Non-integrable geometries, torsion, and holonomy
III: Curvature properties of connections with skew torsion

Prof. Dr. habil. Ilka Agricola
Philipps-Universität Marburg

Torino, Carnival Differential Geometry school
Connections with parallel skew torsion

$(M, g)$ Riemannian mnfd, $\nabla$ a connection with skew torsion $T \in \Lambda^3(M)$

- Know already large families of such manifolds where $\nabla T = 0$ holds: nearly Kähler mnfds, nearly parallel $G_2$-mnfds, Sasaki mnfds, naturally reductive homogeneous spaces.

**Dfn.** For any $T \in \Lambda^3(M)$, define $(e_1, \ldots, e_n$ a local ONF)

$$\sigma_T := \frac{1}{2} \sum_{i=1}^{n} (e_i \perp T) \wedge (e_i \perp T) = \bigotimes_{X,Y,Z} g(T(X,Y), T(Z,V)) \quad (= 0 \text{ if } n \leq 4)$$

[Exa: For $T = \alpha e_{123} + \beta e_{456}, \sigma_T = 0$; for $T = (e_{12} + e_{34}) e_5, \sigma_T = -e_{1234}$]

$\sigma_T$ measures the ‘degeneracy’ of $T$ and appears in many import. rel.:

* 1st Bianchi identity
* $T^2 = -2\sigma_T + \|T\|^2$ in the Clifford algebra
* If $\nabla T = 0$: $dT = 2\sigma_T$, $\nabla^g T = \frac{1}{2} \sigma_T$, $\delta T = 0$. . .
  either $\sigma_T = 0$ or $\text{hol} \nabla \subset \text{iso}(T)$ is non-trivial
Flat metric connections with antisymmetric torsion

Suppose $\nabla$ is metric and has antisymmetric torsion $T \in \Lambda^3(M)$,

$$\nabla_X Y = \nabla^g_X Y + \frac{1}{2} T(X,Y).$$

Q: What are the manifolds with a flat metric connection with antisymmetric torsion?

We shall [this section: A-Fr, 2010]

- discuss a family of easy, yet interesting examples
- discuss a less simple, isolated example
- show that these are all such manifolds (up to coverings and products)

Before the proof, I shall sketch the different approaches to the problem and its history.

Assume simply connected where needed.
Flat connections

Dfn. \( \nabla \) is called \textit{flat}, if \( \mathcal{R}(X, Y) = 0 \) for all \( X, Y \)

\[ \iff \nabla : TM \to \text{End}(TM), \; X \mapsto \nabla_X \text{ is Lie algebra homomorphism} \]

\[ \iff \text{By Ambrose-Singer Thm } (\gamma \in C(p), P_\gamma : T_pM \to T_pM \text{ par.tr.}): \]

\[ 0 = \text{hol}(\nabla, p) = \langle P_\gamma^{-1} \circ \mathcal{R}(P_\gamma V, P_\gamma W) \circ P_\gamma \rangle \subset \mathfrak{so}(T_pM), \]

i.e. \( \text{Hol}(p; \nabla) \) is a discrete group

\[ \iff \text{parallel transport is path-independent} \]

\[ \Rightarrow (M, g) \text{ is parallelisable and therefore spin} \]
If $\nabla = \nabla^g$ the LC connection, Frobenius’ Theorem implies:

**Thm.** If $\nabla^g$ is flat, there exists in the vicinity of every $p \in M$ a chart s.t. the coefficients of the Riemannian metric are

$$g = \text{diag}(1, 1, \ldots, 1).$$

– the proof relies on Cartan’s structure equations and breaks down for connections with torsion.

Hence, $(M, g)$ looks locally like $\mathbb{R}^n$ with the euclidian metric.

Globally, of course more is possible, for example for $n = 2$:
Example 1: Lie groups

Let $M = G$ be a connected Lie group, $\mathfrak{g} = T_e G$, with a biinvariant metric.

**Easy:** $\nabla^g_X Y = \frac{1}{2} [X, Y]$.

** Ansatz:** $T$ proportional to $[\cdot, \cdot]$, i.e. $\nabla_X Y = \lambda [X, Y]$.

- **torsion:** $T^\nabla (X, Y) = (2\lambda - 1) [X, Y]$, hence $T \in \Lambda^3(G)$

- **curvature:**

  $\mathcal{R}^\nabla (X, Y) Z = \lambda (1 - \lambda) [Z, [X, Y]] = \begin{cases} 
  \frac{1}{4} [Z, [X, Y]] & \text{for LC conn. (} \lambda = \frac{1}{2} \text{)} \\
  0 & \text{for } \lambda = 0, 1
\end{cases}$

  [±-connection, Cartan-Schouten, 1926]

- ±-connection satisfies $\sigma_T = 0$ and $\nabla T = 0$ (hence $dT = 0$).
Example 2: \( S^7 \)

- only parallelisable sphere that is not a Lie group (but almost. . . )

Consider spin representation \( \kappa^\mathbb{C} : \text{Spin}(7) \to \text{End}(\Delta^\mathbb{C}_7), \quad \Delta^\mathbb{C}_7 \cong \mathbb{C}^8. \)

In dim.7, this turns out to be complexification of 8-dim. real rep.,

\[ \kappa : \text{Spin}(7) \to \text{End}(\Delta_7), \quad \Delta_7 \cong \mathbb{R}^8. \]

\( \kappa \) is in fact a repr. of the Clifford algebra over \( \mathbb{R}^7 \) (\( \text{Spin}(7) \subset \text{Cl}(\mathbb{R}^7) ! \)),

\[ \kappa : \mathbb{R}^7 \subset \text{Cl}(\mathbb{R}^7) \to \text{End}(\Delta_7). \]

Choose \( e_1, \ldots, e_7 \) an ON basis of \( \mathbb{R}^7 \), and set \( \kappa_i = \kappa(e_i) \).

- Embed \( S^7 \subset \Delta_7 \) as spinors of length 1,

- define VFs on \( S^7 \) by \( V_i(x) = \kappa_i \cdot x \) for all \( x \in S^7 \subset \Delta^7 \)
Properties of the VF
s \( V_i(x) = \kappa(e_i) \cdot x \)

**Thm.** (1) These vector fields realize a ON trivialization of \( S^7 \),

[computation rules for Clifford multipl.]

(2) the connection \( \nabla \) defined by \( \nabla V_i = 0 \) is metric, flat, and with torsion

\[
T(V_i, V_j, V_k)(x) = -\langle [V_i, V_j], V_k \rangle = 2\langle \kappa_i \kappa_j \kappa_k x, x \rangle \in \Lambda^3(S^7),
\]

(3) \( \nabla T \neq 0 \) (check that \( T \) does not have constant coefficients), \( \sigma_T \neq 0 \)

(4) \( \nabla \) is a \( G_2 \) connection of Fernandez-Gray type \( \mathcal{X}_1 \oplus \mathcal{X}_3 \oplus \mathcal{X}_4 \).
Classification

**Goal:** Show that any irreducible, complete, and simply connected $M$ with a flat, metric connection with antisymmetric torsion $T \in \Lambda^3(M)$ is one of these examples.

- 1926: Cartan-Schouten “On manifolds with absolute parallelism” – wrong proof.
- 1968: d’Atri-Nickerson “On the existence of special orthonormal frames” – when does $(M, g)$ admit an ONF of Killing vectors?

This is mainly an equivalent problem:

$V$ is Killing VF $\iff g(\nabla^g_X V, Y) + g(X, \nabla_Y V) = 0$ \hspace{1cm} (*)

If $V$ is parallel for $\nabla$ with torsion $T$, then $\nabla^g_X V = -\frac{1}{2} T(X, V)$, hence

$\left( (*) \iff g(T(X, V), Y) + g(X, T(Y, V)) = 0 \iff T \in \Lambda^3(M) \right)$


**Q:** Both proofs rely on classification of symmetric spaces. Direct proof?
Sketch of proof

(1) General identities: [common to all authors]

- \( \text{Ric}^g(X, Y) = \frac{1}{4} \sum_i \langle T(X, e_i), T(Y, e_i) \rangle, \quad (\Rightarrow \text{Ric}^g(X, X) \geq 0) \)
- \( K^g(X, Y) = \frac{\|T(X,Y)\|^2}{4[\|X\|^2\|Y\|^2 - \langle X, Y \rangle^2]} \geq 0 \) (sectional curvature)
- \( \delta T = 0 \) (= antisymmetric part of \( \text{Ric}^\nabla \))

(2) General tools: \( \sigma_T = \frac{1}{2} \sum_i (e_i \cdot T) \wedge (e_i \cdot T) \in \Lambda^4(M) \) satisfies

- \( T^2 = -2\sigma_T + \|T\|^2 \) (as endomorphisms on \( \Delta_7 \))
- \( \nabla T = 0 \) implies \( dT = 2\sigma_T \) [recall: true for \( G \), wrong for \( S^7 \)]
- All spinors with constant coeff. are parallel \( \Rightarrow 3dT = 2\sigma_T \) (SL formula)
- Bianchi I:
  \[
  X, Y, Z \implies \mathcal{R}(X, Y, Z, V) = dT(X, Y, Z, V) - \sigma^T(X, Y, Z, V) + (\nabla_V T)(X, Y, Z)_{10}
  \]
(3) Rescaling of connection: [implicit in Cartan]

Consider the rescaled connection $\nabla^{1/3}$,

$$
\nabla^{1/3} X Y = \nabla^g X Y + \frac{1}{6} T(X, Y)
$$

$\nabla^{1/3}$ plays a prominent role for Dirac operators with torsion

**Thm.**

- $\nabla^{1/3} T = 0$ ($\iff \nabla V T = -\frac{1}{3} V \perp \sigma_T \iff \nabla^g V T = \frac{1}{6} V \perp \sigma_T$)

In particular, $\|T\|$ and the scalar curvature are constant, and for any tensor field $T$ polynomial in $T$:

$$
\nabla T = -2 \nabla^g T; \text{ in particular: } \nabla T = 0 \iff \nabla^g T = 0
$$

- $\nabla^{1/3} R^g = 0$

By the Ambrose-Singer Thm, $M$ is a naturally reductive space (in particular, homogeneous).
(4) Splitting principle:

**Thm.** Let $M = M_1 \times M_2$ be a manifold with a flat metric connection $\nabla$ with torsion $T \in \Lambda^3(M)$. Then $T = T_1 + T_2$ with $T_i \in \Lambda^3(M_i)$.

(5) Type of $M$:

**Thm.** Let $e_1, \ldots, e_n$ be a ONF of $\nabla$-parallel VFs. Then:

- $\mathcal{R}^g(e_i, e_j)e_k = -\frac{1}{4}[[e_i, e_j], e_k] \ [\Rightarrow \ M \ is \ Einstein]$
- $e_m\langle [e_i, e_j], e_k \rangle = -(\nabla_{e_m} T)(e_i, e_j, e_k) = -\frac{1}{3}\sigma_T(e_i, e_j, e_k, e_m) \ (\ast)$

**Cor.** $e_i(R_{jklm}) = 0$, hence $\nabla^g \mathcal{R}^g = 0$ and, by (2), $\nabla \mathcal{R}^g = 0$ and

$$(\nabla_X - \nabla^g_X) \mathcal{R}^g = [X \lhd T, \mathcal{R}^g] = 0 \ (\ast\ast)$$

**Cor.** $(M, g)$ is a compact symmetric Einstein space.

1st case: $\sigma_T = 0. \ (\ast) \Rightarrow \ all \ \langle [e_i, e_j], e_k \rangle = const \Rightarrow M \ is \ Lie \ group$
2nd case: $\sigma_T \neq 0 \ (n > 4)$. Consider the Lie algebra

$$ g_T := \text{Lie}\langle X \perp T | X \in T_p M \rangle \subset \Lambda^2 T_p M \cong so(T_p M). $$

By the splitting principle, may assume: $g_T$ acts irreducibly on $T_p M$. Let $G_T \subset SO(n)$ be the corresponding Lie group.

$$ \Rightarrow (G_T, T_p M, T) \text{ is an irred. STHS!} $$

Thm (STHT). There are only two possible cases:

(1) $G_T$ does not act not transitively on $S$:

$T(X, Y) =: [X, Y]$ defines a Lie bracket and $M$ is a Lie group,

(2) or $G_T$ acts transitively on $S$:

then $g_T = so(T_p M)$. 
Cor. If $M$ is not a Lie group, $\mathfrak{g}_T = \mathfrak{so}(T_pM)$ and

$$(* *) \Rightarrow R^g = c \cdot \text{Id} \Rightarrow K^g(X, Y) = c \cdot \text{Id}$$

$\Rightarrow M$ is a sphere

$\Rightarrow$ formula for $K^g(X, Y)$ states that $T$ defines a vector cross product

$$\Rightarrow \boxed{M = S^7}$$

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– After description of flat mnfds: what does Einstein mean for skew torsion? –
Einstein manifolds – the classical case

A Riemannian mnfd $(M^n, g)$ is called Einstein if $\text{Ric}^g = c \cdot g$, $c \in C(M)$.

- Fact: $c = \text{Scal}^g / n$ and has to be constant (for $n \geq 3$)

- Einstein metrics are vacuum solutions of eq. of general relativity

- If $M^n$ is compact $\Rightarrow$ Einstein metrics are critical points of the total scalar curvature functional $\int_M \text{Scal}(g) d\text{vol}_g$.

A few general results:

- dim $= 4$ : If $M^4$ compact, oriented admits an Einstein metric

  $\Rightarrow \chi(M) \geq \frac{3}{2} |\tau(M)|$. [Hitchin/Thorpe, 1969/74]

- $(M, g)$ Einstein, complete, $\text{Scal}^g > 0 \Rightarrow$ compact and $\pi_1(M)$ finite

- dim $\geq 5$ : No (further) topological obstructions are known.
Link to special geometric structures and differential eqs.

Many known Einstein metrics carry additional geometric structure:

**Ex.1** $\mathbb{C}P^3 = SU(4)/S(U(1) \times U(3))$ and $\mathbb{F}^3 = SU(3)/T^2$:

= 3-symmetric spaces = twistor spaces of $S^4$ resp. $\mathbb{C}P^2$

$\Rightarrow \exists 2$ Einstein metrics: 1 Kähler & 1 nearly-Kähler

**Ex.2** $V_2(\mathbb{R}^4) = SO(4)/SO(2) = T_1S^3 \Rightarrow 1$ Sasaki-Einstein metric

Common properties of $\mathbb{C}P^3, \mathbb{F}^3$ and $V_2(\mathbb{R}^4)$:

- spin manifolds which carry Killing spinors (KS) $\psi : \nabla^g_X \psi = kX \cdot \psi$.

- KS $\psi$ realize equality case in **Friedrich’s eigenvalue estimate** for the Riemannian Dirac operator $D^g$ on compact spin mnfds: [Friedrich, 80]

$$\lambda^2(D^g) \geq \frac{n}{4(n-1)} \min_{x \in M^n} \text{Scal}^g$$
**Comparison of curvatures**

**Starting point:** Compare $\nabla$ and $\nabla^g$ curvatures:

Difference tensor: $S(X, Y) := \sum_{i,j=1}^{n} T(e_i, X, e_j)T(e_i, Y, e_j)$ (symmetric)

Curvature: $\text{Ric}^\nabla(X, Y) = \text{Ric}^g(X, Y) - \frac{1}{4}S(X, Y) - \frac{1}{2}\delta T(X, Y)$

$s^\nabla = s^g - \frac{3}{2}\|T\|^2$

- $\delta T$ measures the skew symmetric part of $\text{Ric}^\nabla$,
  (recall: $\nabla T = 0 \Rightarrow \delta T = 0$)

- denote by $S(\text{Ric}^\nabla)$ the symmetric part of the Ricci tensor
Einstein manifolds – the skew torsion case

Variational principle: The critical points of

$$\int_M [s\nabla - 2\Lambda] \, d\text{vol}_g$$

are pairs \((g, T)\) satisfying \(S(\text{Ric}^{\nabla}) = (s\nabla / 2 - \Lambda)g\). As in the Riemannian case, taking the trace then implies \(s\nabla / 2 - \Lambda = s\nabla / n\).

Dfn. \((M, g, T)\) is

– ‘Einstein with skew torsion’ if the connection \(\nabla\) with torsion \(T\) satisfies \(S(\text{Ric}^{\nabla}) = (s\nabla / n)g\),

– ‘Einstein with parallel skew torsion’ if it satisfies in addition \(\nabla T = 0\).

Exa. \(M = S^3\) with standard metric: Einstein, \(\text{Scal}^g = 6\), parallelizable. \(f : S^3 \rightarrow \mathbb{R}\) any non-constant function, \(T := 2f e^1 \wedge e^2 \wedge e^3\). Then

- \(\nabla\) Einstein with skew torsion, scalar curvature: \(s\nabla = 6(1 - f(x)^2)\): not constant, any sign possible (even on compact mnfds).
Einstein manifolds with skew torsion: Topology

**Q:** What is a good condition on torsion $T$ that ensures the same properties as in the Riemannian case?

**Thm.** Assume $(M, g, T)$ is Einstein with parallel skew torsion. Then

1) $\text{Scal}^\nabla$ and $\text{Scal}^g$ are constant

2) If $M$ complete connected and $\text{Scal}^\nabla > 0$, then $M$ is compact and $\pi_1(M)$ is finite

**Proof.**

1): Clever computation with divergences of $\text{Ric}^g, \text{Ric}^\nabla$ and $ds^\nabla, ds^g$.

2): Check conditions of Bonnet-Myers Thm, i.e. $\text{Ric}^g(X, X) \geq c\|X\|^2$ for some $c > 0$ and all $X \in TM$. But

$$\text{Ric}^g(X, X) = \text{Ric}^\nabla(X, X) + \frac{1}{4}S(X, X) = \frac{s^\nabla}{n}\|X\|^2 + \frac{1}{4}S(X, X) \geq \frac{s}{n}\|X\|^2$$
**Dfn.** Call $T$ of ‘Einstein type’ if $S = c \cdot g$:

If $\nabla^g$ is Riemannian Einstein, $\nabla$ will then be Einstein with skew torsion.

**Lemma.** Write $T = \sum_{ijk} T_{ijk} e_{ijk}$. $T$ is of Einstein type iff

- no term of the form $T_{ija} e_{ija} + T_{ijb} e_{ijb}$ with $a = b$ occurs;
- if $i$ and $j$ are two indices in $1, \ldots, n$ then the number of occurrences of $i$ and $j$ in $T$ coincides;
- if $\{i, j, k\}$ and $\{a, b, c\}$ are two sets of indices then $T^2_{ijk} = T^2_{abc}$.

→ easy procedure for producing further examples of $\nabla$-Einstein metrics for manifolds that are parallelizable and carry an Einstein metric

Normal forms of 3-forms under $GL(n, \mathbb{R})$: Schouten 1931, Westwick 1981:

Riemannian Einstein manifolds $(M, g)$ will never be Einstein with skew torsion in dimensions 4 and 5.
Normal forms of 3-forms \((n \leq 7)\)

I. \(e_{123} \text{ [ET } n = 3\text{]}\)

II. \(e_3(e_{15} + e_{24})\)

III. \(e_{126} + e_{135} + e_{234}\)

IV. \(e_{123} + e_{456} \text{ [ET } n = 6\text{]}\)

V. \(e_{135} + e_{146} + e_{256} + e_{234} \text{ [ET } n = 6\text{]}\)

VI. \(e_1(e_{23} + e_{45}) + e_{267}\)

VII. \(e_{123} + e_{456} + e_7(e_2 + e_5)(e_3 + e_6)\)

VIII. \(e_1(e_{23} + e_{45} + e_{67})\)

IX. ** VIII + e_{246},

X. VII + e_{147}

XI. VIII + e_2(e_{46} - e_{57})

XII. \(e_1(e_{45} + e_{67}) + e_2(e_{46} - e_{57}) + e_3(e_{47} + e_{56}),\)

XIII. ** XII − e_{123} \text{ [ET } n = 7\text{]}\)
Outlook:

- $(M, g, J)$ 4-dim. compact Hermitian non-Kähler mnfd, Einstein with parallel skew torsion, its universal cover is isometric to $\mathbb{R} \times S^3$.

- For $n = 4$, $\exists$ alternative approach through decomposition of curvature tensor; the Hitchin-Thorpe ineq. still holds (for compact. oriented) and for parallel torsion, the notions coincide \textsuperscript{[Ferreira, 2011]}

- All nearly Kähler mnfds ($n = 6$) and nearly parallel $G_2$ mnfds ($n = 7$) are Einstein with parallel skew torsion ($\text{Scal}^\nabla > 0$);

- Any Einstein-Sasaki mnfd admits a deformation of the metric that is Einstein with parallel skew torsion and $\text{Ric}^\nabla = 0$

\textsuperscript{[⇒ many homogeneous examples of $\nabla$-Ricci flat manifolds which are not flat, as opposed to the Riemannian case!]} 

- Every 7-dim. 3-Sasaki mnfd carries 3 different connections that turn it into an Einstein manifold with parallel skew torsion; it admits a deformation of the metric that carries an Einstein structure with parallel skew torsion. \textsuperscript{[strongly related to canonical connection]}
Literature

