{G_2}{Sp(1)}\), \(M^{31} = \frac{F_4}{Sp(3)}\)

**Thm.** An almost metric contact manifold \((M^{2n+1}, g, \eta)\) admits a connection \(\nabla\) with skew-symmetric torsion and preserving the structure if and only if \(\xi\) is a Killing vector field and the tensor \(N(X, Y, Z) := g(N(X, Y), Z)\) is totally skew-symmetric. In this case, the connection is unique, and its torsion form is given by the formula

\[
T = \eta \wedge d\eta + d^\phi F + N - \eta \wedge \xi \perp N.
\]

[Fr-Ivanov, 2002]
A large class of almost metric contact manifolds thus admits a char. connection $\nabla$, and for these: $\text{Hol}_0(\nabla) \subset U(n) \subset SO(2n + 1)$.

**A special class:** Sasaki manifolds: Riemannian manifolds $(M^{2n+1}, g)$ equipped with a contact form $\eta$, its dual vector field $\xi$ and an endomorphism $\varphi: TM^7 \to TM^7$ s.t.:

- $\eta \wedge (d\eta)^n \neq 0$, \hspace{1em} $\eta(\xi) = 1$, \hspace{1em} $g(\xi, \xi) = 1$
- $g(\varphi X, \varphi Y) = g(X, Y)$ and $\varphi^2 = -\text{Id}$ on $\langle \eta \rangle^\perp$,
- $\nabla^g_X \xi = -\varphi X$, \hspace{1em} $(\nabla^g_X \varphi)(Y) = g(X, Y) \cdot \xi - \eta(Y) \cdot X$.

For Sasaki manifolds, the formula is particularly simple,

$$g(\nabla^c_X Y, Z) = g(\nabla^g_X Y, Z) + \frac{1}{2} \eta \wedge d\eta(X, Y, Z),$$

and $\nabla T = 0$ holds. \hspace{2em} [Kowalski-Wegrzynowski, 1987]
$G_2$ structures in dimension 7

Fix $G_2 \subset SO(7)$, so$(7) = g_2 \oplus m^7 \cong g_2 \oplus \mathbb{R}^7$.

Intrinsic torsion $\Gamma$ lies in $\mathbb{R}^7 \otimes m^7 \cong \mathbb{R}^1 \oplus g_2 \oplus S_0(\mathbb{R}^7) \oplus \mathbb{R}^7 =: \bigoplus_{i=1}^{4} \chi_i$

$\Rightarrow$ four classes of geometric $G_2$ structures \cite{Fernandez-Gray, '82}

- Decomposition of 3-forms: $\Lambda^3(\mathbb{R}^7) = \mathbb{R}^1 \oplus S_0(\mathbb{R}^7) \oplus \mathbb{R}^7$.

$G_2$ is the isotropy group of a generic element of $\omega \in \Lambda^3(\mathbb{R}^7)$:

$$G_2 = \{ A \in SO(7) | A \cdot \omega = \omega \}.$$

**Thm.** A 7-dimensional Riemannian mfd $(M^7, g, \omega)$ with a fixed $G_2$ structure $\omega \in \Lambda^3(M^7)$ admits a characteristic connection $\nabla$

$\iff$ the $g_2$ component of $\Gamma$ vanishes

$\iff$ There exists a VF $\beta$ with $\delta \omega = -\beta \lrcorner \omega$

The torsion of $\nabla$ is then $T = -\ast d\omega - \frac{1}{6} (d\omega, \ast \omega)\omega + \ast (\beta \lrcorner \omega)$, and $\nabla$ admits (at least) one parallel spinor. \cite{Fr-Ivanov, 2002}
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<th>name</th>
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| parallel $G_2$-manifold                        | $\{0\}$ | a) $\nabla^g \omega = 0$  
b) $\exists$ a $\nabla^g$-parallel spinor |
| nearly parallel $G_2$-manifold                 | $\mathcal{X}_1$ | a) $d\omega = \lambda \ast \omega$ for some $\lambda \in \mathbb{R}$  
b) $\exists$ real Killing spinor |
| almost parallel or closed (or calibrated symplectic) $G_2$-m. | $\mathcal{X}_2$ | $d\omega = 0$ |
| balanced $G_2$-manifold                        | $\mathcal{X}_3$ | $\delta \omega = 0$ and $d\omega \wedge \omega = 0$ |
| locally conformally parallel $G_2$-m.          | $\mathcal{X}_4$ | $d\omega = \frac{3}{4} \theta \wedge \omega$ and  
$\quad \quad \quad \quad \quad \quad \quad d \ast \omega = \theta \wedge \ast \omega$ for some $\theta$ |
| cocalibrated (or semi-parallel or cosymplectic ) $G_2$-manifold | $\mathcal{X}_1 \oplus \mathcal{X}_3$ | $\delta \omega = 0$ |
| locally conformally (almost) parallel $G_2$-manifold | $\mathcal{X}_2 \oplus \mathcal{X}_4$ | $d\omega = \frac{3}{4} \theta \wedge \omega$ |
| $G_2T$-manifold                               | $\mathcal{X}_1 \oplus \mathcal{X}_3 \oplus \mathcal{X}_4$ | a) $d \ast \omega = \theta \wedge \ast \omega$ for some $\theta$  
b) $\exists$ char. connection $\nabla^c$ |
Easiest examples:

- $S^7 = \frac{\text{Spin}(7)}{G_2}$, $M_{k,l}^{AW} = \frac{\text{SU}(3)}{U(1)_{k,l}}$, $V_{5,2} = \frac{\text{SO}(5)}{\text{SO}(3)}$, \ldots

- Explicit constructions of $G_2$ structures:

[Friedrich-Kath, Fernandez-Gray, Fernandez-Ugarte, Aloff-Wallach, Boyer-Galicki, \ldots]

- Every orientable hypersurface in $\mathbb{R}^8$ carries a cocalibrated $G_2$-structure

- $S^1$-PFB over 6-dim. Kähler manifolds, nearly Kähler manifolds. \ldots
Example \[\text{[Fernandez-Ugarte, '98]}\]

\(N^6\): 3-dimensional complex solvable group, \(M^7 := N^6 \times \mathbb{R}^1\). There exists a left invariant metric and a left invariant \(G_2\)-structure on \(M^7\) such that the structural equations are:

\[
de_3 = e_{13} - e_{24}, \quad de_4 = e_{23} + e_{14}, \quad de_5 = -e_{15} + e_{26}, \quad de_6 = -e_{25} - e_{16},
\]

all other \(de_i = 0\).

\(M^7\) has a \(G_2\)-invariant characteristic connection \(\nabla^c\) and

- \(T = 2 \cdot e_{256} - 2 \cdot e_{234}, \quad \delta(T) = 0\).
- \(\text{Scal}^c = -16\).
- There are two \(\nabla^c\)-parallel spinors, and both satisfy \(T^c \cdot \Psi = 0\).
An interesting subclass of $G_2$-mnfds: 7-dim. 3-Sasaki mnfds

$M^7$: 3-Sasaki mnfd, corresponds to $SU(2) \subset G_2 \subset SO(7)$.

- 3 orth. Sasaki structures $\eta_i \in T^*M^7$, $[\eta_1, \eta_2] = 2 \eta_3$, $[\eta_2, \eta_3] = 2 \eta_1$, $[\eta_3, \eta_1] = 2 \eta_2$ and $\varphi_3 \circ \varphi_2 = -\varphi_1$ etc. on $\langle \eta_2, \eta_3 \rangle$

- Known: A 3-Sasaki mnfd is always Einstein and has 3 Riemannian Killing spinors, define $T^v := \langle \xi_1, \xi_2 \xi_3 \rangle$, $T^h = (T^v)\perp$

- each Sasaki structures $\eta_i$ induces a characteristic connection $\nabla^i$, but $\nabla^1 \neq \nabla^2 \neq \nabla^3$?!? \Rightarrow Ansatz: $T = \sum_{i,j=1}^3 \alpha_{ij} \eta_i \wedge d\eta_j + \gamma \eta_1 \wedge \eta_2 \wedge \eta_3$

**Thm.** Every 7-dimensional 3-Sasaki mnfd admits a $\mathbb{P}^2$-family of metric connections with skew torsion and parallel spinors. Its holonomy is $G_2$.

[A-Fr, 2003]

**Thm.** There exists a cocalibrated $G_2$-structure with char. connection $\nabla^c$ with parallel spinor $\psi$ on $M^7$ with the properties:

- $\nabla^c$ preserves $T^v$ and $T^h$, and $\nabla^c T = 0$
- $\xi_i \cdot \psi$ are the 3 Riemannian Killing spinors on $M^7$

[A-Fr, 2010]
Example: Naturally reductive spaces

- Homogeneous non symmetric spaces provide a rich source for manifolds with characteristic connection

Let \( M = G/H \) be reductive, i.e. \( \exists \) \( m \subset g \) s.t. \( g = h \oplus m \) and \([h, m] \subset m\); isotropy repr. \( \text{Ad} : H \to \text{SO}(m) \). \( \langle , \rangle \) a pos. def. scalar product on \( m \).

The PFB \( G \to G/H \) induces a distinguished connection on \( G/H \), the so-called canonical connection \( \nabla^1 \). Its torsion is

\[
T^1(X, Y, Z) = - \langle [X, Y]_m, Z \rangle
\]

\( (= 0 \text{ for } M \text{ symmetric}) \)

Dfn. The metric \( \langle , \rangle \) is called naturally reductive if \( T^1 \) defines a 3-form,

\[
\langle [X, Y]_m, Z \rangle + \langle Y, [X, Z]_m \rangle = 0 \text{ for all } X, Y, Z \in m.
\]

They generalize symmetric spaces: \( \nabla^1 T^1 = 0, \nabla^1 R^1 = 0 \).
Naturally reductive spaces – basic facts

**Thm.** A Riemannian manifold equipped with a [regular] homogeneous structure, i.e. a metric connection $\nabla$ with torsion $T$ and curvature $\mathcal{R}$ such that $\nabla \mathcal{R} = 0$ and $\nabla T = 0$, is locally isometric to a homogeneous space. [Ambrose-Singer, 1958, Tricerri 1993]

Hence: Naturally reductive spaces have a metric connection $\nabla$ with skew torsion such that $\nabla T = \nabla \mathcal{R} = 0$

**N.B.** Well-known: Some mnfds carry several nat.red.structures, for exa.

$$S^{2n+1} = \text{SO}(2n+2)/\text{SO}(2n+1) = \text{SU}(n+1)/\text{SU}(n),$$

$$S^6 = G_2/\text{SU}(3), \ S^7 = \text{Spin}(7)/G_2, \ S^{15} = \text{Spin}(9)/\text{Spin}(7).$$

But, another consequence of the STHT:

**Thm.** If $(M, g)$ is not loc. isometric to a sphere or a Lie group, then its admits at most one naturally reductive homogeneous structure. [Olmos-Reggiani, 2012]
Classical construction of naturally reductive spaces

**General construction:**
Consider $M = G/H$ with restriction of the Killing form to $m$:

$$\beta(X, Y) := -\text{tr}(X^t Y), \quad \langle X, Y \rangle = \beta(X, Y) \text{ for } X, Y \in m.$$ 

Suppose that $m$ is an orthogonal sum $m = m_1 \oplus m_2$ such that

$$[\mathfrak{h}, m_2] = 0, \quad [m_2, m_2] \subset m_2.$$ 

Then the new metric, depending on a parameter $s > 0$

$$\tilde{\beta}_s = \beta|_{m_1} \oplus s \cdot \beta|_{m_2}$$

is naturally reductive for $s \neq 1$ w.r.t. the realisation as

$$M = (G \times M_2)/(H \times M_2) =: \overline{G/H}.$$ 

[Chavel, 1969; Ziller / D’Atri, 1979]

This description gets quickly rather tedious – thus, we shall usually describe nat. reductive spaces through their connections with parallel torsion and curvature.
**Example: Lie groups**

Let $M = G$ be a connected Lie group, $\mathfrak{g} = T_eG$.

the metric $g$ on $G$ is biinvariant

\[\Leftrightarrow L_a, R_a \text{ are isometries } \forall a \in G\]

\[\Rightarrow \text{ad}V \in \mathfrak{so}(\mathfrak{g}), \text{ i.e. } g(\text{ad}(V)X, Y) + g(X, \text{ad}(V)Y) = 0 \quad (\ast)\]

**Easy:** $\nabla^g_X Y = \frac{1}{2}[X, Y]$.

**Ansatz:** $T$ proportional to $[,]$, i.e. $\nabla_X Y = \lambda[X, Y]

- **torsion:** $T^\nabla(X, Y) = (2\lambda - 1)[X, Y]$, hence $T \in \Lambda^3(G) \Leftrightarrow (\ast)$

- **curvature:**

\[
\mathcal{R}^\nabla(X, Y)Z = \lambda(1-\lambda)[Z, [X, Y]] = \begin{cases} 
\frac{1}{4}[Z, [X, Y]] & \text{for LC conn.}(\lambda = \frac{1}{2}) \\
0 & \text{for } \lambda = 0, 1 
\end{cases}
\]

[±-connection, Cartan-Schouten, 1926]
Interpretation in the framework of homogeneous spaces

Take $\tilde{G} = G \times G$ with involution $\theta(a, b) = (b, a)$.

- $K := \tilde{G}^{\theta} = \{(a, a) \in \tilde{G}\} = \Delta G$ with Lie alg. $\mathfrak{k} = \{(X, X) \mid X \in \mathfrak{g}\} \subset \tilde{\mathfrak{g}}$

To make $\tilde{G}/\Delta G$ symmetric, one usually chooses as complement of $\mathfrak{k}$ in $\mathfrak{g}$

$$m_{sym} := \{(X, -X) \mid X \in \mathfrak{g}\},$$

for it satisfies $[m_{sym}, m_{sym}] \subset \mathfrak{k}$. But every space

$$m_t := \{X_t := (tX, (t - 1)X) \mid X \in \mathfrak{g}\}, \quad t \in \mathbb{R},$$

also defines a reductive complement, $[\mathfrak{k}, m_t] \subset m_t$. 
Fact: Every reductive homogeneous space has a canonical connection $\nabla^c$ induced from the PFB $\tilde{G} \to \tilde{G}/\Delta G$ (the $\nabla^c$-parallel tensors are exactly the $\tilde{G}$-invariant ones), $[\ , \ ] = [\ , \ ]_t + [\ , \ ]_m$

\[ \nabla^c T^c = 0, \quad T^c(X, Y) = -[X, Y]_m, \]
\[ \nabla^c R^c = 0, \quad R^c(X, Y) = -[[X, Y]_t, Z]. \]

This turns $G$ into a naturally reductive space.

One checks for $X_t = (tX, (t - 1)X), \ Y_t = (tY, (t - 1)Y) \in m_t$

\[ [X_t, Y_t] = (t^2[X, Y], (t - 1)^2[X, Y]) : \text{ hence } " \in m''_t \iff t = 0, 1 \]

i.e. $R^c = 0$ for $t = 0, 1$ – these are again the $\pm$-connections of Cartan-Schouten.

In particular, $\nabla T = 0$ for these connections on Lie groups.
Connections on homogeneous spaces – Wang’s Theorem

Wang’s Thm. [see Kobayashi-Nomizu]

Let $M^n = G/H$ be reductive, $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$. Then there is a bijection between $GL(n)$-invariant connections $\nabla$ on $M^n$ and maps $\Lambda : \mathfrak{m} \to \mathfrak{gl}(n)$ satisfying

$$\Lambda(Ad h)X = Ad(h)\Lambda(X)Ad(h)^{-1}$$

for all $h \in H$ and $X \in \mathfrak{m}$.

[Idea: $\Lambda$ is the evaluation of the connection form at $eH$]

Comments:

• If $\Lambda : \mathfrak{m} \to \mathfrak{so}(n)$, the corresponding connection is metric
• $\Lambda = 0$ is a solution, corresponds to the canonical connection
• Torsion: $T(X, Y) = \Lambda(X)Y - \Lambda(Y)X - [X, Y]_m$
• Curvature: $R(X, Y)Z = \Lambda(X)\Lambda(Y)Z - \Lambda(Y)\Lambda(X)Z - Ad([X, Y]_\mathfrak{h})Z$
**Example: The Berger sphere** $M^5 = SU(3)/SU(2)$

$su(3) \subset M_3(\mathbb{C})$, $su(2) \cong \{ \begin{bmatrix} 0 & 0 \\ 0 & B \end{bmatrix} : B \in su(2) \}$, $m_0 := \{ \begin{bmatrix} 0 & -\bar{v}^t \\ v & 0 \end{bmatrix} : v \in \mathbb{C}^2 \}$

Hence, we get a reductive decomposition

$$su(3) = su(2) \oplus m, \ m = m_0 \oplus \langle \eta \rangle \text{ with } \eta = \frac{1}{\sqrt{3}} \text{diag}(-2i, i, i)$$

Basis of $m_0$: $e_1, \ldots e_4$ corresponding to $v = (1, 0)$, $(i, 0)$, $(0, 1)$, $(0, i)$.

Deform the Killing form $\beta(X, Y) = -\text{tr}(XY)/2$ of $su(3)$ on $m$ to the family of metrics

$$g_\gamma := \beta|_{m_0} \oplus \frac{1}{\gamma} \beta|_{\langle \eta \rangle}, \ \gamma > 0.$$

- $\tilde{\eta} = \eta/\sqrt{\gamma} =: e_5$ and $\varphi := e_{12} + e_{34}$ defines an $\alpha$-Sasakian on $M^5$, its characteristic connection is described by $\Lambda : m \rightarrow so(m)$

$$\Lambda(e_i) = 0 \text{ for } i = 1, \ldots, 4, \ \Lambda(e_5) = (\sqrt{3/\gamma} - \sqrt{3\gamma})(E_{12} + E_{34}).$$

- Torsion $T = \tilde{\eta} \wedge d\tilde{\eta} = \sqrt{3/\gamma}(e_{12} + e_{34}) \wedge e_5$
Link to Dirac operators

Without torsion:

- Classical Schrödinger-Lichnerowicz formula on Riemannian spin manifolds
- Parthasarathy formula on symmetric spaces: \((D^g)^2 = \Omega + \frac{1}{8} \text{Scal}^g\), where \(\Omega\) : Casimir operator

With torsion: Assume \((M^n, g)\) is manifold with \(G\)-structure and characteristic connection \(\nabla\) with torsion \(T\)

\(\mathcal{D}\): Dirac operator of connection with torsion \(T/3\)

- **Generalized SL formula:** \([A-Fr, 2003]\)

\[
\mathcal{D}^2 = \Delta_T + \frac{1}{4} \text{Scal}^g + \frac{1}{8} ||T||^2 - \frac{1}{4} T^2
\]

[\(1/3\) rescaling: Slebarski (1987), Bismut (1989), Kostant, Goette (1999), IA (2002)]

- Similarly, \(\mathcal{D}^2 = \Omega + \text{const}\) on naturally reductive homogeneous spaces \([Kostant 1999, A 2002]\)
Almost hermitian manifolds and their Dirac operators

Almost hermitian manifolds

Parthasarathy:
\[(D^g)^2 = \Omega + \frac{1}{8}\text{Scal}^g\]

Kostant:
\[\mathcal{D}^2 = \Omega + \text{const}\]

\[T = 0\] (integrable)

\[T \neq 0\] (non integrable)
Literature


