Philipps Universität Marburg

Non-integrable geometries, torsion, and holonomy
III: Curvature properties of connections with skew torsion

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Connections with parallel skew torsion

(M,g) Riemannian mnfd, ∇ a connection with skew torsion $T \in \Lambda^3(M)$

• Know already large families of such manifolds where $\nabla T = 0$ holds: nearly Kähler mnfds, nearly parallel G_2-mnfds, Sasaki mnfds, naturally reductive homogeneous spaces. . .

Dfn. For any $T \in \Lambda^3(M)$, define ($e_1, \ldots, e_n$ a local ONF)

$$\sigma_T := \frac{1}{2}\sum_{i=1}^{n}(e_i \lrcorner T) \wedge (e_i \lrcorner T) = \overset{X,Y,Z}{\mathfrak{S}} g(T(X,Y),T(Z,V)) \quad (= 0 \text{ if } n \leq 4)$$

[Exa: For $T = \alpha\, e_{123} + \beta\, e_{456}$, $\sigma_T = 0$; for $T = (e_{12} + e_{34})e_5$, $\sigma_T = -e_{1234}$]

σ_T measures the 'degeneracy' of T and appears in many import. rel.:

* 1st Bianchi identity
* $T^2 = -2\sigma_T + \|T\|^2$ in the Clifford algebra
* If $\nabla T = 0$: $dT = 2\sigma_T$, $\nabla^g T = \frac{1}{2}\sigma_T$, $\delta T = 0$. . .

  either $\sigma_T = 0$ or $\mathfrak{hol}^{\nabla} \subset \mathfrak{iso}(T)$ is non-trivial

Flat metric connections with antisymmetric torsion

Suppose ∇ is metric and has antisymmetric torsion $T \in \Lambda^3(M)$,

$$\nabla_X Y = \nabla^g_X Y + \tfrac{1}{2} T(X, Y).$$

Q: What are the manifolds with a flat metric connection with antisymmetric torsion?

We shall [this section: A-Fr, 2010]

- discuss a family of easy, yet interesting examples
- discuss a less simple, isolated example
- show that these are all such manifolds (up to coverings and products)

Before the proof, I shall sketch the different approaches to the problem and its history.

Assume simply connected where needed.

Flat connections

Dfn. ∇ is called *flat*, if $\mathcal{R}(X, Y) = 0$ for all X, Y

$\Leftrightarrow$ $\nabla : TM \to \mathrm{End}(TM),\ X \mapsto \nabla_X$ is Lie algebra homomorphism

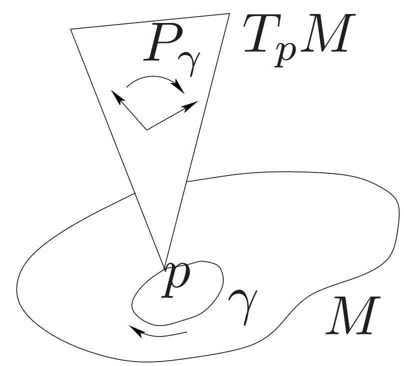
$\Leftrightarrow$ By Ambrose-Singer Thm ($\gamma \in C(p), P_\gamma : T_pM \to T_pM$ par.tr.):

$$0 = \mathfrak{hol}(\nabla, p) = \langle P_\gamma^{-1} \circ \mathcal{R}(P_\gamma V, P_\gamma W) \circ P_\gamma \rangle \subset \mathfrak{so}(T_pM),$$

i. e. $\mathrm{Hol}(p; \nabla)$ is a discrete group

$\Leftrightarrow$ parallel transport is path-independent

$\Rightarrow$ (M, g) is parallelisable and therefore spin



If $\nabla = \nabla^g$ the LC connection, Frobenius' Theorem implies:

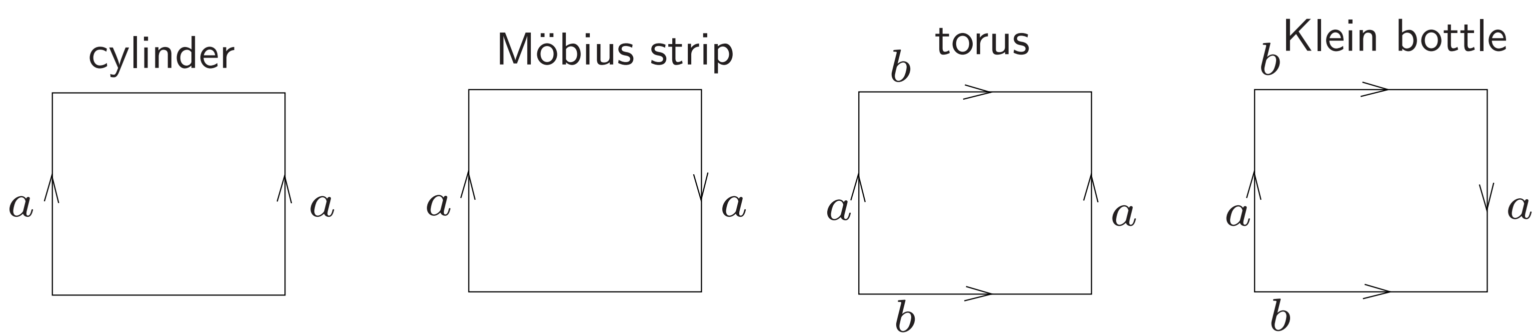
Thm. If ∇^g is flat, there exists in the vicinity of every $p \in M$ a chart s. t. the coefficients of the Riemannian metric are

$$g = \mathrm{diag}(1, 1, \ldots, 1).$$

– the proof relies on Cartan's structure equations and breaks down for connections with torsion.

Hence, (M, g) looks locally like $\mathbb{R}^n$ with the euclidian metric.

Globally, of course more is possible, for example for $n = 2$:



Example 1: Lie groups

Let $M = G$ be a connected Lie group, $\mathfrak{g} = T_e G$, with a biinvariant metric.

Easy: $\nabla^g_X Y = \frac{1}{2}[X, Y]$.

Ansatz: T proportional to $[,]$, i. e. $\nabla_X Y = \lambda[X, Y]$

- torsion: $T^\nabla(X, Y) = (2\lambda - 1)[X, Y]$, hence $T \in \Lambda^3(G)$
- curvature:

$$\mathcal{R}^\nabla(X, Y)Z = \lambda(1-\lambda)[Z, [X, Y]] = \begin{cases} \frac{1}{4}[Z, [X, Y]] & \text{for LC conn.} (\lambda = \frac{1}{2}) \\ 0 & \text{for } \lambda = 0, 1 \end{cases}$$

[±-connection, Cartan-Schouten, 1926]

- ±-connection satisfies $\sigma_T = 0$ and $\nabla T = 0$ (hence $dT = 0$).

Example 2: S^7

- only parallelisable sphere that is not a Lie group (but almost. . .)

Consider spin representation $\kappa^{\mathbb{C}}: \ \mathrm{Spin}(7) \to \mathrm{End}(\Delta_7^{\mathbb{C}}), \quad \Delta_7^{\mathbb{C}} \cong \mathbb{C}^8$.

In dim.7, this turns out to be complexification of 8-dim. real rep.,

$$\kappa: \ \mathrm{Spin}(7) \to \mathrm{End}(\Delta_7), \quad \Delta_7 \cong \mathbb{R}^8.$$

κ is in fact a repr. of the Clifford algebra over $\mathbb{R}^7$ $(\mathrm{Spin}(7) \subset \mathrm{Cl}(\mathbb{R}^7)!)$,

$$\kappa: \ \mathbb{R}^7 \subset \mathrm{Cl}(\mathbb{R}^7) \to \mathrm{End}(\Delta_7).$$

Choose $e_1, \ldots, e_7$ an ON basis of $\mathbb{R}^7$, and set $\kappa_i = \kappa(e_i)$.

- Embed $S^7 \subset \Delta_7$ as spinors of length 1,
- define VFs on S^7 by $V_i(x) = \kappa_i \cdot x$ for all $x \in S^7 \subset \Delta^7$

Properties of the VFs $V_i(x) = \kappa(e_i) \cdot x$

Thm. (1) These vector fields realize a ON trivialization of S^7,

[computation rules for Clifford multipl.]

(2) the connection ∇ defined by $\nabla V_i = 0$ is metric, flat, and with torsion

$$T(V_i, V_j, V_k)(x) \;=\; -\langle [V_i, V_j], V_k \rangle \;=\; 2\langle \kappa_i \kappa_j \kappa_k x, x \rangle \in \Lambda^3(S^7),$$

(3) $\nabla T \neq 0$ (check that T does not have constant coefficients), $\sigma_T \neq 0$

(4) ∇ is a G_2 connection of Fernandez-Gray type $\mathcal{X}_1 \oplus \mathcal{X}_3 \oplus \mathcal{X}_4$.

Classification

Goal: Show that any irreducible, complete, and simply connected M with a flat, metric connection with antisymmetric torsion $T \in \Lambda^3(M)$ is one of these examples.

- 1926: Cartan-Schouten "On manifolds with absolute parallelism" – wrong proof.

- 1968: d'Atri-Nickerson "On the existence of special orthonormal frames" – when does (M, g) admit an ONF of Killing vectors?

  This is mainly an equivalent problem:

  $$V \text{ is Killing VF} \Leftrightarrow g(\nabla^g_X V, Y) + g(X, \nabla_Y V) = 0 \qquad (*)$$

  If V is parallel for ∇ with torsion T, then $\nabla^g_X V = -\frac{1}{2}T(X, V)$, hence

  $$(*) \Leftrightarrow g(T(X, V), Y) + g(X, T(Y, V)) = 0 \Leftrightarrow T \in \Lambda^3(M)$$

- 1972: J. Wolf "On the geometry and classification of absolute parallelisms" – 2 long papers in J. Diff.Geom.

Q: Both proofs rely on classification of symmetric spaces. Direct proof?

Sketch of proof

(1) General identities:

[common to all authors]

- $\mathrm{Ric}^g(X,Y) = \frac{1}{4}\sum_i \langle T(X,e_i), T(Y,e_i)\rangle$, ($\Rightarrow$ $\mathrm{Ric}^g(X,X) \geq 0$)
- $K^g(X,Y) = \frac{\|T(X,Y)\|^2}{4[\|X\|^2\|Y\|^2 - \langle X,Y\rangle^2]} \geq 0$ (sectional curvature)
- $\delta T = 0$ ($=$ antisymmetric part of Ric^∇)

(2) General tools: $\sigma_T = \frac{1}{2}\sum_i (e_i \lrcorner T) \wedge (e_i \lrcorner T) \in \Lambda^4(M)$ satisfies

- $T^2 = -2\sigma_T + \|T\|^2$ (as endomorphisms on Δ_7)
- $\nabla T = 0$ implies $dT = 2\sigma_T$ [recall: true for G, wrong for S^7]
- All spinors with constant coeff. are parallel $\Rightarrow 3dT = 2\sigma_T$ (SL formula)
- Bianchi I:

$$\underset{X,Y,Z}{\mathfrak{S}} \mathcal{R}(X,Y,Z,V) = dT(X,Y,Z,V) - \sigma^T(X,Y,Z,V) + (\nabla_V T)(X,Y,Z)$$

(3) Rescaling of connection: [implicit in Cartan]

Consider the rescaled connection $\nabla^{1/3}$,

$$\nabla^{1/3}{}_X Y = \nabla^g_X Y + \frac{1}{6} T(X,Y)$$

– $\nabla^{1/3}$ plays a prominent role for Dirac operators with torsion

Thm.

- $\nabla^{1/3} T = 0$ $\left(\Leftrightarrow \nabla_V T = -\frac{1}{3} V \lrcorner \sigma_T \Leftrightarrow \nabla^g_V T = \frac{1}{6} V \lrcorner \sigma_T\right)$

In particular, $\|T\|$ and the scalar curvature are constant, and for any tensor field $\mathcal{T}$ polynomial in T:

$$\nabla \mathcal{T} = -2\nabla^g \mathcal{T}; \text{ in particular: } \nabla \mathcal{T} = 0 \Leftrightarrow \nabla^g \mathcal{T} = 0$$

- $\nabla^{1/3} \mathcal{R}^g = 0$

By the Ambrose-Singer Thm, M is a naturally reductive space (in particular, homogeneous).

(4) Splitting principle:

Thm. Let $M = M_1 \times M_2$ be a mnfd with a flat metric connection ∇ with torsion $T \in \Lambda^3(M)$. Then $T = T_1 + T_2$ with $T_i \in \Lambda^3(M_i)$.

(5) Type of M:

Thm. Let $e_1, \ldots, e_n$ be a ONF of ∇-parallel VFs. Then:

- $\mathcal{R}^g(e_i, e_j)e_k = -\frac{1}{4}[[e_i, e_j], e_k]$ [$\Rightarrow$ M is Einstein]

- $e_m\langle[e_i, e_j], e_k\rangle = -(\nabla_{e_m}T)(e_i, e_j, e_k) = -\frac{1}{3}\sigma_T(e_i, e_j, e_k, e_m)$ $\quad (*)$

Cor. $e_i(R_{jklm}) = 0$, hence $\nabla^g\mathcal{R}^g = 0$ and, by (2), $\nabla\mathcal{R}^g = 0$ and

$$(\nabla_X - \nabla^g_X)\mathcal{R}^g = [X \lrcorner T, \mathcal{R}^g] = 0 \qquad (**)$$

Cor. (M, g) is a compact symmetric Einstein space.

1st case: $\sigma_T = 0$. $(*)$ $\Rightarrow$ all $\langle[e_i, e_j], e_k\rangle$ = const $\Rightarrow$ M is Lie group

2nd case: $\sigma_T \neq 0$ $(n > 4)$. Consider the Lie algebra

$$\mathfrak{g}_T \;:=\; \mathrm{Lie}\langle X \lrcorner T | X \in T_pM \rangle \subset \Lambda^2 T_pM \;\cong\; \mathfrak{so}(T_pM).$$

By the splitting principle, may assume: $\mathfrak{g}_T$ acts irreducibly on T_pM. Let $G_T \subset \mathrm{SO}(n)$ be the corresponding Lie group.

$\Rightarrow (G_T, T_pM, T)$ is an irred. STHS!

Thm (STHT). There are only two possible cases:

(1) G_T does not act not transitively on S:

$T(X,Y) =: [X,Y]$ defines a Lie bracket and M is a Lie group,

(2) or G_T acts transitively on S:

then $\mathfrak{g}_T = \mathfrak{so}(T_pM)$.

Cor. If M is not a Lie group, $\mathfrak{g}_T = \mathfrak{so}(T_pM)$ and

$$(**) \Rightarrow \mathcal{R}^g = c \cdot \mathrm{Id} \Rightarrow K^g(X,Y) = c \cdot \mathrm{Id}$$

$\Rightarrow$ M is a sphere

$\Rightarrow$ formula for $K^g(X,Y)$ states that T defines a vector cross product

$$\Rightarrow \quad \boxed{M = S^7}$$

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– After description of flat mnfds: what does Einstein mean for skew torsion? –

Einstein manifolds – the classical case

A Riemannian mnfd (M^n, g) is called Einstein if $\mathrm{Ric}^g = c \cdot g$, $c \in C(M)$.

- Fact: $c = \mathrm{Scal}^g/n$ and has to be constant (for $n \geq 3$)
- Einstein metrics are vacuum solutions of **eq. of general relativity**
- If M^n is compact $\Rightarrow$ Einstein metrics are critical points of the total scalar curvature functional $\displaystyle\int_M \mathrm{Scal}(g) d\mathrm{vol}_g$.

A few general results:

- $\dim = 4$ **:** If M^4 compact, oriented admits an Einstein metric $\Rightarrow \chi(M) \geq \frac{3}{2}|\tau(M)|$. [Hitchin/Thorpe, 1969/74]
- (M, g) Einstein, complete, $\mathrm{Scal}^g > 0 \Rightarrow$ compact and $\pi_1(M)$ finite
- $\dim \geq 5$ **:** No (further) topological obstructions are known.

Link to special geometric structures and differential eqs.

Many known Einstein metrics carry additional geometric structure:

Ex.1 $\mathbb{C}P^3 = SU(4)/S(U(1)\times U(3))$ and $\mathbb{F}^3 = SU(3)/T^2$:

= 3-symmetric spaces = twistor spaces of S^4 resp. $\mathbb{C}P^2$

$\Rightarrow \exists$ **2** Einstein metrics: **1 Kähler** & **1 nearly-Kähler**

Ex.2 $V_2(\mathbb{R}^4) = SO(4)/SO(2) = T_1 S^3 \Rightarrow$ **1 Sasaki-Einstein metric**

Common properties of $\mathbb{C}P^3, \mathbb{F}^3$ and $V_2(\mathbb{R}^4)$:

- spin manifolds which carry Killing spinors (KS) ψ : $\nabla^g_X \psi = kX\cdot\psi$.

- KS ψ realize equality case in **Friedrich's eigenvalue estimate** for the Riemannian Dirac operator D^g on compact spin mnfds: [Friedrich, 80]

$$\lambda^2(D^g) \ \geq\ \frac{n}{4(n-1)} \min_{x\in M^n} \mathrm{Scal}^g$$

Comparison of curvatures

Starting point: Compare ∇ and ∇^g curvatures:

Difference tensor: $S(X,Y) := \sum_{i,j=1}^{n} T(e_i, X, e_j) T(e_i, Y, e_j)$ (symmetric)

Curvature: $\mathrm{Ric}^{\nabla}(X,Y) = \mathrm{Ric}^g(X,Y) - \frac{1}{4}S(X,Y) - \frac{1}{2}\delta T(X,Y)$

$$s^{\nabla} = s^g - \frac{3}{2}\|T\|^2$$

- δT measures the skew symmetric part of Ric^{∇},
(recall: $\nabla T = 0 \Rightarrow \delta T = 0$)

- denote by $S(\mathrm{Ric}^{\nabla})$ the symmetric part of the Ricci tensor

Einstein manifolds – the skew torsion case

Variational principle: The critical points of [This section: A-Ferreira, 2012]

$$\int_M [s^\nabla - 2\Lambda]\, d\mathrm{vol}_g$$

are pairs (g, T) satisfying $S(\mathrm{Ric}^\nabla) = (s^\nabla/2 - \Lambda)g$. As in the Riemannian case, taking the trace then implies $s^\nabla/2 - \Lambda = s^\nabla/n$.

Dfn. (M, g, T) is

– 'Einstein with skew torsion' if the connection ∇ with torsion T satisfies $S(\mathrm{Ric}^\nabla) = (s^\nabla/n)g$,

– 'Einstein with parallel skew torsion' if it satisfies in addition $\nabla T = 0$.

Exa. $M = S^3$ with standard metric: Einstein, $\mathrm{Scal}^g = 6$, parallelizable. $f : S^3 \longrightarrow \mathbb{R}$ any non-constant function, $T := 2fe^1 \wedge e^2 \wedge e^3$. Then

• ∇ Einstein with skew torsion, scalar curvature: $s^\nabla = 6(1 - f(x)^2)$: not constant, any sign possible (even on compact mnfds).

Einstein manifolds with skew torsion: Topology

Q: What is a good condition on torsion T that ensures the same properties as in the Riemannian case?

Thm. Assume (M, g, T) is Einstein with *parallel* skew torsion. Then

1) Scal^∇ and Scal^g are constant

2) If M complete connected and $\mathrm{Scal}^\nabla > 0$, then M is compact and $\pi_1(M)$ is finite

Proof.

1): Clever computation with divergences of $\mathrm{Ric}^g, \mathrm{Ric}^\nabla$ and ds^∇, ds^g.

2): Check conditions of Bonnet-Myers Thm, i. e. $\mathrm{Ric}^g(X, X) \geq c\|X\|^2$ for some $c > 0$ and all $X \in TM$. But

$$\mathrm{Ric}^g(X, X) = \mathrm{Ric}^\nabla(X, X) + \frac{1}{4}S(X, X) = \frac{s^\nabla}{n}\|X\|^2 + \frac{1}{4}S(X, X) \geq \frac{s^\nabla}{n}\|X\|^2$$

Dfn. Call T of 'Einstein type' if $S = c \cdot g$:

If ∇^g is Riemannian Einstein, ∇ will then be Einstein with skew torsion.

Lemma. Write $T = \sum_{ijk} T_{ijk} e_{ijk}$. T is of Einstein type iff

- no term of the form $T_{ija} e_{ija} + T_{ijb} e_{ijb}$ with $a = b$ occurs;
- if i and j are two indices in $1, \ldots, n$ then the number of occurrences of i and j in T coincides;
- if $\{i, j, k\}$ and $\{a, b, c\}$ are two sets of indices then $T^2_{ijk} = T^2_{abc}$.

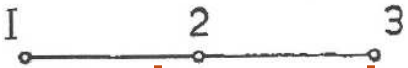
$\rightarrow$ easy procedure for producing further examples of ∇-Einstein metrics for manifolds that are parallelizable and carry an Einstein metric

Normal forms of 3-forms under $\mathrm{GL}(n, \mathbb{R})$: Schouten 1931, Westwick 1981:

Riemannian Einstein manifolds (M, g) will never be Einstein with skew torsion in dimensions 4 and 5.

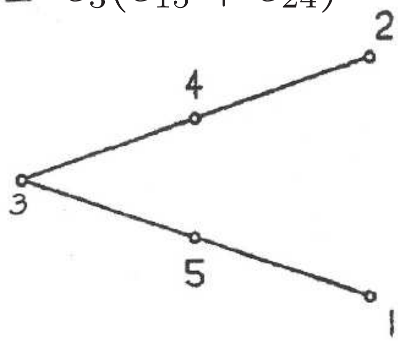
Normal forms of 3-forms ($n \leq 7$)

I

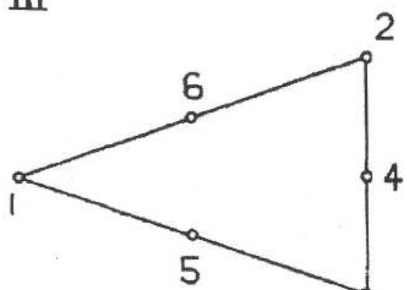


e_{123} [ET $n = 3$]

II $e_3(e_{15} + e_{24})$

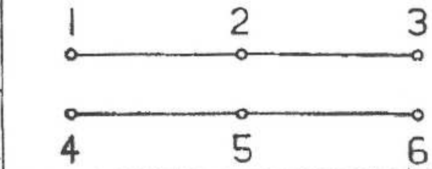


III

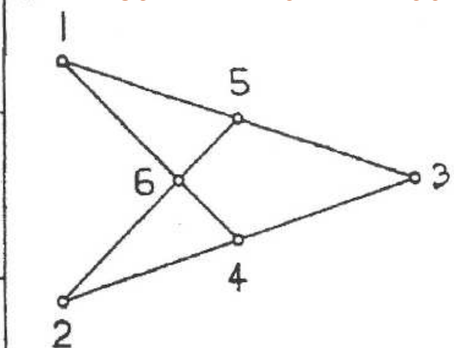


$e_{126} + e_{135} + e_{234}$

IV.* $e_{123} + e_{456}$ [ET $n = 6$]



V.* $e_{135} + e_{146} + e_{256} + e_{234}$ [ET $n = 6$]



VI. $e_1(e_{23} + e_{45}) + e_{267}$

VII. $e_{123} + e_{456} + e_7(e_2 + e_5)(e_3 + e_6)$

VIII. $e_1(e_{23} + e_{45} + e_{67})$

IX.** VIII $+ e_{246}$,

X. VII $+ e_{147}$

XI. VIII $+ e_2(e_{46} - e_{57})$

XII. $e_1(e_{45} + e_{67}) + e_2(e_{46} - e_{57}) + e_3(e_{47} + e_{56})$,

XIII.** XII $-e_{123}$ [ET $n = 7$]

Outlook:

• (M, g, J) 4-dim. compact Hermitian non-Kähler mnfd, Einstein with parallel skew torsion, its universal cover is isometric to $\mathbb{R} \times S^3$.

• For $n = 4$, $\exists$ alternative approach through decomposition of curvature tensor; the Hitchin-Thorpe ineq. still holds (for compact. oriented) and for parallel torsion, the notions coincide [Ferreira, 2011]

• All nearly Kähler mnfds ($n = 6$) and nearly parallel G_2 mnfds ($n = 7$) are Einstein with parallel skew torsion ($\mathrm{Scal}^{\nabla} > 0$);

• Any Einstein-Sasaki mnfd admits a deformation of the metric that is Einstein with parallel skew torsion and $\mathrm{Ric}^{\nabla} = 0$

[⇒ many homogeneous examples of ∇-Ricci flat manifolds which are not flat, as opposed to the Riemannian case!]

• Every 7-dim. 3-Sasaki mnfd carries 3 different connections that turn it into an Einstein manifold with parallel skew torsion; it admits a deformation of the metric that carries an Einstein structure with parallel skew torsion. [strongly related to canonical connection]

Literature

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