Non-integrable geometries, torsion, and holonomy
IV: Classification of naturally reductive homogeneous spaces

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Naturally reductive homogeneous spaces

Traditional approach:

\((M, g)\) a Riemannian manifold, \(M = G/H\) s. t. \(G\) is a group of isometries acting transitively and effectively

**Dfn.** \(M = G/H\) is *naturally reductive* if \(\mathfrak{h}\) admits a reductive complement \(\mathfrak{m}\) in \(g\) s. t.

\[
\langle [X, Y]_\mathfrak{m}, Z \rangle + \langle Y, [X, Z]_\mathfrak{m} \rangle = 0 \quad \text{for all } X, Y, Z \in \mathfrak{m}, \quad (\ast)
\]

where \(\langle - , - \rangle\) denotes the inner product on \(\mathfrak{m}\) induced from \(g\).

The PFB \(G \to G/H\) induces a metric connection \(\nabla\) with torsion

\[
g(T(X, Y), Z) := T(X, Y, Z) = -\langle [X, Y]_\mathfrak{m}, Z \rangle,
\]

the so-called *canonical connection*. It always satisfies \(\nabla T = \nabla R = 0\).

**Observation:** condition \((\ast)\) \(\iff\) \(T\) is a 3-form, i. e. \(T \in \Lambda^3(M)\).
Conversely:

**Thm.** A Riemannian manifold equipped with a [regular] homogeneous structure, i.e. a metric connection $\nabla$ with torsion $T$ and curvature $R$ such that $\nabla R = 0$ and $\nabla T = 0$, is locally isometric to a homogeneous space. [Ambrose-Singer, 1958, Tricerri 1993]

However, a classification in all dimensions is impossible!

**Main pb:** No invariant theory for $\Lambda^3(\mathbb{R}^n)$ under $SO(n)$ for $n \geq 6$

- Use *torsion* (instead of curvature) as basic geometric quantity, find a *G-structure* inducing the nat. red. structure

**In this talk:** General strategy, some general results, classification for $n \leq 6$
[joint work with Ana C. Ferreira, Th. Friedrich]
Set-up: $(M, g)$ Riemannian mnfd, $\nabla$ metric conn., $\nabla^g$ Levi-Civita conn.

$$T(X, Y, Z) = g(\nabla_X Y - \nabla_Y X - [X, Y], Z) \in \Lambda^3(M^n)$$

$$\nabla_X Y = \nabla^g_X Y + \frac{1}{2} T(X, Y, -)$$

$(M, g, T)$ carries nat. red. homog. structure if $\nabla \mathcal{R} = 0$ and $\nabla T = 0$

Obviously:

| nat.red.homog. Riemannian mnfds | (homogeneous) Riemannian mnfds with parallel skew torsion |
Review of some classical results

- all isotropy irreducible homogeneous manifolds are naturally reductive

- the $\pm$-connections on any Lie group with a biinvariant metric are naturally reductive (and, by the way, flat) \[\text{[Cartan-Schouten, 1926]}\]

- construction / classification (under some assumptions) of left-invariant naturally reductive metrics on compact Lie groups \[\text{[D’Atri-Ziller, 1979]}\]

- All 6-dim. homog. nearly Kähler mnfds (w.r.t. their canonical almost Hermitian structure) are naturally reductive. These are precisely: $S^3 \times S^3$, $\mathbb{CP}^3$, the flag manifold $F(1, 2) = U(3)/U(1)^3$, and $S^6 = G_2/SU(3)$.

- Known classifications:

  - dimension 3 \[\text{[Tricerri-Vanhecke, 1983]}\], dimension 4 \[\text{[Kowalski-Vanhecke, 1983]}\], dimension 5 \[\text{[Kowalski-Vanhecke, 1985]}\]

These proceed by finding normal forms for the curvature operator, more details to follow later.
An important tool: the 4-form $\sigma_T$

**Dfn.** For any $T \in \Lambda^3(M)$, define $(e_1, \ldots, e_n$ a local ONF)

$$\sigma_T := \frac{1}{2} \sum_{i=1}^{n} (e_i \downarrow T) \wedge (e_i \downarrow T) = \sum_{X,Y,Z} g(T(X,Y), T(Z,V))$$

- $\sigma_T$ measures the ‘degeneracy’ of $T$ and, if non degenerate, induces the geometric structure on $M$
- $\sigma_T$ appears in many important relations:
  
  * 1st Bianchi identity: $\mathcal{R}(X, Y, Z, V) = \sigma_T(X, Y, Z, V)$
  * $T^2 = -2\sigma_T + \|T\|^2$ in the Clifford algebra
  * If $\nabla T = 0$: $dT = 2\sigma_T$ and $\nabla^g T = \frac{1}{2} \sigma_T$
    
    either $\sigma_T = 0$ or $\text{hol} \nabla \subset \text{iso}(T)$ is non-trivial
\( \sigma_T \) and the Nomizu construction

**Idea:** for \( M = G/H \), reconstruct \( g \) from \( \mathfrak{h} \), \( T \), \( \mathcal{R} \) and \( V \cong T_x M \)

**Set-up:** \( \mathfrak{h} \) a real Lie algebra, \( V \) a real f.d. \( \mathfrak{h} \)-module with \( \mathfrak{h} \)-invariant pos. def. scalar product \( \langle \cdot, \cdot \rangle \), i. e. \( \mathfrak{h} \subset \mathfrak{so}(V) \cong \Lambda^2 V \)

\( \mathcal{R} : \Lambda^2 V \to \mathfrak{h} \) an \( \mathfrak{h} \)-equivariant map, \( T \in (\Lambda^3 V)^{\mathfrak{h}} \) an \( \mathfrak{h} \)-invariant 3-form,

Define a Lie algebra structure on \( g := \mathfrak{h} \oplus V \) by \( (A, B \in \mathfrak{h}, X, Y \in V) \):

\[
[A + X, B + Y] := ([A, B]_\mathfrak{h} - \mathcal{R}(X, Y)) + (AY - BX - T(X, Y))
\]

Jacobi identity for \( g \) \iff

- \( X, Y, Z \)
  - \( \mathcal{S} \quad \mathcal{R}(X, Y, Z, V) = \sigma_T(X, Y, Z, V) \) (1st Bianchi condition)
- \( X, Y, Z \)
  - \( \mathcal{S} \quad \mathcal{R}(T(X, Y), Z) = 0 \) (2nd Bianchi condition)
**Observation:** If \((M, g, T)\) satisfies \(\nabla T = 0\), then \(\mathcal{R} : \Lambda^2(M) \to \Lambda^2(M)\) is symmetric (as in the Riemannian case).

Consider \(\mathcal{C}(V) := \mathcal{C}(V, -\langle, \rangle)\): Clifford algebra, (recall: \(T^2 = -2\sigma_T + \|T\|^2\))

**Thm.** If \(\mathcal{R} : \Lambda^2 V \to \mathfrak{h} \subset \Lambda^2 V\) is symmetric, the first Bianchi condition is equivalent to \(T^2 + \mathcal{R} \in \mathbb{R} \subset \mathcal{C}(V)\) \((\iff 2\sigma_T = \mathcal{R} \subset \mathcal{C}(V))\), and the second Bianchi condition holds automatically.

Exists in the literature in various formulations: based on an algebraic identity (Kostant); crucial step in a formula of Parthasarathy type for the square of the Dirac operator (A, '03); previously used by Schoemann 2007 and Fr. 2007, but without a clear statement nor a proof.

**Practical relevance:** allows to evaluate the 1st Bianchi identity in one condition!
Splitting theorems

Dfn. For $T$ 3-form, define

- kernel: $\text{ker } T = \{ X \in TM \mid X \perp T = 0 \}$
- Lie algebra generated by its image: $\mathfrak{g}_T := \text{Lie} \langle X \perp T \mid X \in V \rangle$

$\mathfrak{g}_T$ is not related in any obvious way to the isotropy algebra of $T$!

Thm 1. Let $(M, g, T)$ be a c. s. c. Riemannian mfld with parallel skew torsion $T$. Then $\text{ker } T$ and $(\text{ker } T)^\perp$ are $\nabla$-parallel and $\nabla^g$-parallel integrable distributions, $M$ is a Riemannian product s.t.

$$(M, g, T) = (M_1, g_1, T_1 = 0) \times (M_2, g_2, T_2), \quad \text{ker } T_2 = \{0\}$$

Thm 2. Let $(M, g, T)$ be a c. s. c. Riemannian mfld with parallel skew torsion $T$ s.t. $\sigma_T = 0$, $TM = T_1 \oplus \ldots \oplus T_q$ the decomposition of $TM$ in $\mathfrak{g}_T$-irreducible, $\nabla$-par. distributions. Then all $T_i$ are $\nabla^g$-par. and integrable, $M$ is a Riemannian product, and the torsion $T$ splits accordingly

$$(M, g, T) = (M_1, g_1, T_1) \times \ldots \times (M_q, g_q, T_q)$$
A structure theorem for vanishing $\sigma_T$

**Thm.** Let $(M^n, g)$ be an irreducible, c.s.c. Riemannian manifold with parallel skew torsion $T \neq 0$ s.t. $\sigma_T = 0$, $n \geq 5$. Then $M^n$ is a simple compact Lie group with biinvariant metric or its dual noncompact symmetric space.

Key ideas: $\sigma_T = 0 \Rightarrow$ Nomizu construction yields Lie algebra structure on $TM$
use $g_T$; use STHT to show that $G_T$ is simple and acts on $TM$ by its adjoint rep.
prove that $g_T = \mathfrak{iso}(T) = \mathfrak{hol}^g$, hence acts irreducibly on $TM$, hence $M$ is an irred.
symmetric space by Berger’s Thm

**Exa.** Fix $T \in \Lambda^3(\mathbb{R}^n)$ with constant coefficients s.t. $\sigma_T = 0$. Then the flat space $(\mathbb{R}^n, g, T)$ is a reducible Riemannian manifold with parallel skew torsion and $\sigma_T = 0 \rightarrow$ assumption ‘$M$ irreducible’ is crucial! (the Riemannian manifold is decomposable, but the torsion is not)
Classification of nat. red. spaces in $n = 3$

[Tricerri-Vanhecke, 1983]

Then $\sigma_T = 0$, and the Nomizu construction can be applied directly to obtain in a few lines:

**Thm.** Let $(M^3, g, T \neq 0)$ be a 3-dim. c.s.c. Riemannian mnfld with a naturally reductive structure. Then $(M^3, g)$ is one of the following:

- $\mathbb{R}^3$, $S^3$ or $H^3$;
- isometric to one of the following Lie groups with a suitable left-invariant metric:
  
  $$SU(2), \quad \widetilde{SL}(2, \mathbb{R}), \quad \text{or the 3-dim. Heisenberg group } H^3$$

**N.B.** A general classification of mnfds with par. skew torsion is meaningless – any 3-dim. volume form of a metric connection is parallel.
Proof: \( T = \lambda e_{123}; \) \( M \) is either Einstein (\( \rightarrow \) space form) or \( \mathfrak{hol}^\nabla \) is one-dim., i.e. \( \mathfrak{hol}^\nabla = \mathbb{R} \cdot \Omega \) and \( \mathcal{R} = \alpha \Omega \odot \Omega \).

By the Nomizu construction, \( e_1, e_2, e_3 \), and \( \Omega \) are a basis of \( \mathfrak{g} \) with commutator relations

\[
[e_1, e_2] = -\alpha \Omega - \lambda e_3 =: \tilde{\Omega}, \quad [e_1, e_3] = \lambda e_2, \quad [e_2, e_3] = -\lambda e_1, \quad [\Omega, e_1] = e_2, \quad [\Omega, e_2] = -e_1, \quad [\Omega, e_3] = 0.
\]

The 3-dimensional subspace \( \mathfrak{h} \) spanned by \( e_1, e_2, \) and \( \tilde{\Omega} \) is a Lie subalgebra of \( \mathfrak{g} \) that is transversal to the isotropy algebra \( \mathfrak{k} \) (since \( \lambda \neq 0 \)). Consequently, \( M^3 \) is a Lie group with a left invariant metric. One checks that \( \mathfrak{h} \) has the commutator relations

\[
[e_1, e_2] = \tilde{\Omega}, \quad [\tilde{\Omega}, e_1] = (\lambda^2 - \alpha)e_2, \quad [e_2, \tilde{\Omega}] = (\lambda^2 - \alpha)e_1.
\]

For \( \alpha = \lambda^2 \), this is the 3-dimensional Heisenberg Lie algebra, otherwise it is \( \mathfrak{su}(2) \) or \( \mathfrak{sl}(2, \mathbb{R}) \) depending on the sign of \( \lambda^2 - \alpha \).
Classification of nat. red. spaces in $n = 4$

**Thm.** $(M^4, g, T \neq 0)$ a c. s. c. Riem. 4-mnfld with parallel skew torsion.

1) $V := \ast T$ is a $\nabla^g$-parallel vector field.

2) $\text{Hol}(\nabla^g) \subset \text{SO}(3)$, hence $M^4$ is isometric to a product $N^3 \times \mathbb{R}$, where $(N^3, g)$ is a 3-manifold with a parallel 3-form $T$.

- $T$ has normal form $T = e_{123}$, so $\dim \ker T = 1$ and 2) follows at once from our 1st splitting thm: but the existence of $V$ explains directly & geometrically the result in a few lines.

- Thm shows that the next result does not rely on the curvature or the homogeneity. Since a Riemannian product is is nat. red. iff both factors are nar. red., we conclude:

**Cor.** A 4-dim. nat. reductive Riemannian manifold with $T \neq 0$ is locally isometric to a Riemannian product $N^3 \times \mathbb{R}$, where $N^3$ is a 3-dimensional naturally reductive Riemannian manifold.

[Kowalski-Vanhecke, 1983]
Classification of nat. red. spaces in $n = 5$

Assume $(M^5, g, T \neq 0)$ is Riemannian mnfd with parallel skew torsion

- $\exists$ a local frame s.t (for constants $\lambda, \varrho \in \mathbb{R}$)

\[ T = -(\varrho e_{125} + \lambda e_{345}), \quad *T = -(\varrho e_{34} + \lambda e_{12}), \quad \sigma_T = \varrho \lambda e_{1234} \]

- **Case A:** $\sigma_T = 0$ ($\iff \varrho \lambda = 0$): apply 2nd splitting thm, $M^5$ is then loc. a product $N^3 \times N^2$ (if nat. red., $N^2$ has constant Gaussian curvature)

- **Case B:** $\sigma_T \neq 0$, two subcases:
  - * Case B.1: $\lambda \neq \varrho$, $\text{Iso}(T) = \text{SO}(2) \times \text{SO}(2)$
  - * Case B.2: $\lambda = \varrho$, $\text{Iso}(T) = \text{U}(2)$

**Recall:** Given a $G$-structure on $(M, g)$, a *characteristic connection* is a metric connection with skew torsion preserving the $G$-structure (if existent, it’s unique)
\( n = 5: \) The induced contact structure

**Case B:** \( \sigma_T \neq 0 \)

**Dfn.** A metric almost contact structure \((\varphi, \eta)\) on \((M^{2n+1}, g)\) is called \((N: \text{Nijenhuis tensor}, F(X,Y) := g(X, \varphi Y))\)

- quasi-Sasakian if \( N = 0 \) and \( dF = 0 \)
- \( \alpha \)-Sasakian if \( N = 0 \) and \( d\eta = \alpha F \) (Sasaki: \( \alpha = 2 \))

**Thm.** Let \((M^5, g, T)\) be a Riemannian 5-mnfld with parallel skew torsion \( T \) such that \( \sigma_T \neq 0 \). Then \( M \) is a quasi-Sasakian manifold and \( \nabla \) is its characteristic connection.

The structure is \( \alpha \)-Sasakian iff \( \lambda = \varrho \) (case B.2), and it is Sasakian if \( \lambda = \varrho = 2 \).

**Construction:** \( V := *\sigma_T \neq 0 \) is a \( \nabla \)-parallel Killing vector field of constant length

\[ \equiv \text{contact direction } \eta = e_5 \text{ (up to normalisation)} \]

Check: \( T = \eta \wedge d\eta \), define \( F = -(e_{12} + e_{34}) \), then prove that this works.
For $\lambda = \varrho$ (case B.2), no classification for parallel skew torsion is possible (many non-homogeneous Sasakian mnfds are known). But for

**Case B.1: $\lambda \neq \varrho$**

**Thm.** Let $(M^5, g, T)$ be Riemannian 5-manifold with parallel skew torsion s.t. $T$ has the normal form

$$T = -(\varrho e_{125} + \lambda e_{345}), \quad \varrho \lambda \neq 0 \text{ and } \varrho \neq \lambda.$$

Then $\nabla R = 0$, i.e. $M$ is locally naturally reductive, and the family of admissible torsion forms and curvature operators depends on 4 parameters.

[Use Clifford criterion to relate $R$ and $\sigma_T$]

Now one can apply the Nomizu construction to obtain the classification:
\( n = 5 \): Classification II

**Thm.** A c.s.c. Riemannian 5-mnfld \((M^5, g, T)\) with parallel skew torsion \(T = -(\rho e_{125} + \lambda e_{345})\) with \(\rho\lambda \neq 0\) is isometric to one of the following naturally reductive homogeneous spaces:

If \(\lambda \neq \rho\) (B.1):

a) The 5-dimensional Heisenberg group \(H^5\) with a two-parameter family of left-invariant metrics,

b) A manifold of type \((G_1 \times G_2)/SO(2)\) where \(G_1\) and \(G_2\) are either \(SU(2), SL(2, \mathbb{R})\), or \(H^3\), but not both equal to \(H^3\) with one parameter \(r \in \mathbb{Q}\) classifying the embedding of \(SO(2)\) and a two-parameter family of homogeneous metrics.

If \(\lambda = \rho\) (B.2): One of the spaces above or \(SU(3)/SU(2)\) or \(SU(2, 1)/SU(2)\) (the family of metrics depends on two parameters).

[Kowalski-Vanhecke, 1985]
Example: The \((2n + 1)\)-dimensional Heisenberg group

\[
H^{2n+1} = \left\{ \begin{bmatrix} 1 & x^t & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} ; \ x, y \in \mathbb{R}^n, z \in \mathbb{R} \right\} \cong \mathbb{R}^{2n+1}, \text{ local coordinates} \ x_1, \ldots, x_n, y_1, \ldots, y_n, z
\]

- Metric: parameters \( \lambda = (\lambda_1, \ldots, \lambda_n) \), all \( \lambda_i > 0 \)

\[
g_\lambda = \sum_{i=1}^{n} \frac{1}{\lambda_i} (dx_i^2 + dy_i^2) + \left[ dz - \sum_{j=1}^{n} x_j dy_j \right]^2
\]

- Contact str.: \( \eta = dz - \sum_{i=1}^{n} x_i dy_1, \varphi = \sum_{i=1}^{n} \left[ dx_i \otimes \left( \frac{\partial}{\partial y_i} + x_i \frac{\partial}{\partial z} \right) - dy_i \otimes \frac{\partial}{\partial x_i} \right] \)

- Characteristic connection \( \nabla \): torsion: \( T = \eta \wedge d\eta = -\sum_{i=1}^{n} \lambda_i \eta \wedge \alpha_i \wedge \beta_i \)

Curvature: \( R = \sum_{i \leq j}^{n} \lambda_i \lambda_j (\alpha_i \wedge \beta_i)^2 \) \[\text{read: symm. tensor product of 2-forms}\]

**Nice property:** For \( n \geq 2 \), \( H^{2n+1} \) admits Killing spinors with torsion, i.e. solutions of \( \nabla_X \psi = \alpha \psi \) (but no Riemannian Killing spinors, i.e. no sol. for \( \nabla = \nabla^g \) / \( \not\exists \) Einstein metric) \[\text{[A-Becker-Bender, 2012]}\]
The case $n = 6$

Assume $\ker T = 0$ from beginning. Distinction $\sigma_T =, \neq 0$ is too crude.

$\ast \sigma_T$: a 2-form $\equiv$ skew-symm. endomorphism, classify by its rank! ($= 0, 2, 4, 6$ / Case A, B, C, D)

**Geometry:** Can $\ast \sigma_T$ be interpreted as an almost complex structure?

**Exa.** Recall: $\Lambda^3(\mathbb{R}^6)^{\mathfrak{so}(n)} \cong W_1^{(2)} \oplus W_3^{(12)} \oplus W_4^{(6)}$: types of almost complex structures with characteristic connection

On $S^3 \times S^3$, there exist 3-forms with the following subcases:

<table>
<thead>
<tr>
<th>Type</th>
<th>$W_1 \oplus W_3$</th>
<th>$W_1$</th>
<th>$W_3 \oplus W_4$</th>
<th>$-$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{rk} \ast \sigma_T$</td>
<td>6</td>
<td>6</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>$\text{iso}(T)$</td>
<td>$\mathfrak{so}(3)$</td>
<td>$\mathfrak{su}(3)$</td>
<td>$T^2$</td>
<td>$\mathfrak{so}(3) \times \mathfrak{so}(3)$</td>
</tr>
</tbody>
</table>

$W_1 \oplus W_3$: torsion $T = \alpha e_{135} + \alpha' e_{246} + \beta (e_{245} + e_{236} + e_{146})$.

$W_3 \oplus W_4$: torsion $T = (e_{12} - e_{34}) \wedge (\sigma e_5 + \nu e_6) + \tau (e_{12} - e_{34}) \wedge e_5$. 
**Case A: \( \sigma_T = 0 \)**

This covers, for example, torsions of form \( \mu e_{123} + \nu e_{456} \). This is basically all by our 2nd splitting thm:

**Thm.** A c.s.c. Riemannian 6-mnfld with parallel skew torsion \( T \) s.t. \( \sigma_T = 0 \) and \( \ker T = 0 \) splits into two 3-dimensional manifolds with parallel skew torsion,

\[
(M^6, g, T) = (N^3_1, g_1, T_1) \times (N^3_2, g_2, T_2)
\]

**Cor.** Any 6-dim. nat. red. homog. space with \( \sigma_T = 0 \) and \( \ker T = 0 \) is locally isometric to a product of two 3-dimensional nat. red. homog. spaces.
The case $n = 6$ II

Case B: $\text{rk} (\ast \sigma_T) = 2$

A priori, it is not possible to define an almost complex structure.

**Thm.** Let $(M^6, g, T)$ be a 6-mnfd with parallel skew torsion s.t. $\ker T = 0$, $\text{rk} (\ast \sigma_T) = 2$. Then $\nabla R = 0$, i.e. $M$ is nat. red., and there exist constants $a, b, c, \alpha, \beta \in \mathbb{R}$ s.t.

$$T = \alpha (e_{12} + e_{34}) \wedge e_5 + \beta (e_{12} - e_{34}) \wedge e_6$$

$$R = a (e_{12} + e_{34})^2 + c (e_{12} + e_{34}) \odot (e_{12} - e_{34}) + b (e_{12} - e_{34})^2$$

with the relation $a + b = - (\alpha^2 + \beta^2)$.

Now perform Nomizu construction to conclude:

**Thm.** A c.s.c. Riemannian 6-mnfd with parallel skew torsion $T$ and $\text{rk} (\ast \sigma_T) = 2$ is the product $G_1 \times G_2$ of two Lie groups equipped with a family of left invariant metrics. $G_1$ and $G_2$ are either $S^3 = \text{SU}(2)$, $\tilde{\text{SL}}(2, \mathbb{R})$, or $H^3$. 

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The case \( n = 6 \) III

**Case B:** \( \text{rk} (\ast \sigma_T) = 4 \)

**Thm.** For the torsion form of a metric connection with parallel skew torsion \((\ker T = 0)\), the case \( \text{rk} (\ast \sigma_T) = 4 \) cannot occur.

[but: such forms exist if \( \nabla T \neq 0 \)! – these results explain why a classification is possible without knowing the orbit class. of \( \Lambda^3(\mathbb{R}^6) \) under \( \text{SO}(6) \)]
The case $n = 6$ IV

Case C: $\text{rk} (\ast \sigma_T) = 6$

Thm. Such a 6-mnfd with parallel skew torsion admits an almost complex structure $J$ of Gray-Hervella class $W_1 \oplus W_3$.

All three eigenvalues of $\ast \sigma_T$ are equal, hence $\ast \sigma_T$ is proportional to $\Omega$, the fundamental form of $J$. It’s either nearly Kähler ($W_1$), or it is naturally reductive and $\frak{hol}_\nabla = \frak{so}(3)$.

Why no $W_4$ part? if $\sigma_T = \ast \Omega$, then $d\sigma_T = d \ast \Omega$; but $d\sigma_T = (ddT)/2 = 0$, hence $\delta \Omega = 0$.

N.B. If class $W_1$ ($M^6$ nearly Kähler mnfd): the only homogeneous ones are $S^6$, $S^3 \times S^3$, $\mathbb{CP}^3$, $F(1, 2)$. [Butruille, 2005]

It is not known whether there exist non-homogeneous nearly Kähler mnfds.

Again, we have an explicit formula for torsion and curvature, then perform the Nomizu construction ( . . . and survive).
Example: $\text{SL}(2, \mathbb{C})$ viewed as a 6-dimensional real mnfd

- Write $\mathfrak{sl}(2, \mathbb{C}) = \mathfrak{su}(2) \oplus i \mathfrak{su}(2)$; Killing form $\beta(X, Y)$ is neg. def. on $\mathfrak{su}(2)$, pos. def. on $i \mathfrak{su}(2)$

- $M^6 = G/H = \text{SL}(2, \mathbb{C}) \times \text{SU}(2)/\text{SU}(2)$ with $H = \text{SU}(2)$ embedded diag (recall that $\mathfrak{hol}^\nabla = \mathfrak{so}(3)$; want that isotropy rep. = holonomy rep.)

- $\mathfrak{m}_\alpha$ red. compl. of $\mathfrak{h}$ inside $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{su}(2)$ depending on $\alpha \in \mathbb{R} - \{1\}$, $\mathfrak{h} = \{(B, B) : B \in \mathfrak{su}(2)\}$, $\mathfrak{m}_\alpha := \{(A+\alpha B, B) : A \in i \mathfrak{su}(2), B \in \mathfrak{su}(2)\}$.

- Riemannian metric:
  $$g_\lambda((A_1 + \alpha B_1, B_1), (A_2 + \alpha B_2, B_2)) := \beta(A_1, A_2) - \frac{1}{\lambda^2} \beta(B_1, B_2), \quad \lambda > 0$$

- In suitable ONB: almost hermitian str.: $\Omega := x_{12} + x_{34} + x_{56}$ with torsion $T = N + d\Omega \circ J = \left[2\lambda(1 - \alpha) + \frac{4}{\lambda(1 - \alpha)}\right] x_{135} + \frac{2}{\lambda(1 - \alpha)}[x_{146} + x_{236} + x_{245}]$.

- Curvature: has to be a map $\mathcal{R} : \Lambda^2(M^6) \rightarrow \mathfrak{hol}^\nabla \subset \mathfrak{so}(6)$, here: mainly projection on $\mathfrak{hol}^\nabla = \mathfrak{so}(3)$.

- $\nabla T = \nabla \mathcal{R} = 0$, i.e. naturally reductive for all $\alpha, \lambda$; type $W_1 \oplus W_3$ or $W_3$.
The case \( n = 6 \) V

Final result of Nomizu construction:

**Thm.** A c. s. c. Riemannian 6-mnfd with parallel skew torsion \( T \), \( \text{rk} (*\sigma_T) = 6 \) and \( \ker T = 0 \) that is *not* isometric to a nearly Kähler manifold is one of the following Lie groups with a suitable family of left-invariant metrics:

- The nilpotent Lie group with Lie algebra \( \mathbb{R}^3 \times \mathbb{R}^3 \) with commutator 
  \[
  [(v_1, w_1), (v_2, w_2)] = (0, v_1 \times v_2),
  \]

- the direct or the semidirect product of \( S^3 \) with \( \mathbb{R}^3 \),

- the product \( S^3 \times S^3 \),

- the Lie group \( \text{SL}(2, \mathbb{C}) \) viewed as a 6-dimensional real mnfd.

- prove that manifold is indeed a Lie group,

- identify its abstract Lie algebra by degeneracy / EV of its Killing form,

- find 3-dim. subalgebra defining a 3-dim. quotient and prove that the 6-dim. Lie alg. is its isometry algebra;

for example, \( \text{SL}(2, \mathbb{C}) \) appears because it’s the isometry group of hyperbolic space \( \mathbb{H}^3 \)
Literature

I. Agricola, A. C. Ferreira, Th. Friedrich, Classification of naturally reductive homogeneous spaces in dimensions $n \leq 6$, preprint


