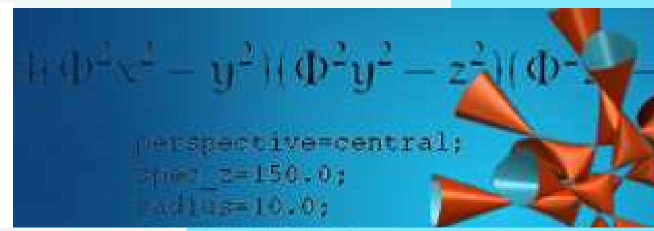
Philipps Universität Marburg

Non-integrable geometries, torsion, and holonomy

II b): Geometric structures modelled on some rank two symmetric spaces



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dim.	5	8	14	26	comments
symm. model	$\mathrm{SU}(3)/\ \mathrm{SO}(3)$	$\mathrm{SU}(3)$	$\mathrm{SU}(6)/\mathrm{Sp}(3)$	E_6/F_4	rk=2, non herm.
family	AI	A_2	AII	EIV	
isotropy rep.	$H_5 = \mathrm{SO}(3)$ on $S_0(\mathbb{R}^3)$	$H_8 = \mathrm{SU}(3)$ on $\mathrm{Her}_0(\mathbb{C}^3)$	$H_{14} = \mathrm{Sp}(3)$ on $\mathrm{Her}_0(\mathbb{H}^3)$	$H_{26} = F_4$ on $\mathrm{Her}_0(\mathbb{O}^3)$	
educing symm. 3-tensor Υ	Υ from $\mathrm{tr}\, M$	Υ	$\Upsilon, \Upsilon\circ$ conj	$\Upsilon, \Upsilon\circ$ conj	these are all examples
geom. descr.			special quat. str. on $\mathbb{C}^6$		?
# irreps in $\Lambda^3(\mathbb{R}^n)$	2	4	4	3	all mult. free
$\exists$ char. conn	$\Leftrightarrow$	$\Leftarrow$ ($\Leftrightarrow$ with add. cond)	$\Leftrightarrow$	$\Leftarrow$ ($\Leftrightarrow$ with add. cond)	nearly Kähler: $\Leftarrow$

Thm. The reduced holonomy $\mathrm{Hol}_0(M; \nabla^g)$ of the LC connection ∇^g is either that of a **symmetric space** or

$$\mathrm{Sp}(n)\mathrm{Sp}(1)\ [\mathrm{qK}],\ \mathrm{U}(n)\ [\mathrm{K}],\ \underbrace{\mathrm{SU}(n)\ [\mathrm{CY}],\ \mathrm{Sp}(n)\ [\mathrm{hK}],\ G_2,\ \mathrm{Spin}(7)}_{\mathrm{Ric}=0}.$$

[Berger / Simons, ≥ 1955]

– in this part: geometries modelled on **symmetric spaces.**

A look back to 1938: Cartan's work on isoparametric hypersurfaces

Dfn. M^{m-1} immersed into $\mathbb{R}^m, S^m$, or H^m is called an *isoparametric hypersurface* if its principal curvatures are constant. [$\Rightarrow$ const. mean curv.]

Set $p := \#$ of different principal curvatures

Thm. In $S^{n-1} \subset \mathbb{R}^n$: [Cartan 1938-40]

- If $p = 1$: M^{n-2} is a hypersphere in S^{n-1}
- If $p = 2$: $M^{n-2} = S^p(r) \times S^p(s)$ for $p + q = n - 2,\ r^2 + s^2 = 1$
- If $p = 3$: M^{n-2} is a tube of constant radius over a generalized Veronese emb. of $\mathbb{KP}^2$ into S^{n-1} for $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$
  - Thus, for $p = 3$, n must be $5, 8, 14$, or 26 !

Construction: use harmonic homog. polynomial F of degree p on $\mathbb{R}^n$ satisfying

$$\|\text{grad}\, F\|^2 = p^2 \|x\|^{2p-2}$$

The level sets of $F\big|_{S^{n-1}}$ define an isoparametric hypersurface family.

For $p = 3$, Cartan described explicitly the polynomial F.

Link to geometry:

F can be understood as a symmetric rank p tensor Υ, and each level set M will be invariant under the stabilizer of Υ!

Connection to rank 2 symmetric spaces

Fundamental oberservation: If $M^{n-2} \subset S^{n-1} = \mathrm{SO}(n)/SO(n-1)$ is an orbit of $G \subset \mathrm{SO}(n)$, then it is isoparametric (because it is homogeneous).

classif. of all $G \subset \mathrm{SO}(n)$ s.t. codim$\big|_{S^{n-1}}$ (princ. G-orbit)=1 or, equiv., codim$\big|_{\mathbb{R}^n}$=2	$\Rightarrow$	classif. of homogeneous isopar. hypersurfaces in S^{n-1}

Needed: a classification of all irred. reps. of $G \subset \mathrm{SO}(n)$ on $\mathbb{R}^n$ with codimension 2 principal orbits.

Thm. These are exactly the isotropy representations of rank 2 symmetric spaces. [Hsiang^2 / Lawson, 1970/71]

Proof produces a list, and it turns out to coincide with the list of isotropy representations.

Takagi & Takahashi (1972) made the relation more precise:

Thm. Let $M^n = G/H$ cpct symmetric space, $\mathrm{rk} = 2$, $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$.

• An H-orbit M of a unit vector in $S^{n-1} \subset \mathfrak{p}$ is an isoparametric hypersurface.

• normal great circles $\leftrightarrow$ $\mathfrak{a} \cap S^{n-1}$, focal points $\leftrightarrow$ singular elements in $\mathfrak{a}$

• the principal curvatures and their mult. are computed from the root data, for example: The order of the Weyl group is $2p$.

$\Rightarrow$ only $p = 1, 2, 3, 4, 6$ are possible

$\Rightarrow$ there are **4 symmetric spaces yielding isoparametric hypersurfaces with $p = 3$:**

$$\mathrm{SU}(3)/SO(3),\ \mathrm{SU}(3),\ \mathrm{SU}(6)/\mathrm{Sp}(3),\ E_6/F_4$$

Description of their isotropy representations

Let $\mathbb{R}^n$ be $(n = 5, 8, 14, 26)$

- $\mathrm{Her}_0(\mathbb{K}^3)$ Hermitian trace-free endomorphisms on $\mathbb{K}^3$, $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$

with the conjugation action of $H_n = \mathrm{SO}(3), \mathrm{SU}(3), \mathrm{Sp}(3)$, or F_4, resp.

Define for $X, Y, Z \in \mathbb{R}^n$ a symmetric 3-tensor by polarisation from $\mathrm{tr}:$

$$\begin{aligned} \Upsilon(X,Y,Z) \quad &:= \quad 2\sqrt{3}[\mathrm{tr}\, X^3 + \mathrm{tr}\, Y^3 + \mathrm{tr}\, Z^3] - tr(X+Y)^3 \\ &\quad -\mathrm{tr}\,(X+Z)^3 - \mathrm{tr}\,(Y+Z)^3 + \mathrm{tr}\,(X+Y+Z)^3. \end{aligned}$$

For $\mathbb{K} = \mathbb{H}, \mathbb{O}$, a second tensor is obtained as $\tilde{\Upsilon}(X,Y,Z) := \Upsilon(\bar{X}, \bar{Y}, \bar{Z})$
– it is not conjugate to Υ under $\mathrm{SO}(n)$.

Thm. For $n = 5, 8, 14, 26$: $H_n = \{A \in \mathrm{SO}(n) \ : \ A^*\Upsilon = \Upsilon\}$

and for any basis $V_1, \dots V_n$ of $\mathbb{R}^n \cong \mathrm{Her}_0(\mathbb{K}^3)$

- Υ is totally symmetric,
- Υ is trace-free, i. e. $\sum_i \Upsilon(X, V_i, V_i) = 0$,
- Υ satisfies the identity (g: metric)

$$\sum_{X,Y,Z}^{c} \sum_i \Upsilon(X, Y, V_i)\Upsilon(Z, U, V_i) = \sum_{X,Y,Z}^{c} g(X, Y)g(Z, U)$$

In particular: Υ determines g!

N.B. For $n = 14, 26$, the non-commutativity of $\mathbb{K}$ implies existence of two determinants, $\det_1, \det_2$. But $3 \det_1(X) = \mathrm{tr}\, X^3$, hence polarisation from $\det$ would yield the same tensor(s).

For $n = 8, 14$, $\exists$ an alternative tensor reducing $\mathrm{SO}(n)$ to H_n:

- $n = 8$: a 3-form, $n = 14$: a 5-form (129 terms. . .)

H_n-structures on Riemannian manifolds

Dfn. For $n = 5, 8, 14, 26$:

A n-mnfd with a H_n-structure is a Riemannian mnfd (M^n, g) with a reduction of the frame bundle $\mathcal{R}(M^n)$ to H_n.

$\Rightarrow$ has automatically a 3-tensor Υ with the properties above!

Thm. An integrable H_n-structure ($\Leftrightarrow \nabla^g \Upsilon = 0$) is isometric to one of the symmetric spaces G_n/H_n, i. e.

$$\mathrm{SU}(3)/SO(3),\ \mathrm{SU}(3),\ \mathrm{SU}(6)/\mathrm{Sp}(3),\ E_6/F_4,$$

or one of their non-compact dual symmetric spaces. [Nurowski, 2007]

Questions:

- topological conditions for existence of H_n-structure ?
- non-symmetric examples of H_n-mnfds?

Topological conditions: the case $H_5 = \mathrm{SO}(3)$

$\exists$ two nonequivalent embeddings $\mathrm{SO}(3) \to \mathrm{SO}(5)$:

- $*$ as upper diagonal block matrices: '$\mathrm{SO}(3)_{\mathrm{st}}$'
- $*$ by the irreducible 5-dim. representation of $\mathrm{SO}(3)$: '$\mathrm{SO}(3)_{\mathrm{ir}}$'

Question: Conditions for $\mathrm{SO}(3)_{st}$- or $\mathrm{SO}(3)_{ir}$-structures ?

Dfn. Kervaire semi-characteristics:

$$k(M^5) \quad := \quad \sum_{i=0}^{2} \dim_{\mathbb{R}}\big(H^{2i}(M^5;\mathbb{R})\big) \mod 2 \ ,$$

$$\hat{\chi}_2(M^5) \quad := \quad \sum_{i=0}^{2} \dim_{\mathbb{Z}_2}\big(H_i(M^5;\mathbb{Z}_2)\big) \mod 2 \ .$$

Thm. $k(M^5) - \hat{\chi}_2(M^5) = w_2(M^5) \cup w_3(M^5)$. In particular, if M^5 is spin, then $k(M^5) = \hat{\chi}_2(M^5)$. [Lusztig-Milnor-Peterson 1969]

$SO(3)_{st}$**-structure** ($\Leftrightarrow$ $\exists$ two global lin. indep. vector fields)

Thm. A compact oriented 5-mnfd admits an $\mathrm{SO}(3)_{st}$-structure iff $w_4(M^5) = 0,\ k(M^5) = 0.$ [Thomas 1967; Atiyah 1969]

$SO(3)_{ir}$**-structures** [IA-Becker-Bender-Fr, 2010]

Example. $M^5 = \mathrm{SU}(3)/\mathrm{SO}(3)$ has an $\mathrm{SO}(3)_{ir}$-structure.

Some topological properties of this space:

- M^5 is simply connected and a rational homology sphere.
- M^5 does not admit any Spin- or $\mathrm{Spin}^{\mathbb{C}}$-structure.
- $k(M^5) = 1$ and $\hat{\chi}_2(M^5) = 0$

In particular, $M^5 = \mathrm{SU}(3)/\mathrm{SO}(3)$ does not admit any $\mathrm{SO}(3)_{st}$-structure!

Prop. M^5 admits an $\mathrm{SO}(3)_{ir}$-structure iff there exists a 3-dim. real bundle E^3 such that $T(M^5) = S^2_0(E^3)$.

Thm. Suppose that $T(M^5) = S^2_0(E^3)$. Then

- $p_1(M^5) = 5 \cdot p_1(E^3)$; in particular, $p_1(M^5)/5 \in H^4(M^5;\mathbb{Z})$ is integral.
- $w_1(M^5) = w_4(M^5) = w_5(M^5) = 0$.
- $w_2(M^5) = w_2(E^3)$ and $w_3(M^5) = w_3(E^3)$.

Example. $\mathbb{RP}^5$ has none of both $\mathrm{SO}(3)$-str., since $w_4(\mathbb{R}P^5) \neq 0$.

Conjecture: M^5 admits an $\mathrm{SO}(3)_{ir}$-structure iff

$$w_4(M^5) \ = \ 0 \ , \quad \hat{\chi}_2(M^5) = 0 \ , \quad \frac{p_1(M^5)}{5} \in H^4(M^5;\mathbb{Z}) \ .$$

('$\Rightarrow$' follows from previous Thm)

Can only prove:

Thm. A compact, s.c. spin mnfd admitting a $\mathrm{SO}(3)_{ir}$- or $\mathrm{SO}(3)_{st}$-str. is parallelizable.

Cor. S^5 has none of both $\mathrm{SO}(3)$-structures.

Example. The connected sums $(2l+1)\#(S^2 \times S^3)$ are s.c., spin and admit a $\mathrm{SO}(3)_{st}$-structure.

A rather sophisticated construction yields:

Thm. There exist mnfds $p\,\mathbb{CP}^2 \# q\,\overline{\mathbb{CP}^2}$ such that *every* S^1-bundle over them admits a SO_{ir}-structure. (for example: $(p,q) = (21,1)$, $(43,3)$, $(197,17)\ldots$)

Topological conditions: the case $H_{14} = \mathrm{Sp}(3)$

. . . very hard. From $H^*(B\mathrm{Sp}(3), \mathbb{Z}) = \mathbb{Z}[q_4, q_8, q_{12}]$ (with $q_i \in H^i$), one deduces: Every cpct 14-dimensional mnfd with a $\mathrm{Sp}(3)$-structure satisfies

- $\chi(M) = 0$ and $w_i(M) = 0$ except for $i = 4, 8, 12$

In particular, it is orientable and spin; for exa. S^{14} has no $\mathrm{Sp}(3)$-structure.

Open problem: sufficient and necessary conditions ?!?

Some non-compact examples: use isom. $\mathrm{Spin}(5) \cong \mathrm{Sp}(2) \subset \mathrm{Sp}(3)$ and the decomposition $\mathbb{R}^{14} \overset{\mathrm{Spin}(5)}{=} \mathbb{R} \oplus \mathbb{R}^5 \oplus \Delta_5$ (the 5-dim. spin rep.)

Every S^1-bundle M^{14} over one of the following

- spin bundle of a 5-dim. spin mnfd X^5 ($=$ 8-dim VB)

- associated bundle $\mathcal{R}(Y^8) \times_{\mathrm{Spin}(5)} \mathbb{R}^5$ over an 8-dim. mnfd Y^8 with an $\mathrm{Sp}(2)$-structure (hyper-Kähler, quaternionic-Kähler etc.)

carries a $\mathrm{Sp}(3)$-structure. [IA-Fr, 2011]

Possible types of H_n-structures

Decompose $\Lambda^3(\mathbb{R}^n)$ under H_n-action:

- $n = 5$: $\Lambda^3(\mathbb{R}^5) \cong \Lambda^2(\mathbb{R}^5) \cong \mathfrak{so}(5) = \mathfrak{so}(3)_{\text{ir}} \oplus V^7$
- $n = 8$: $\Lambda^3(\mathbb{R}^8) \cong \mathbb{R} \oplus \mathfrak{su}(3) \oplus V^{20} \oplus V^{27}$
- $n = 14$: $\Lambda^3(\mathbb{R}^{14}) \cong \mathfrak{sp}(3) \oplus V^{70} \oplus V^{84} \oplus V^{189}$
- $n = 26$: $\Lambda^3(\mathbb{R}^{26}) \cong V^{273} \oplus V^{1053} \oplus V^{1274}$.

Recall:

Thm. A geometric G-structure $\mathcal{R} \subset \mathcal{F}(M^n)$ admits a metric G-connection with antisymmetric torsion iff Γ lies in the image of Θ,

$$\Theta : \Lambda^3(M^n) \to T^*(M^n) \otimes \mathfrak{m}, \quad \Theta(T) := \sum_{i=1}^n e_i \otimes \text{pr}_{\mathfrak{m}}(e_i \lrcorner T).$$

[Fr, 2003]

So mnfds whose intrinsic torsion has parts in $\mathbb{R}^n \otimes \mathfrak{m}$ that are not in the image of Θ cannot admit a characteristic connection. Uniqueness?

Characteristic connections

Recall:

Thm. If $G \not\subset \mathrm{SO}(n)$ acts irreducibly and not by its adjoint rep. on $\mathbb{R}^n \cong T_pM^n$, then $\ker\Theta = \{0\}$, and hence the characteristic connection of a G-structure on a Riemannian manifold (M^n, g) is, if existent, unique.

[A-Fr-Höll, 2013]

• $n = 5$: injectivity of Θ can be established by elementary methods

[Fr 2003, Bobienski-Nurowski 2006]

• $n = 8$: this is an adjoint action, so the thm cannot be applied, and indeed the characteristic connection is not unique [Puhle, 2012]

• $n = 14, 26$: The thm is applicable, $\ker\Theta = \{0\}$ so the characteristic connection is unique (when existent).

Remark. If the H_n-manifold (M, g) admits a characteristic connection ∇ with torsion $T \in \Lambda^3(M^n)$, it satisfies $\nabla\Upsilon = 0$ by the general holonomy principle. A short calculation then shows $\nabla^g_V\Upsilon(V, V, V) = 0$.

Homogeneous examples: the case $H_5 = \mathrm{SO}(3)$

Exa 1: 'twisted' Stiefel mnfd $V_{2,4}^{\mathrm{ir}} = \mathrm{SO}(3) \times \mathrm{SO}(3)/\mathrm{SO}(2)_{\mathrm{ir}}$

Recall: classical Stiefel manifold $V_{2,4}^{\mathrm{st}} = \mathrm{SO}(4)/\mathrm{SO}(2)$:

Carries an $\mathrm{SO}(3)_{\mathrm{st}}$ structure, an Einstein-Sasaki metric, 2 Riemannian Killing spinors [Jensen 75, Fr 1981]

Consider now $H := \mathrm{SO}(2) \subset \mathrm{SO}(3)_{\mathrm{ir}}$,

$H \ni A \longmapsto (A, A^2) \in \mathrm{SO}(3) \times \mathrm{SO}(3) =: G$, $V_{2,4}^{\mathrm{ir}} := \mathrm{SO}(3) \times \mathrm{SO}(3)/\mathrm{SO}(2)_{\mathrm{ir}}$.

- isotropy rep.: $\lambda : \mathrm{SO}(2) \to \mathrm{SO}(5)$, $\lambda(A) = \mathrm{diag}(1, A, A^2)$
- decompose $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, $\mathfrak{m} = \mathfrak{n} \oplus \mathfrak{m}_1 \oplus \mathfrak{m}_2$ of dims 1, 2, 2
- new metric: $g_{\alpha,\beta,\gamma} = \alpha \cdot g\big|_{\mathfrak{n}} \oplus \beta \cdot g\big|_{\mathfrak{m}_1} \oplus \gamma \cdot g\big|_{\mathfrak{m}_2}$, $\alpha, \beta, \gamma > 0$

Thm. $V_{2,4}^{\mathrm{ir}} = \mathrm{SO}(3) \times \mathrm{SO}(3)/SO(2)_{\mathrm{ir}}$ with $g_{\alpha\beta\gamma}$ satisfies:

• If $\alpha\beta + 4\,\gamma\alpha - 25\,\beta\gamma = 0$, the $\mathrm{SO}(3)_{\mathrm{ir}}$ structure admits a char. connection and the torsion $T^{\alpha\beta\gamma}$ of its characteristic connection $\nabla^{\alpha\beta\gamma}$ is

$$T^{\alpha\beta\gamma} = \frac{2\sqrt{\alpha}}{5\beta} e_1 \wedge e_2 \wedge e_3 - \frac{\sqrt{\alpha}}{5\gamma} e_1 \wedge e_4 \wedge e_5 .$$

• Its holonomy is $\mathrm{SO}(2)_{\mathrm{ir}}$ and its torsion is parallel, $\nabla^{\alpha\beta\gamma} T^{\alpha\beta\gamma} = 0$.

• The metric of the $\mathrm{SO}(3)_{\mathrm{ir}}$ structure is naturally reductive if and only if $\alpha = 5\beta = 5\gamma$.

• $\exists_1$ Einstein metric, not nat. reductive (for complicated values of α, β, γ)

• $\exists$ two invariant almost contact metric structures, characterized by

$$\xi \cong \eta = e_1, \quad \varphi_{\pm} = -E_{23} \pm E_{45}, \quad dF_{\pm} = 0.$$

Both admit a unique characteristic connection with the torsion above.

• The contact structure is Sasakian (but never Einstein) if and only if $\alpha = 25\beta^2 = 100\gamma^2$; it is in addition an $\mathrm{SO}(3)_{\mathrm{ir}}$ structure for $(\alpha, \beta, \gamma) = (\frac{25}{36}, \frac{1}{6}, \frac{1}{12})$.

– this is a very well-behaved example.

N.B. $V_{2,4}^{\mathrm{ir}}$ has a non-compact partner, $\tilde{V}_{2,4}^{\mathrm{ir}} := \mathrm{SO}(2,1) \times \mathrm{SO}(3)/\mathrm{SO}(2)_{\mathrm{ir}}$

• very similar, but the metric of the $\mathrm{SO}(3)_{\mathrm{ir}}$ structure admitting a char. connection is never naturally reductive and never Einstein.

Exa 2: $W^{\mathrm{ir}} = \mathbb{R} \times (\mathrm{SL}(2,\mathbb{R}) \ltimes \mathbb{R}^2)/\mathrm{SO}(2)_{\mathrm{ir}}$

Construction: $G = \mathbb{R} \times (\mathrm{SL}(2,\mathbb{R}) \ltimes \mathbb{R}^2)$; $X, E^{\pm}$ standard basis of $\mathfrak{sl}(2,\mathbb{R})$

- choose basis for $\mathfrak{g} = \mathbb{R} \oplus \mathfrak{sl}(2,\mathbb{R}) \oplus \mathbb{R}^2$ that depends on $\mu \in \mathbb{R}$,

$$\bar{e}_0^{\mu} = E_+ - E_- + \mu,\ \bar{e}_1^{\mu} = 1 - \mu(E_+ - E_-),\ \text{remaining el'ts standard.}$$

$\bar{e}_0^{\mu}$ generates a one-dimensional $\mathrm{SO}(2) \cong H_\mu \subset G$, with same isotropy repr. as in previous example

- $\mu = 0$ corresponds to the standard embedding $\mathfrak{so}(2) \to \mathfrak{sl}(2,\mathbb{R})$
- decompose again $\mathfrak{m} = \mathfrak{n}^{\mu} \oplus \mathfrak{m}_1 \oplus \mathfrak{m}_2$ with same Ansatz for metric

Thm.

- $\forall\ \beta > 0$ and $\alpha, \gamma > 0$ s.t. $\alpha \geq 12\gamma$, the $\mathrm{SO}(3)_{\mathrm{ir}}$ structure admits a char. connection for the two embeddings of $\mathrm{SO}(2) \cong H_\mu \to \mathrm{SO}(5)$

$$\mu = (2\sqrt{3\gamma})^{-1}[\sqrt{\alpha} \pm \sqrt{\alpha - 12\gamma}]$$

• the torsion $T^{\alpha\beta\gamma}$ of its characteristic connection $\nabla^{\alpha\beta\gamma}$ is

$$T^{\alpha\beta\gamma} \;=\; -\frac{2\sqrt{3}}{\sqrt{\gamma}}(e_1 \wedge e_2 \wedge e_3 + e_1 \wedge e_4 \wedge e_5).$$

• Its holonomy is $\mathrm{SO}(3)_{\mathrm{ir}} \subset \mathrm{SO}(5)$. Its torsion is *not* parallel, but it is divergence-free, $\delta T^{\alpha\beta\gamma} = 0$.

• The metric of the $\mathrm{SO}(3)_{\mathrm{ir}}$ str. is never naturally reductive and never Einstein.

• $\nexists$ a compatible contact structure.

Consequence:

• $\mathrm{SO}(3)_{\mathrm{ir}}$ structures are conceptionally really different from contact structures; they define a new type of geometry on 5-manifolds.

• It can happen that the torsion is not parallel.

Homogeneous examples: the case $H_{14} = \mathrm{Sp}(3)$

Exa 1: Higher Aloff-Wallach mnfd $M^{14} = \mathrm{SU}(4)/S^1$

Embed S^1 as $\mathrm{diag}(e^{-it}, e^{-it}, e^{it}, e^{-it}) \subset \mathrm{SU}(4)$.

- $\mathfrak{su}(4) = \mathbb{R} \oplus \mathfrak{m}^{14}$, $\mathfrak{m} = \bigoplus_{i=1}^{4} V_i \oplus \bigoplus_{j=1}^{6} W_j$, $\dim V_i = 2$, $\dim W_j = 1$.
- new metric g depending on $\alpha_1, \ldots, \alpha_{10}$

Thm.

- $\exists$ a 3-dim. space of metrics that are nearly integrable $\mathrm{Sp}(3)$-structures
- Ric has then 3 EV's of mult. 4 and twice EV 0. In particular, the metric is never Einstein.
- the $\mathrm{Sp}(3)$- structure is always of general type, i. e. its torsion has contributions in all summands of $\Lambda^3(M)$. For some metrics, the torsion is parallel.

Exa 2: the homogeneous space $M^{14} = \mathrm{SU}(5)/\mathrm{Sp}(2)$

as a mnfd, same as $\mathrm{SU}(6)/\mathrm{Sp}(3)$, but not symmetric

- $\mathfrak{su}(5) = \mathfrak{sp}(2) \oplus \mathfrak{m}^{14}$, $\mathfrak{m}^{14} = \mathbb{R} \oplus \mathbb{R}^5 \oplus \Delta_5$ (recall $\mathrm{Sp}(2) \cong \mathrm{Spin}(5)$)
- 3 deformation parameters in the metric

Thm.

- all metrics are nearly integrable $\mathrm{Sp}(3)$-structures
- the characteristic connection has full holonomy $\mathrm{Sp}(3)$.
- the $\mathrm{Sp}(3)$-structure can be of general type or of type $\mathfrak{sp}(3)$, V^{189}, the torsion is sometimes parallel.
- Ric has then 3 EV's of mult. 1, 5, 8. In particular, the metric is never Einstein.

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