

ON THE TOPOLOGY AND THE GEOMETRY OF $SO(3)$ -MANIFOLDS

ILKA AGRICOLA, JULIA BECKER-BENDER, AND THOMAS FRIEDRICH

ABSTRACT. Consider the nonstandard embedding of $SO(3)$ into $SO(5)$ given by the 5-dimensional irreducible representation of $SO(3)$, henceforth called $SO(3)_{\text{ir}}$. In this note, we study the topology and the differential geometry of 5-dimensional Riemannian manifolds carrying such an $SO(3)_{\text{ir}}$ structure, i. e. with a reduction of the frame bundle to $SO(3)_{\text{ir}}$.

1. INTRODUCTION

We consider the nonstandard embedding of $SO(3)$ into $SO(5)$ given by the 5-dimensional irreducible representation of $SO(3)$, henceforth called $SO(3)_{\text{ir}}$. In this note, we investigate the topology and the differential geometry of 5-dimensional Riemannian manifolds carrying such an $SO(3)_{\text{ir}}$ structure, i. e. with a reduction of the frame bundle to $SO(3)_{\text{ir}}$. These spaces are the non-integrable analogues of the symmetric space $SU(3)/SO(3)$ and its non-compact dual $SL(3, \mathbb{R})/SO(3)$. While the general frame work for the investigation of such structures was outlined in [Fri03b], first concrete results were obtained by M. Bobiński and P. Nurowski (general theory, [BN07]) as well as S. G. Chiossi and A. Fino ($SO(3)_{\text{ir}}$ structures on 5-dimensional Lie groups, [CF07]).

In the first part of the paper, we describe the topological properties of the two different types of $SO(3)$ structures. While classical results by E. Thomas and M. Atiyah are available for the standard diagonal embedding of $SO(3)_{\text{st}} \rightarrow SO(5)$, the case of $SO(3)_{\text{ir}}$ is first investigated in this paper. We show that the symmetric space $SU(3)/SO(3)$ admits a $SO(3)_{\text{ir}}$ structure, but no $SO(3)_{\text{st}}$ structure. We prove necessary relations for the characteristic classes of a 5-manifold with a topological $SO(3)_{\text{ir}}$ structure: its first Pontrjagin class $p_1(M)$ has to be divisible by five, the Stiefel-Whitney classes $w_1(M)$, $w_4(M)$, $w_5(M)$ vanish etc. Moreover, a simply-connected $SO(3)_{\text{ir}}$ -manifold that is spin is automatically parallelizable. We construct explicit examples of S^1 -fibrations over a 4-dimensional base that admit a topological $SO(3)_{\text{ir}}$ structure.

In the second part, the differential geometry of some homogeneous examples is studied in detail. We will focus on a ‘twisted’ Stiefel manifold $V_{2,4}^{\text{ir}} = SO(3) \times SO(3)/SO(2)_{\text{ir}}$, its non compact partner $\tilde{V}_{2,4}^{\text{ir}} = SO(2, 1) \times SO(3)/SO(2)_{\text{ir}}$ and the space $W^{\text{ir}} = \mathbb{R} \times (SL(2, \mathbb{R}) \times \mathbb{R}^2)/SO(2)_{\text{ir}}$. On each of these, a family of metrics depending on three deformation parameters α, β, γ is considered; in the case W^{ir} it is in addition necessary to consider a family of possible embeddings of $SO(2)_{\text{ir}}$ into $\mathbb{R} \times (SL(2, \mathbb{R}) \times \mathbb{R}^2)$, as the ones leading to $SO(3)_{\text{ir}}$ structures are far from trivial. The standard Stiefel manifold is known to admit an Einstein-Sasaki metric that was crucial for the understanding of Riemannian Killing spinors. In contrast, we show that the twisted Stiefel manifold admits a nearly integrable $SO(3)_{\text{ir}}$ structure with parallel torsion (it can even be naturally reductive for some parameters of the metric), a compatible Sasaki structure whose contact connection coincides with the $SO(3)_{\text{ir}}$ connection, but none of the Sasaki structures is Einstein (however, an Einstein metric is shown to exist). All in all, the twisted Stiefel manifold is an example of a rather well-behaved $SO(3)_{\text{ir}}$ -manifold. The manifold W^{ir} carries an $SO(3)_{\text{ir}}$ structure that disproves several conjectures on $SO(3)_{\text{ir}}$ -manifolds that one might be tempted

Date: October 5, 2010.

2000 Mathematics Subject Classification. Primary 53 C 25; Secondary 81 T 30.

Key words and phrases. flat connections, skew-symmetric torsion.

Supported by the Junior Research Group “Special Geometries in Mathematical Physics” of the VolkswagenStiftung.

to conclude from the previous example. It carries a $\mathrm{SO}(3)_{\mathrm{ir}}$ structure only for two possible embeddings of $\mathrm{SO}(2)_{\mathrm{ir}}$ that depend on the parameters α, β, γ of the metric. The torsion of the $\mathrm{SO}(3)_{\mathrm{ir}}$ connection turns out to be non parallel, the space is never Einstein and never naturally reductive, and there does not exist a compatible contact structure whose contact connection would coincide with the $\mathrm{SO}(3)_{\mathrm{ir}}$ connection. In particular, this shows that a $\mathrm{SO}(3)_{\mathrm{ir}}$ structure is conceptionally really different from a contact structure; it thus defines a new type of geometry on 5-manifolds.

2. GENERAL REMARKS ON $\mathrm{SO}(3)$ STRUCTURES

The Lie group $\mathrm{SO}(3)$ admits two inequivalent embeddings into $\mathrm{SO}(5)$. The standard embedding is as upper diagonal matrices, $\mathrm{SO}(3)_{\mathrm{st}} \subset \mathrm{SO}(5)$, $A \mapsto \mathrm{diag}(A, 1, 1)$, while the second embedding corresponds to the unique faithful irreducible 5-dimensional representation of $\mathrm{SO}(3)$ and will henceforth be denoted by $\mathrm{SO}(3)_{\mathrm{ir}} \subset \mathrm{SO}(5)$. A realization of this representation which is particular adapted to the spirit of this note is by conjugation on symmetric trace free endomorphisms of \mathbb{R}^3 , denoted by $S_0^2(\mathbb{R}^3)$,

$$\varrho(h)X := hXh^{-1} \text{ for } h \in \mathrm{SO}(3), X \in S_0^2(\mathbb{R}^3) \cong \mathbb{R}^5.$$

If we choose the following basis for $S_0^2(\mathbb{R}^3)$,

$$X = \sum_{i=1}^5 x_i e_i = \begin{bmatrix} \frac{x_1}{\sqrt{3}} - x_5 & x_4 & x_2 \\ x_4 & \frac{x_1}{\sqrt{3}} + x_5 & x_3 \\ x_2 & x_3 & -2\frac{x_1}{\sqrt{3}} \end{bmatrix},$$

and denote by E_{ij} the endomorphism sending e_i to e_j , e_j to $-e_i$ and everything else to zero, the Lie algebras of the two embeddings above are spanned by the bases

$$\mathfrak{so}(3)_{\mathrm{st}} = \langle s_1 := E_{23}, s_2 := E_{31}, s_3 := E_{12} \rangle, \quad \mathfrak{so}(3)_{\mathrm{ir}} = \langle X_1, X_2, X_3 \rangle,$$

$$X_1 = \varrho(s_1) = \sqrt{3}E_{13} + E_{42} + E_{53}, X_2 = \varrho(s_2) = \sqrt{3}E_{21} + E_{34} + E_{52}, X_3 = \varrho(s_3) = E_{23} + 2E_{45}.$$

By definition, a $\mathrm{SO}(3)_{\mathrm{st}}$ resp. $\mathrm{SO}(3)_{\mathrm{ir}}$ structure on a 5-manifold is a reduction of its frame bundle to a subgroup $\mathrm{SO}(3) \subset \mathrm{SO}(5)$ isomorphic to $\mathrm{SO}(3)_{\mathrm{st}}$ resp. $\mathrm{SO}(3)_{\mathrm{ir}}$. The first example of a manifold with a $\mathrm{SO}(3)_{\mathrm{ir}}$ structure is the Riemannian symmetric space $\mathrm{SU}(3)/\mathrm{SO}(3)$ with its natural Sasaki-Einstein metric, see [BN07] for a detailed description.

One crucial observation of [BN07] is that $\mathfrak{so}(3)_{\mathrm{ir}}$ may be characterized as being the isotropy group of of a symmetric $(3,0)$ -tensor Υ on \mathbb{R}^5 . Basically, this symmetric tensor is one of the coefficients of the characteristic polynomial of $X \in S_0^2(\mathbb{R}^3)$, more precisely,

$$\det(X - \lambda \mathrm{Id}) = -\lambda^3 + g(X, X)\lambda - \frac{2\sqrt{3}}{9} \Upsilon(X, X, X).$$

The coordinates are chosen in such a way that the bilinear form g takes the simple expression $g(X, X) = \sum_1^5 x_1^2$, while Υ is the homogeneous polynomial

$$\Upsilon(X, X, X) = x_1^3 + \frac{3}{2}x_1(x_2^2 + x_3^2 - 2x_4^2 - 2x_5^2) + \frac{3\sqrt{3}}{2}(x_2^2 - x_3^2)x_5 - 3\sqrt{3}x_2x_3x_4.$$

In fact, a $\mathrm{SO}(3)_{\mathrm{ir}}$ structure on a 5-dimensional Riemannian manifold (M^5, g) can equally be characterised as being a rank 3 tensor field Υ for which the associated linear map $TM \rightarrow \mathrm{End}(TM)$, $v \mapsto \Upsilon_v$ defined by $(\Upsilon_v)_{ij} = \Upsilon_{ijk}v_k$ satisfies

- (1) it is totally symmetric: $g(u, \Upsilon_v w) = g(w, \Upsilon_v u) = g(u, \Upsilon_w v)$,
- (2) it is trace-free: $\mathrm{tr} \Upsilon_v = 0$,
- (3) it reconstructs the metric: $\Upsilon_v^2 v = g(v, v)v$.

Recall that for a G structure, a metric connection ∇ is called a *characteristic connection* if it is a G connection whose torsion is totally antisymmetric [Fri03b].

Theorem 2.1 ([BN07]). *A $\mathrm{SO}(3)_{\mathrm{ir}}$ structure (M, g, Υ) can only admit a characteristic connection if it is nearly integrable, i. e. if the tensor Υ satisfies $(\nabla_v^g \Upsilon)(v, v, v) = 0$ for all vector fields v . In this case, the torsion of the characteristic connection is of algebraic type $\Lambda^3(\mathbb{R}^5) \cong \Lambda^2(\mathbb{R}^5) \cong \mathfrak{so}(5) = \mathfrak{so}(3)_{\mathrm{ir}} \oplus \mathfrak{n}$.*

The analogy to the definition of a nearly Kähler manifold is evident. However, contrary to nearly Kähler manifold (see [Kir77], [AFS05]), the torsion T of the characteristic connection of a nearly integrable $\mathrm{SO}(3)_{\mathrm{ir}}$ structure is not always parallel. Examples will be discussed in Section 4.

3. TOPOLOGICAL EXISTENCE CONDITIONS

Necessary and sufficient conditions for the existence of a $\mathrm{SO}(3)_{\mathrm{st}}$ structure on an oriented 5-manifold were investigated in the late 1960ies. In fact, the existence of such a structure is equivalent to the existence of two global linearly independent vector fields. Recalling that the Kervaire semi-characteristic is defined by

$$k(M^5) := \sum_{i=0}^2 \dim_{\mathbb{R}} H^{2i}(M^5; \mathbb{R}) \pmod{2},$$

one has the following classical result:

Theorem 3.1 ([Th68], [At70]). *A 5-dimensional compact oriented manifold admits two global linearly independent vector fields if and only if*

$$w_4(M^5) = 0 \text{ and } k(M^5) = 0.$$

There is a second semi-characteristic,

$$\hat{\chi}_2(M^5) := \sum_{i=0}^2 \dim_{\mathbb{Z}_2} H_i(M^5; \mathbb{Z}_2) \pmod{2}.$$

The Lusztig-Milnor-Peterson formula [LMP69] establishes the link between these two semi-characteristics,

$$k(M^5) - \hat{\chi}_2(M^5) = w_2(M^5) \cup w_3(M^5).$$

In particular, if M^5 is spin or $w_3(M^5) = 0$, then $k(M^5) = \hat{\chi}_2(M^5)$.

In [BN07] and [Bob06], it was claimed that the existence of a $\mathrm{SO}(3)_{\mathrm{ir}}$ structure is equivalent to the existence of a $\mathrm{SO}(3)_{\mathrm{st}}$ structure and the divisibility of the first integral Pontrjagin class by five. However, the symmetric space $\mathrm{SU}(3)/\mathrm{SO}(3)$ is known to have a $\mathrm{SO}(3)_{\mathrm{ir}}$ structure, but it does not have a $\mathrm{SO}(3)_{\mathrm{st}}$ structure.

Example 3.1. The symmetric space $M^5 := \mathrm{SU}(3)/\mathrm{SO}(3)$ admits a $\mathrm{SO}(3)_{\mathrm{ir}}$ structure, but no $\mathrm{SO}(3)_{\mathrm{st}}$ structure.

Proof. Let us start by computing the isotropy representation of M^5 . We choose as a basis of $\mathfrak{so}(3)$ the elements $a_1 := E_{23}$, $a_2 := E_{31}$, $a_3 := E_{12}$ and complete it to a basis of $\mathfrak{su}(3)$ by choosing

$$b_1 = i \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad b_2 = i \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad b_3 = i \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},$$

$$b_4 = i \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad b_5 = \frac{i}{\sqrt{3}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}.$$

In this basis, the isotropy representation $\lambda : \mathfrak{so}(3) \rightarrow \mathfrak{so}(5)$ is given by

$$\lambda(a_1) = E_{12} + E_{34} - \sqrt{3} E_{35}, \quad \lambda(a_2) = -E_{13} + E_{24} + \sqrt{3} E_{25}, \quad \lambda(a_3) = -2 E_{14} + E_{23}.$$

This representation is irreducible, hence isomorphic to the 5-dimensional irreducible representation of $\mathfrak{so}(3)$. M^5 cannot admit a $\mathrm{SO}(3)_{\mathrm{st}}$ structure as claimed. Indeed, M^5 is a rational homology sphere and a computation of the \mathbb{Z}_2 -cohomology yields the following result:

$$H^1(M^5; \mathbb{Z}_2) = H^4(M^5; \mathbb{Z}_2) = 0, \quad H^2(M^5; \mathbb{Z}_2) = H^3(M^5; \mathbb{Z}_2) = \mathbb{Z}_2.$$

Consequently, we obtain $k(M^5) = 1$ and $\hat{\chi}_2(M^5) = 0$. Moreover, M^5 does not admit any $\mathrm{spin}^{\mathbb{C}}$ structure (see [Fr02], page 50). \square

Our description of the irreducible representation of $\mathrm{SO}(3)$ on $S_0^2(\mathbb{R}^3)$ implies the following characterization:

Lemma 3.1. *A 5-manifold M^5 admits a $\mathrm{SO}(3)_{\mathrm{ir}}$ structure if and only if there exists a 3-dimensional oriented vector bundle $E^3 \rightarrow M^5$ such that the tangent bundle TM^5 is isomorphic to the bundle of symmetric trace free endomorphisms of E^3 , $TM^5 \cong S_0^2(E^3)$.*

This characterization allows to formulate some necessary topological conditions for the existence of $\mathrm{SO}(3)_{\mathrm{ir}}$ structures.

Theorem 3.2. *Suppose that M^5 admits a $\mathrm{SO}(3)_{\mathrm{ir}}$ structure and that E^3 is a vector bundle over M^5 as just described. Then the following relations hold:*

- (1) $p_1(M^5) = 5p_1(E^3) \in H^4(M^5; \mathbb{Z})$.
In particular, the first Pontrjagin class $p_1(M^5)$ is divisible by five.
- (2) $w_1(M^5) = 0$, $w_4(M^5) = 0$, $w_5(M^5) = 0$,
- (3) $w_2(M^5) = w_2(E^3)$, $w_3(M^5) = w_3(E^3)$.

Proof. We use the Borel-Hirzebruch formalism and consider, for a connected and compact Lie group G and its maximal torus $T \subset G$, the maps $H^*(BT)^W = H^*(BG) \rightarrow H^*(M^5)$ induced by the classifying maps of the bundles $T(M^5)$ and E^3 . Denote by ω_1, ω_2 the weights of $\mathrm{SO}(5)$ and by ω the weight of $\mathrm{SO}(3)$. Then we have

$$1 + p_1(TM^5) = 1 + (\omega_1^2 + \omega_2^2), \quad 1 + p_1(E^3) = 1 + \omega^2.$$

The inclusion $\mathrm{SO}(3)_{\mathrm{ir}} \subset \mathrm{SO}(5)$ induces the map $\omega \rightarrow (\omega_1, 2\omega_2)$ and we obtain

$$p_1(M^5) = \omega_1^2 + \omega_2^2 = \omega^2 + (2\omega)^2 = 5\omega^2 = 5p_1(E^3).$$

If the Stiefel-Whitney classes of E^3 are given by the elementary symmetric functions

$$w(E^3) = (1 + x_1)(1 + x_2)(1 + x_3),$$

then the classes of $S_0^2(E^3)$ are computed by

$$w(S_0^2(E^3)) = w(S(E^3)) = \prod_{1 \leq i \leq j \leq 3} (1 + x_i + x_j).$$

The bundle E^3 is oriented, $x_1 + x_2 + x_3 = w_1(E^3) = 0$. A direct computation mod 2 yields now the results

$$w_1(S(E^3)) = w_4(S(E^3)) = w_5(S(E^3)) = 0,$$

and

$$\begin{aligned} w_2(S(E^3)) &= (x_1 + x_2 + x_3)^2 + x_1x_2 + x_1x_3 + x_2x_3 = w_2(E^3), \\ w_3(S(E^3)) &= (x_1 + x_2 + x_3)(x_1x_2 + x_1x_3 + x_2x_3) + x_1x_2x_3 = w_3(E^3). \end{aligned}$$

\square

Remark 3.1. The construction and classification of oriented 3-dimensional vector bundles over compact, oriented 5-manifolds in terms of topological data is difficult. But assume that we have such a bundle E^3 over M^5 and $w_2(E^3) = w_2(M^5)$, $5p_1(E^3) = p_1(M^5)$ holds. Moreover, assume that $H^4(M^5; \mathbb{Z}) = H_1(M^5; \mathbb{Z})$ has no 2-torsion. Then $w_2(S_0^2(E^3)) = w_2(T(M^5))$, $p_1(S_0^2(E^3)) = p_1(T(M^5))$ and the real vector bundles $S_0^2(E^3)$ and $T(M^5)$ are stable equivalent (see [DW59] and [Th68]).

Wu's formulas linking the Stiefel-Whitney classes of an oriented 5-manifold M^5 read as

$$\begin{aligned} w_3(M^5) &= Sq^1(w_2(M^5)), & w_4(M^5) &= w_2(M^5) \cup w_2(M^5), \\ w_2(M^5) \cup w_3(M^5) &= Sq^1(w_2(M^5) \cup w_2(M^5)) + Sq^2(w_3(M^5)). \end{aligned}$$

In particular, we obtain

Corollary 3.1. *If M^5 admits a $SO(3)_{\text{ir}}$ or a $SO(3)_{\text{st}}$ structure, then*

$$w_2(M^5) \cup w_2(M^5) = 0, \quad w_3(M^5) = Sq^1(w_2(M^5)), \quad w_2(M^5) \cup w_3(M^5) = Sq^2(w_3(M^5))$$

holds.

Example 3.2. The real projective space $\mathbb{R}P^5$ cannot have either kind of a $SO(3)$ structure, for in both cases the vanishing of $w_4(\mathbb{R}P^5)$ would be a necessary condition. Indeed, its Stiefel-Whitney class $w_4(\mathbb{R}P^5) \neq 0$ is non-trivial.

The necessary conditions expressed via the Pontrjagin class as well as the Stiefel-Whitney classes do not imply the existence of a $SO(3)_{\text{ir}}$ structure. In fact, there is a further obstruction in $H^5(M^5; \mathbb{Z}_2)$, probably the vanishing of $\hat{\chi}_2(M^5)$. Here we prove only a weaker statement.

Theorem 3.3. *Let M^5 be a compact, simply-connected spin manifold admitting a $SO(3)_{\text{ir}}$ or a $SO(3)_{\text{st}}$ structure. Then M^5 is parallelizable. In particular, the sphere S^5 does not admit a $SO(3)_{\text{ir}}$ nor a $SO(3)_{\text{st}}$ structure.*

Remark 3.2. The Theorem is well known for the standard embedding. Indeed, if $w_2(M^5) = 0$ and the simply-connected M^5 admits a $SO(3)_{\text{st}}$ structure, then $p_1(M^5) = 0$ as well as $k(M^5) = \hat{\chi}_2(M^5) = 0$. These conditions imply that M^5 is parallelizable (see [Th68]).

In order to prove this theorem, we need the following

Lemma 3.2. *Let $i : SO(3) \rightarrow SO(5)$ be the standard or the irreducible embedding of the group $SO(3)$ into $SO(5)$. Then the induced homomorphism*

$$i_* : \pi_4(SO(3)) = \mathbb{Z}_2 \longrightarrow \pi_4(SO(5)) = \mathbb{Z}_2$$

is trivial.

Proof of Lemma 3.2. We remark that both homotopy groups are isomorphic to \mathbb{Z}_2 ,

$$\begin{aligned} \pi_4(SO(3)) &= \pi_4(\text{Spin}(3)) = \pi_4(S^3) = \mathbb{Z}_2, \\ \pi_4(SO(5)) &= \pi_4(\text{Spin}(5)) = \pi_4(\text{Sp}(2)) = \pi_4(\text{Sp}(1)) = \pi_4(S^3) = \mathbb{Z}_2. \end{aligned}$$

The second line is a consequence of the classical isomorphism $\text{Spin}(5) = \text{Sp}(2)$ and the fibration $\text{Sp}(2)/\text{Sp}(1) = S^7$. First consider the standard embedding $i_{\text{st}} : SO(3) \rightarrow SO(5)$. Then $SO(5)/i_{\text{st}}(SO(3)) = V_{5,2}$ is the Stiefel manifold and we obtain the exact sequence

$$\dots \longrightarrow \pi_4(SO(3)) = \mathbb{Z}_2 \longrightarrow \pi_4(SO(5)) = \mathbb{Z}_2 \longrightarrow \pi_4(V_{5,2}) \longrightarrow \pi_3(SO(3)) = \mathbb{Z} \longrightarrow \dots$$

Since $\pi_4(V_{5,2}) = \mathbb{Z}_2$ (see [Pae56]) we conclude that $\pi_4(SO(5)) \rightarrow \pi_4(V_{5,2})$ is surjective, i.e. $(i_{\text{st}})_* : \pi_4(SO(3)) \rightarrow \pi_4(SO(5))$ is trivial. If $i_{\text{ir}} : SO(3) \rightarrow SO(5)$ is the irreducible embedding then we denote by $X^7 = SO(5)/i_{\text{ir}}(SO(3)) = \text{Sp}(2)/i_{\text{ir}}(\text{Sp}(1))$ the corresponding homogeneous space (the so called Berger space). Its homotopy groups are known,

$$\pi_1(X^7) = \pi_2(X^7) = 0, \quad \pi_3(X^7) = \mathbb{Z}_{10}.$$

The exact sequence of the homotopy groups of that fibration yields that $(i_{\text{ir}})_* : \pi_3(\text{Sp}(1)) = \mathbb{Z} \rightarrow \pi_3(\text{Sp}(2)) = \mathbb{Z}$ is multiplication by 10. Consequently, the map $i_{\text{ir}} : S^3 = \text{Sp}(1) \rightarrow \text{Sp}(2)$ represents ten times the generator $[h] \in \pi_3(\text{Sp}(2))$, $[i_{\text{ir}}] = 10 \cdot [h]$. A similar argument proves that $[i_{\text{st}}] = 2 \cdot [h]$ holds. Fix a map $g : S^3 = \text{Sp}(1) \rightarrow \text{Sp}(1) = S^3$ of degree 5. Then we obtain

$$[i_{\text{st}} \circ g] = 10 \cdot [h] = [i_{\text{ir}}],$$

i. e. the maps $i_{\text{st}} \circ g, i_{\text{ir}} : \text{Sp}(1) \rightarrow \text{Sp}(2)$ are homotopic. The induced map $(i_{\text{ir}})_* = (i_{\text{st}})_* \circ g_*$ of the embedding i_{ir} is given by the induced maps of i_{st} and of g . Finally we see that $(i_{\text{ir}})_* : \pi_4(SO(3)) \rightarrow \pi_4(SO(5))$ is again trivial. \square

Proof of Theorem 3.3. Suppose that M^5 admits a $\mathrm{SO}(3)_{\mathrm{ir}}$ structure. Then E^3 is a 3-dimensional, oriented bundle with a spin structure. Its frame bundle P_{E^3} is given by a classifying map $M^5 \rightarrow B\mathrm{Spin}(3) = \mathbb{H}\mathbb{P}^\infty$. The manifold M^5 is 5-dimensional and, consequently, the classifying map is a map into S^4 . Since $H^4(M^5; \mathbb{Z}) = H_1(M^5; \mathbb{Z}) = 0$ there are at most two homotopy classes of maps from M^5 into S^4 (see [Span66, Ch. 8, S. 5, Thm. 15]). The frame bundle P_{E^3} is trivial over the 4-skeleton of M^5 and the two bundles are given by their obstruction classes in $H^5(M^5; \pi_4(\mathrm{SO}(3))) = \pi_4(\mathrm{SO}(3)) = \mathbb{Z}_2$. However, the map $i_* : \pi_4(\mathrm{SO}(3)) \rightarrow \pi_4(\mathrm{SO}(5))$ is trivial. This implies that the bundles $E^3 \oplus \theta^2$ (in case of the standard embedding) or $S_0^2(E^3)$ (in case of the irreducible embedding) are trivial. Finally, M^5 is parallelizable. \square

Example 3.3. The connected sums $(2l+1)\#(S^2 \times S^3)$ are simply-connected, spin and they admit a $\mathrm{SO}(3)_{\mathrm{st}}$ structure (the Kervaire semi-characteristic k vanishes). Therefore they are parallelizable.

The subgroup $\mathrm{SO}(2)_{\mathrm{ir}} := \{(A, A^2, 1) : A \in \mathrm{SO}(2)\}$ is contained in $\mathrm{SO}(3)_{\mathrm{ir}} \subset \mathrm{SO}(5)$. A 5-manifold M^5 admits a $\mathrm{SO}(2)_{\mathrm{ir}}$ structure (and, in particular, a $\mathrm{SO}(3)_{\mathrm{ir}}$ structure) if and only if there exists a complex line bundle E such that $T(M^5) = E \oplus E^2 \oplus \theta^1$. Suppose that M^5 is a S^1 -fibration over a 4-manifold X^4 . Then the tangent bundle of X^4 should split into $T(X^4) = E \oplus E^2$. Let us discuss the latter condition. In this way we are able to construct whole families of 5-manifolds admitting a topological $\mathrm{SO}(3)_{\mathrm{ir}}$ structure.

Proposition 3.1. *Let X^4 be a smooth, compact, oriented 4-dimensional manifold. Then the following conditions are equivalent:*

- (1) *The tangent bundle splits into $T(X^4) = E \oplus E^2$.*
- (2) *There exists an element $c \in H^2(X^4; \mathbb{Z})$ such that*

$$p_1(X^4) = 5c^2, \quad \chi(X^4) = 2c^2, \quad \text{and} \quad c \equiv w_2(X^4) \pmod{2}.$$

- (3) *There exists an element $c \in H^2(X^4; \mathbb{Z})$ such that*

$$\chi(X^4) = 2c^2, \quad c \equiv w_2(X^4) \pmod{2} \quad \text{and} \quad 6\sigma(X^4) = 5\chi(X^4).$$

Proof. If $T(X^4) = E \oplus E^2$ then the first Chern class $c = c_1(E)$ of the line bundle E satisfies all the conditions. Conversely, suppose that there exists an element $c \in H^2(X^4; \mathbb{Z})$ with the described properties. Then we consider the line bundle E defined by the condition $c = c_1(E)$. The Euler, the Pontrjagin and the Stiefel-Whitney classes of the real bundles $T(X^4)$ and $E \oplus E^2$ coincide and $H^4(X^4; \mathbb{Z})$ has no 2-torsion. Consequently, the 4-dimensional real vector bundles $T(X^4)$ and $E \oplus E^2$ are isomorphic (see [DW59], [Th68]). \square

Compact spin manifolds X^4 with finite fundamental group and admitting a decomposition $T(X^4) = E \oplus E^2$ do not exist. Indeed, denote by U the intersection form of $S^2 \times S^2$ and by Γ_8 the non-trivial, positive definite quadratic form of rank 8. The intersection form of the universal covering \tilde{X}^4 is isomorphic to $p \cdot U \oplus q \cdot \Gamma_8$ (see [Serre70], chapter 5, Theorem 5). But

$$\sigma(\tilde{X}^4) = 8q, \quad \chi(\tilde{X}^4) = 2p + 8q + 2$$

and $6\sigma(\tilde{X}^4) = 5\chi(\tilde{X}^4)$ yields $8q = 10p + 10 > 8p$. Finally we obtain $p < q$ and the manifold cannot be smooth (the 11/8 conjecture, see [Fu01]).

Any indefinite and odd quadratic form over \mathbb{Z} can be realized as the intersection form of a smooth, compact and simply-connected 4-manifold X^4 . Any such form is isomorphic to the sum of two trivial forms (see [Serre70], chapter 5, Theorem 4), $H^2(X^4; \mathbb{Z}) = s \cdot \langle 1 \rangle \oplus t \cdot \langle -1 \rangle$. Then we obtain

$$\chi(X^4) = 2 + s + t, \quad \sigma(X^4) = s - t$$

and the condition $6\sigma(X^4) = 5\chi(X^4)$ implies $s - 11t = 10$. Consider the generators e_1, \dots, e_s and f_1, \dots, f_t of the quadratic form with $e_\alpha^2 = 1$ as well as $f_\beta^2 = -1$, $\alpha = 1, \dots, s$ and $\beta = 1, \dots, t$.

An admissible class $c \equiv w_2(X^4)$ is a linear combination with odd coefficients,

$$c = a_1 e_1 + \dots + a_s e_s + b_1 f_1 + \dots + b_t f_t$$

and the equation $2c^2 = \chi(X^4)$ becomes $a_1^2 + \dots + a_s^2 - b_1^2 - \dots - b_t^2 = 6 + 6t$. The system

$$s - 11t = 10, \quad a_1^2 + \dots + a_s^2 - b_1^2 - \dots - b_t^2 = 6 + 6t.$$

has solutions in odd number a_α, b_β . A first solution is $s = 21, t = 1, a_1 = \dots = a_{21} = 1, b_1 = 3$ and the corresponding manifold is homeomorphic to $21 \mathbb{C}\mathbb{P}^2 \# \bar{\mathbb{C}}\mathbb{P}^2$. A second solution is $s = 43, t = 3, a_1 = \dots = a_{43} = 3, b_1 = b_2 = b_3 = 11$ with the manifold $43 \mathbb{C}\mathbb{P}^2 \# 3 \bar{\mathbb{C}}\mathbb{P}^2$. A third solution is $s = 197, t = 17, a_1 = \dots = a_{197} = 15, b_1 = \dots = b_{17} = 51$ with the manifold $197 \mathbb{C}\mathbb{P}^2 \# 17 \bar{\mathbb{C}}\mathbb{P}^2$. Any S^1 -bundle M^5 over these spaces admits a $\text{SO}(2)_{\text{ir}} \subset \text{SO}(3)_{\text{ir}}$ structure.

Remark 3.3. The Thom-Gysin sequence yields the relations

$$\hat{\chi}_2(M^5) \equiv \dim_{\mathbb{Z}_2} H_2(X^4; \mathbb{Z}_2), \quad k(M^5) \equiv \dim_{\mathbb{R}} H_2(X^4; \mathbb{R}), \quad \hat{\chi}_2(M^5) = k(M^5)$$

for any oriented S^1 -bundle $\pi : M^5 \rightarrow X^4$ over a compact, simply-connected 4-manifold X^4 . Note that $w_3(M^5) = \pi^*(w_3(X^4)) = 0$ holds anyway. M^5 is a spin manifold if and only if the Chern class $c^* \in H^2(X^4; \mathbb{Z})$ of the fibration $\pi : M^5 \rightarrow X^4$ represents the Stiefel-Whitney class of X^4 , $c^* \equiv w_2(X^4) \pmod{2}$. The Pontrjagin class $p_1(M^5)$ vanishes if and only if there exists an element $x \in H^2(X^4; \mathbb{Z})$ such that $p_1(X^4) = c^* \cup x$.

4. HOMOGENEOUS EXAMPLES AND THEIR GEOMETRIC PROPERTIES

Homogeneous manifolds with a $\text{SO}(3)_{\text{ir}}$ structure have been classified using Cartan's method of integration by Bobiński and Nurowski, see [Bob06]. In this section, it is our goal to describe their geometric properties.

Given that a non-discrete subgroup $H \subset \text{SO}(3)_{\text{ir}}$ can only have dimension 1 or 3, a homogeneous space $M^5 = G/H$ can only be the quotient of a group G of dimension 8 or 6. The case $\dim G = 8$ is not so interesting, as these are precisely the symmetric spaces $\text{SU}(3)/\text{SO}(3)$, $\text{SL}(3, \mathbb{R})/\text{SO}(3)$ and \mathbb{R}^5 . We shall therefore concentrate our attention on homogeneous spaces $M^5 = G^6/\text{SO}(2)$, with $\text{SO}(2) \subset \text{SO}(3)_{\text{ir}}$. The case of 5-dimensional Lie groups with $\text{SO}(3)_{\text{ir}}$ structure has been discussed in detail by Chiossi and Fino, see [CF07].

4.1. The 'twisted' Stiefel manifold $V_{2,4}^{\text{ir}} = \text{SO}(3) \times \text{SO}(3)/\text{SO}(2)_{\text{ir}}$. We consider the inclusion

$$H := \text{SO}(2) \ni A \mapsto (A, A^2) \in \text{SO}(3) \times \text{SO}(3) =: G$$

and the corresponding homogeneous space $V_{2,4}^{\text{ir}} := \text{SO}(3) \times \text{SO}(3)/\text{SO}(2)$. If we choose as standard basis of the Lie algebra $\mathfrak{so}(3)$ the elements s_1, s_2 and s_3 defined in Section 2 and as basis of $\mathfrak{g} = \mathfrak{so}(3) \oplus \mathfrak{so}(3)$ the elements $a_i = (s_i, 0), b_i = (0, s_i), i = 1, 2, 3$, the Lie algebra \mathfrak{h} of $\text{SO}(2)$ is given by

$$\mathfrak{h} = \mathbb{R} \cdot \tilde{e}_0 \text{ with } \tilde{e}_0 = (s_3, 2s_3) = (a_3 + 2b_3).$$

We further define

$$\tilde{e}_1 = b_3 - 2a_3, \quad \tilde{e}_2 = a_1, \quad \tilde{e}_3 = a_2, \quad \tilde{e}_4 = b_1, \quad \tilde{e}_5 = b_2.$$

As a reductive complement \mathfrak{m} of \mathfrak{h} , we may then choose

$$\mathfrak{m} = \mathfrak{n} \oplus \mathfrak{m}_1 \oplus \mathfrak{m}_2, \quad \mathfrak{n} = \mathbb{R} \cdot \tilde{e}_1, \quad \mathfrak{m}_1 = \langle \tilde{e}_2, \tilde{e}_3 \rangle, \quad \mathfrak{m}_2 = \langle \tilde{e}_4, \tilde{e}_5 \rangle.$$

One checks that the isotropy representation $\lambda = \text{Ad}|_H : \text{SO}(2) \rightarrow \text{SO}(5)$ is given by $\lambda(A) = \text{diag}(1, A, A^2)$ and has differential $d\lambda(\tilde{e}_0) = E_{23} + E_{45}$. In particular, one sees that $\lambda(H)$ is indeed a subgroup of $\text{SO}(3)_{\text{ir}} \subset \text{SO}(5)$, but not of $\text{SO}(3)_{\text{st}} \subset \text{SO}(5)$. Thus, the quotient G/H has a $\text{SO}(3)_{\text{ir}}$ structure as claimed. In order to make this property more transparent, we shall write $\text{SO}(2)_{\text{ir}}$ for the chosen embedding of $\text{SO}(2)$ inside $\text{SO}(3) \times \text{SO}(3)$ as well as for its image $\lambda(\text{SO}(2)_{\text{ir}}) \subset \text{SO}(5)$.

In order to define a suitable family of metrics on \mathfrak{m} and thereby of Riemannian metrics on G/H , we first note that not only \mathfrak{m} itself, but each of the spaces $\mathfrak{n}, \mathfrak{m}_1, \mathfrak{m}_2$ is $\lambda(H)$ invariant. Thus,

it makes sense to consider a metric that is a renormalization of the Killing form on each of the factors \mathfrak{n} , \mathfrak{m}_1 and \mathfrak{m}_2 . Since our initial basis consists of orthogonal vectors for the Killing form, we define a 3-parameter family of metrics by

$$g_{\alpha\beta\gamma} = \text{diag}(\alpha, \beta, \beta, \gamma, \gamma), \quad \alpha, \beta, \gamma > 0.$$

We set $e_0 = \tilde{e}_0$ and renormalize our previous basis so that it becomes an orthonormal basis for this new metric,

$$e_1 = \frac{\tilde{e}_1}{\sqrt{\alpha}}, \quad e_2 = \frac{\tilde{e}_2}{\sqrt{\beta}}, \quad e_3 = \frac{\tilde{e}_3}{\sqrt{\beta}}, \quad e_4 = \frac{\tilde{e}_4}{\sqrt{\gamma}}, \quad e_5 = \frac{\tilde{e}_5}{\sqrt{\gamma}}.$$

For later reference, we state all non-vanishing commutator relations:

$$\begin{aligned} [e_0, e_2] &= e_3, & [e_0, e_3] &= -e_2, & [e_0, e_4] &= 2e_5, & [e_0, e_5] &= -2e_4, \\ [e_1, e_2] &= -\frac{2}{\sqrt{\alpha}}e_3, & [e_1, e_3] &= \frac{2}{\sqrt{\alpha}}e_2, & [e_1, e_4] &= \frac{1}{\sqrt{\alpha}}e_5, & [e_1, e_5] &= -\frac{1}{\sqrt{\alpha}}e_4, \\ [e_2, e_3] &= \frac{1}{5\beta}(e_0 - 2\sqrt{\alpha}e_1), & [e_4, e_5] &= \frac{1}{5\gamma}(2e_0 + \sqrt{\alpha}e_1). \end{aligned}$$

Remark 4.1. Before investigating its properties in more detail, let us compare the homogeneous space $V_{2,4}^{\text{ir}} = \text{SO}(3) \times \text{SO}(3)/\text{SO}(2)_{\text{ir}}$ with the classical Stiefel manifold $V_{2,4}^{\text{st}} = \text{SO}(4)/\text{SO}(2)$. Because of $\text{SO}(4) = S^3 \times \text{SO}(3)$, the corresponding Lie algebra embedding $\mathfrak{so}(2) \rightarrow \mathfrak{so}(3) \times \mathfrak{so}(3)$ is just $h \mapsto (h, 0)$ in the classical case, $h \mapsto (h, 2h)$ in the case that we are considering. Hence, we see that that $V_{2,4}^{\text{st}}$ carries an $\text{SO}(3)_{\text{st}}$ structure, hence justifying the superscript. In [Jen75], it was shown that the classical Stiefel manifold $V_{2,4}^{\text{st}}$ carries an Einstein metric. Later, this Einstein metric was recognized to be Sasaki and the existence of two Riemannian Killing spinors was established [Fri80]. Connections with antisymmetric torsion on $V_{2,4}^{\text{st}}$ were investigated in [Agr03]. All in all, this example turned out to be crucial for the understanding of the relations between contact structures and the existence of Killing spinors.

Remark 4.2. Let us display the $\text{SO}(3)_{\text{ir}}$ structure of $V_{2,4}^{\text{ir}}$ in yet another way, namely, as a bundle E^3 satisfying $S_0^2(E^3) \cong TV_{2,4}^{\text{ir}}$ as used in Section 3. The frame bundle of the homogeneous space $V_{2,4}^{\text{ir}}$ is $\mathcal{R} = G \times_{\lambda(H)} \text{SO}(5)$ and its tangent bundle is $TV_{2,4}^{\text{ir}} = G \times_{\lambda(H)} \mathfrak{m}$; therefore, the following vector bundle is well defined,

$$E^3 = G \times_{\lambda(H)} \mathfrak{so}(3)_{\text{ir}},$$

where the action of H is by conjugation on the subspace $\mathfrak{so}(3)_{\text{ir}} \subset \mathfrak{so}(5)$ as always. One then checks that, as H representations, $S_0^2(\mathfrak{so}(3)_{\text{ir}}) \cong \mathfrak{m}$, hence showing $S_0^2(E^3) \cong TV_{2,4}^{\text{ir}}$ as claimed.

Theorem 4.1 (Connection properties). *The twisted Stiefel manifold $V_{2,4}^{\text{ir}} = \text{SO}(3) \times \text{SO}(3)/\text{SO}(2)_{\text{ir}}$ equipped with the family of metrics $g_{\alpha\beta\gamma}$ has the following properties:*

- (1) *For parameters $\alpha, \beta, \gamma > 0$ satisfying $\alpha\beta + 4\gamma\alpha - 25\beta\gamma = 0$, the $\text{SO}(3)_{\text{ir}}$ structure is nearly integrable and the torsion $T^{\alpha\beta\gamma}$ of its characteristic connection $\nabla^{\alpha\beta\gamma}$ is, in a suitable orthonormal basis, given by*

$$T^{\alpha\beta\gamma} = \frac{2\sqrt{\alpha}}{5\beta} e_1 \wedge e_2 \wedge e_3 - \frac{\sqrt{\alpha}}{5\gamma} e_1 \wedge e_4 \wedge e_5.$$

Its holonomy is $\text{SO}(2)_{\text{ir}} \subset \text{SO}(5)$ and its torsion is parallel, $\nabla^{\alpha\beta\gamma} T^{\alpha\beta\gamma} = 0$.

- (2) *The metric of the nearly integrable $\text{SO}(3)_{\text{ir}}$ structure is naturally reductive if and only if $\alpha = 5\beta = 5\gamma$.*

Proof. By a Theorem of Wang [KN96, X.2], invariant metric connections $\nabla^{\alpha\beta\gamma}$ on $V_{2,4}^{\text{ir}} = G/H$ are in bijective correspondence with linear maps $\Lambda_{\mathfrak{m}} : \mathfrak{m} \rightarrow \mathfrak{so}(5)$ that are equivariant under the adjoint representation,

$$(*) \quad \Lambda_{\mathfrak{m}}(hXh^{-1}) = \text{Ad}(h)\Lambda_{\mathfrak{m}}(X)\text{Ad}(h)^{-1} \quad \forall h \in H, X \in \mathfrak{m}.$$

Since $\Lambda_{\mathfrak{m}}$ is basically the connection form, $\nabla^{\alpha\beta\gamma}$ will be a SO_{ir} connection if and only if $\Lambda_{\mathfrak{m}}$ takes values in the structure group $\text{SO}(3)_{\text{ir}}$ of the reduction of the frame bundle. We have $\lambda(e_0) = e_{23} + 2E_{45} = X_3$ in the notation of Section 2 and complete it to a basis of $\mathfrak{so}(3)_{\text{ir}} \subset \mathfrak{so}(5)$ by choosing as additional elements X_1 and X_2 . Thus, $\Lambda_{\mathfrak{m}}(e_i)$ is a priori for each $i = 1, \dots, 4$ a linear combination of the elements X_i , $i = 1, 2, 3$. However, the equivariance condition (*) further restricts the possible values of $\Lambda_{\mathfrak{m}}(e_i)$; one checks that the most general Ansatz for a SO_{ir} connection is $(a, b, c \in \mathbb{R})$

$$\Lambda_{\mathfrak{m}}(e_1) = aX_3, \quad \Lambda_{\mathfrak{m}}(e_2) = bX_1 - cX_2, \quad \Lambda_{\mathfrak{m}}(e_3) = cX_1 + bX_2, \quad \Lambda_{\mathfrak{m}}(e_4) = \Lambda_{\mathfrak{m}}(e_5) = 0.$$

For the possible torsion $T^{\alpha\beta\gamma} \in \Lambda^3(G/H)$, observe that the only λ -invariant 3-forms are $e_1 \wedge e_2 \wedge e_3$ and $e_1 \wedge e_4 \wedge e_5$, thus the torsion has to be of the form $(m, n \in \mathbb{R})$

$$T^{\alpha\beta\gamma} = m e_1 \wedge e_2 \wedge e_3 + n e_1 \wedge e_4 \wedge e_5.$$

Since the torsion of the connection defined by $\Lambda_{\mathfrak{m}}$ is given by [KN96, X.2.3]

$$(*) \quad T(X, Y)_o = \Lambda_{\mathfrak{m}}(X)Y - \Lambda_{\mathfrak{m}}(Y)X - [X, Y]_{\mathfrak{m}}, \quad X, Y \in \mathfrak{m},$$

one concludes by a routine evaluation on all pairs of vectors $e_i \neq e_j$ that

$$b = c = 0, \quad m = a + \frac{2}{\sqrt{\alpha}} = \frac{2\sqrt{\alpha}}{5\beta}, \quad n = 2a - \frac{1}{\sqrt{\alpha}} = -\frac{\sqrt{\alpha}}{5\gamma}.$$

A $\text{SO}(3)_{\text{ir}}$ connection is thus obtained if and only if $\alpha\beta + 4\alpha\gamma - 25\beta\gamma = 0$ and is then defined by

$$\Lambda_{\mathfrak{m}}(e_1) = \left(\frac{2\sqrt{\alpha}}{5\beta} - \frac{2}{\sqrt{\alpha}} \right) X_3, \quad T^{\alpha\beta\gamma} = \frac{2\sqrt{\alpha}}{5\beta} e_1 \wedge e_2 \wedge e_3 - \frac{\sqrt{\alpha}}{5\gamma} e_1 \wedge e_4 \wedge e_5.$$

If one requires further that $T^{\alpha\beta\gamma}(X, Y, Z) = -g([X, Y]_{\mathfrak{m}}, Z)$, one obtains a naturally reductive space and a comparison with the commutator relations yields the stronger condition $\alpha = 5\beta = 5\gamma$. Indeed, under this condition $\Lambda_{\mathfrak{m}} = 0$, and the characteristic connection coincides with the canonical connection.

We now show that the torsion $T^{\alpha\beta\gamma}$ is $\nabla^{\alpha\beta\gamma}$ -parallel. Invariant tensor are parallel with respect to the canonical connection defined by $\Lambda_{\mathfrak{m}}^c = 0$, hence $\nabla_{e_i}^{\alpha\beta\gamma} T = \Lambda_{\mathfrak{m}}(e_i)T$. We first note that on the invariant vector e_1 , trivially $\Lambda_{\mathfrak{m}}(e_1)e_1 = 0$ holds, and after the identification $\mathfrak{so}(\mathfrak{m}) \cong \Lambda^2(\mathfrak{m})$, the action of the connection on 2-forms ω is given

$$(**) \quad \nabla_{e_i}^{\alpha\beta\gamma} \omega = \Lambda_{\mathfrak{m}}(e_i)\omega = \sum_{j=1}^5 (e_j \lrcorner \Lambda_{\mathfrak{m}}(e_i)) \wedge (e_j \lrcorner \omega).$$

Thus, one checks that $\nabla^{\alpha\beta\gamma}(e_2 \wedge e_3) = \nabla^{\alpha\beta\gamma}(e_4 \wedge e_5) = 0$ and, consequently, $\nabla^{\alpha\beta\gamma} T^{\alpha\beta\gamma} = 0$ holds.

Finally, it is worth computing the curvature of the characteristic connection. In general, it is given by [KN96, X.2.3]

$$R(X, Y)_o = [\Lambda_{\mathfrak{m}}(X), \Lambda_{\mathfrak{m}}(Y)] - \Lambda_{\mathfrak{m}}([X, Y]_{\mathfrak{m}}) - \lambda([X, Y]_{\mathfrak{h}}).$$

In the case at hand, one obtains as the only non-vanishing terms

$$R(e_2, e_3) = \frac{1}{5\beta} \left[\frac{4\alpha}{5\beta} - 5 \right] X_3, \quad R(e_4, e_5) = -\frac{2\alpha}{25\beta\gamma} X_3.$$

Let $\mathfrak{m}_0 \subset \mathfrak{m}$ be the space spanned by the non-vanishing curvature transformations. The holonomy algebra of the connection is then

$$\text{hol}(\nabla^{\alpha\beta\gamma}) = \mathfrak{m}_0 + [\Lambda_{\mathfrak{m}}(\mathfrak{m}), \mathfrak{m}_0] + [\Lambda_{\mathfrak{m}}(\mathfrak{m}), [\Lambda_{\mathfrak{m}}(\mathfrak{m}), \mathfrak{m}_0]] + \dots = \mathbb{R} \cdot X_3 = \mathfrak{so}(2)_{\text{ir}} \subset \mathfrak{so}(5). \quad \square$$

Remark 4.3. In the positive quadrant $\{(\alpha, \beta, \gamma) \in \mathbb{R}^3 : \alpha > 0, \beta > 0, \gamma > 0\}$, the metrics $g_{\alpha\beta\gamma}$ defining a nearly integrable $\text{SO}(3)_{\text{ir}}$ structure form a ruled surfaces of rays through the origin. The naturally reductive metrics are exactly one ray (line) on this ruled surface.

Theorem 4.2 (Riemannian Curvature properties).

(1) The Riemannian Ricci tensor of the metric $g_{\alpha\beta\gamma}$ is given by $\text{Ric}(e_1, e_1) = \frac{2\alpha}{25\beta^2} + \frac{\alpha}{50\gamma^2}$,

$$\text{Ric}(e_2, e_2) = \text{Ric}(e_3, e_3) = \frac{1}{\beta} - \frac{2\alpha}{25\beta^2}, \quad \text{Ric}(e_4, e_4) = \text{Ric}(e_5, e_5) = \frac{1}{\gamma} - \frac{\alpha}{50\gamma^2}.$$

(2) The metric is Einstein (but not naturally reductive) for $\alpha = 1$ and

$$\beta = \frac{18}{25(10^{2/3} - 2 \cdot 10^{1/3} + 4)} \cong 0.166177, \quad \gamma = \frac{3(9 + \sqrt{21 + 12 \cdot 10^{1/3} + 3 \cdot 10^{2/3}})}{25(20 - 4 \cdot 10^{1/3} - 10^{2/3})} \cong 0.299009.$$

The Riemannian scalar curvature is $\text{Scal}^g = \left[\frac{5}{3}\right]^3 (100 - 2 \cdot 10^{1/3} - 10^{2/3}/2) \cong 15.60340835$.

(3) The Riemannian holonomy of $(V_{2,4}^{\text{ir}}, g_{\alpha\beta\gamma})$ is $\text{SO}(5)$, i. e. maximal.

Proof. The Levi-Civita connection ∇^g is the unique metric connection whose torsion vanishes, hence a calculation similar to that in the proof of Theorem 4.1 yields that ∇^g corresponds to the map $\Lambda_{\mathfrak{m}}^g : \mathfrak{m} \rightarrow \mathfrak{so}(5)$,

$$\begin{aligned} \Lambda_{\mathfrak{m}}^g(e_1) &= \left[\frac{\sqrt{\alpha}}{5\beta} - \frac{2}{\sqrt{\alpha}} \right] E_{23} + \left[\frac{1}{\sqrt{\alpha}} - \frac{\sqrt{\alpha}}{10\gamma} \right] E_{45}, & \Lambda_{\mathfrak{m}}^g(e_2) &= \frac{\sqrt{\alpha}}{5\beta} E_{13}, \\ \Lambda_{\mathfrak{m}}^g(e_3) &= -\frac{\sqrt{\alpha}}{5\beta} E_{12}, & \Lambda_{\mathfrak{m}}^g(e_4) &= -\frac{\sqrt{\alpha}}{10\gamma} E_{15}, & \Lambda_{\mathfrak{m}}^g(e_5) &= \frac{\sqrt{\alpha}}{10\gamma} E_{14}. \end{aligned}$$

The Riemannian curvature, its Ricci tensor and the Riemannian holonomy then follow from a lengthy, but routine calculation, see the proof of Theorem 4.1 for the general method.

Let us investigate the Einstein condition $g = \kappa \cdot \text{Ric}$. It leads to the equations

$$\kappa = \frac{1}{\gamma} - \frac{1}{50\gamma^2}, \quad \kappa = \frac{2}{25\beta^2} + \frac{1}{50\gamma^2}, \quad \kappa = \frac{1}{\beta} - \frac{2}{25\beta^2},$$

which in turn are equivalent to

$$25\kappa\beta^2 - 25\beta + 2 = 0, \quad 50\kappa\gamma^2 - 50\gamma + 1 = 0, \quad \frac{2}{25\beta^2} + \frac{1}{50\gamma^2} - \kappa = 0.$$

The last condition implies that $\kappa \neq 0$, the metric cannot be Ricci-flat. In this case, the general solution of the first two quadratic equations is

$$\beta_{\pm} = \frac{5 \pm \sqrt{25 - 8\kappa}}{10\kappa}, \quad \gamma_{\pm} = \frac{5 \pm \sqrt{25 - 2\kappa}}{10\kappa}.$$

Inserting all four combinations into the last equation, we obtain four possible conditions for an Einstein metric. One then checks that the combination (β_+, γ_+) yields the Einstein metric given in the theorem (satisfying, in particular, $\kappa < 25/8$), while the other combinations have no admissible solutions. \square

Theorem 4.3 (Contact properties).

(1) $(V_{2,4}^{\text{ir}}, g^{\alpha\beta\gamma})$ carries two invariant normal almost contact metric structures, characterized by

$$\xi \cong \eta = e_1, \quad \varphi_{\pm} = -E_{23} \pm E_{45}, \quad dF_{\pm} = 0.$$

Both normal almost contact metric structures admit a unique characteristic connection with the same characteristic torsion

$$T^c = \eta \wedge d\eta = \frac{2\sqrt{\alpha}}{5\beta} e_1 \wedge e_2 \wedge e_3 - \frac{\sqrt{\alpha}}{5\gamma} e_1 \wedge e_4 \wedge e_5.$$

For a metric $g_{\alpha\beta\gamma}$ defining a nearly integrable $\text{SO}(3)_{\text{ir}}$ structure, this connection coincides with the characteristic connection of the $\text{SO}(3)_{\text{ir}}$ structure.

(2) The invariant normal almost contact metric structure (ξ, η, φ_+) is Sasakian if and only if $\alpha = 25\beta^2 = 100\gamma^2$; it is in addition an $\text{SO}(3)_{\text{ir}}$ structure for $(\alpha, \beta, \gamma) = (\frac{25}{36}, \frac{1}{6}, \frac{1}{12})$.

Proof. An almost contact structure is given by a vector field ξ with dual 1-form η and a $(1, 1)$ -tensor field φ such that $\varphi^2 = -\text{Id} + \eta \otimes \xi$. In order to become an almost contact *metric* structure, a compatibility condition about the underlying Riemannian metric is required,

$$g(\varphi(X), \varphi(Y)) = g(X, Y) - \eta(X)\eta(Y).$$

With the Sasaki case already in mind, e_1 – being a Killing vector field – is a natural choice for $\xi \cong \eta$. Furthermore, it is reasonable in our setting to restrict our attention to invariant structures, i. e. endomorphisms $\varphi : \langle e_1 \rangle^\perp \rightarrow \langle e_1 \rangle^\perp$ that are invariant under the isotropy representation. These are precisely the endomorphisms commuting with $d\lambda(\tilde{e}_0)$, hence φ has to be of the form $\varepsilon_1 E_{23} + \varepsilon_2 E_{45}$, $\varepsilon_{1,2} = \pm 1$. This leads to the two only inequivalent cases described in (1). The almost contact metric structure will be called *normal* if its Nijenhuis tensor N_φ vanishes,

$$N_\varphi(X, Y) = [\varphi(X), \varphi(Y)] - \varphi[X, \varphi(Y)] - \varphi[\varphi(X), Y] + \varphi^2[X, Y] + d\eta(X, Y)\xi.$$

One computes $d\eta = \frac{2\sqrt{\alpha}}{5\beta}e_2 \wedge e_3 - \frac{\sqrt{5}}{5\gamma}e_4 \wedge e_5$ and uses this fact to check that, indeed, $N_{\varphi_\pm} = 0$. The fundamental form of a contact structure is defined by $F(X, Y) := g(X, \varphi(Y))$, hence one obtains in our case

$$F_\pm = e_2 \wedge e_3 \mp e_4 \wedge e_5.$$

By a routine calculation, one shows that $d(e_2 \wedge e_3) = d(e_4 \wedge e_5) = 0$, hence $dF_\pm = 0$ and the general expression for the torsion of an almost contact metric structure [Fri02, Thm 8.2] is reduced to $T^c = \eta \wedge d\eta$, and hence yields the formula stated in (1).

In order to become Sasakian, only $2F = d\eta$ still needs to be satisfied. Thus, F_- can never be Sasakian and F_+ has to satisfy the condition $\alpha = 25\beta^2 = 100\gamma^2$. This finishes the proof. \square

Remark 4.4. Together with Theorem 4.2, we conclude that all Sasaki metrics on $V_{2,4}^{\text{ir}}$ are non-Einstein, contrary to the standard Sasaki metric on $V_{2,4}^{\text{st}}$ (see, for example, [BFGK91, Ch. 4.3] for a detailed discussion of this Einstein-Sasaki structure and its properties).

Geometrically speaking, the set of Sasaki metrics defines a roughly square root shaped curve in the positive quadrant $\{(\alpha, \beta, \gamma) \in \mathbb{R}^3 : \alpha > 0, \beta > 0, \gamma > 0\}$ that intersects the ruled surface of nearly integrable $SO(3)_{\text{ir}}$ metrics in exactly one point, as described in (2).

The twisted Stiefel manifold $V_{2,4}^{\text{ir}} = SO(3) \times SO(3)/SO(2)_{\text{ir}}$ has a non-compact partner, namely, $\tilde{V}_{2,4}^{\text{ir}} := SO(2, 1) \times SO(3)/SO(2)_{\text{ir}}$. We realize $\mathfrak{so}(2, 1)$ as

$$\mathfrak{so}(2, 1) = \langle a_1, a_2, a_3 \rangle \text{ with basis } a_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}, a_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, a_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The commutator relations are $[a_1, a_2] = -a_3$, $[a_2, a_3] = a_1$, $[a_3, a_1] = a_2$. For $\mathfrak{so}(3)$, we use the same basis b_1, b_2, b_3 as before. The embedding of $H = SO(2)$ into $G = SO(2, 1) \times SO(3)$ is also unchanged, namely, with generator $a_3 + 2b_3$, and a good choice for a reductive complement \mathfrak{m} is

$$\mathfrak{m} = \mathfrak{n} \oplus \mathfrak{m}_1 \oplus \mathfrak{m}_2, \quad \mathfrak{n} = \mathbb{R} \cdot (b_3 - 2a_3), \quad \mathfrak{m}_1 = \langle a_1, a_2 \rangle, \quad \mathfrak{m}_2 = \langle b_1, b_2 \rangle.$$

The isotropy representation is given by $\lambda : \mathfrak{so}(2) \rightarrow \mathfrak{so}(5)$, $\lambda(a_3 + 2b_3) = E_{23} + 2E_{45}$, i. e. it turns out to agree with the result in the compact case. A metric $g_{\alpha\beta\gamma}$ is then defined by deformation in the three summands of \mathfrak{m} as before.

Yet, there are a few differences worth noticing. We summarize the results in the following theorem without proof (see [BB09] for details).

Theorem 4.4 (Properties of $\tilde{V}_{2,4}^{\text{ir}} = SO(2, 1) \times SO(3)/SO(2)_{\text{ir}}$). *The reductive homogeneous space $(\tilde{V}_{2,4}^{\text{ir}}, g_{\alpha\beta\gamma})$ has the following properties:*

- (1) *For parameters $\alpha, \beta, \gamma > 0$ satisfying $-\alpha\beta + 4\gamma\alpha + 25\beta\gamma = 0$, the $SO(3)_{\text{ir}}$ structure is nearly integrable and the torsion $T^{\alpha\beta\gamma}$ of its characteristic connection $\nabla^{\alpha\beta\gamma}$ is, in a*

suitable orthonormal basis, given by

$$T^{\alpha\beta\gamma} = -\frac{2\sqrt{\alpha}}{5\beta} e_1 \wedge e_2 \wedge e_3 - \frac{\sqrt{\alpha}}{5\gamma} e_1 \wedge e_4 \wedge e_5.$$

Its holonomy is $\mathrm{SO}(2)_{\mathrm{ir}} \subset \mathrm{SO}(5)$ and its torsion is parallel, $\nabla^{\alpha\beta\gamma} T^{\alpha\beta\gamma} = 0$.

- (2) The metric of the nearly integrable $\mathrm{SO}(3)_{\mathrm{ir}}$ structure is never naturally reductive and never Einstein.

4.2. The manifold $W^{\mathrm{ir}} = \mathbb{R} \times (\mathrm{SL}(2, \mathbb{R}) \ltimes \mathbb{R}^2) / \mathrm{SO}(2)_{\mathrm{ir}}$. In contrast to the compact and non-compact twisted Stiefel manifold, this space will turn out to carry a full $\mathrm{SO}(3)_{\mathrm{ir}}$ structure (i. e. not only a $\mathrm{SO}(2)_{\mathrm{ir}}$ structure) and will prove that the torsion of nearly integrable $\mathrm{SO}(3)_{\mathrm{ir}}$ structure is not necessarily parallel. Let $G = \mathbb{R} \times (\mathrm{SL}(2, \mathbb{R}) \ltimes \mathbb{R}^2)$, and take as standard basis of $\mathfrak{sl}(2, \mathbb{R})$

$$X = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad E_+ = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad E_- = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad [X, E_{\pm}] = \pm 2E_{\pm}, \quad [E_+, E_-] = X.$$

We choose a basis for $\mathfrak{g} = \mathbb{R} \oplus \mathfrak{sl}(2, \mathbb{R}) \oplus \mathbb{R}^2$ that depends on a parameter $\mu \in \mathbb{R}$,

$$\bar{e}_0^\mu = E_+ - E_- + \mu, \quad \bar{e}_1^\mu = 1 - \mu(E_+ - E_-), \quad \bar{e}_2 = (0, 1)^t, \quad \bar{e}_3 = (1, 0)^t, \quad \bar{e}_4 = E_+ + E_-, \quad \bar{e}_5 = X.$$

The element \bar{e}_0^μ generates a one-dimensional subgroup H_μ of G isomorphic to $\mathrm{SO}(2)$, the isotropy representation is $\lambda : \mathfrak{h}_\mu \rightarrow \mathfrak{so}(5)$, $\lambda(\bar{e}_0) = E_{23} + 2E_{45}$. In particular, $\mu = 0$ corresponds to the standard embedding $\mathfrak{so}(2) \rightarrow \mathfrak{sl}(2, \mathbb{R})$; we will see later that, as to be expected, $\mu = 0$ is not admissible for a nearly integrable $\mathrm{SO}(3)_{\mathrm{ir}}$ structure. The subspace

$$\mathfrak{m} = \mathfrak{n}^\mu \oplus \mathfrak{m}_1 \oplus \mathfrak{m}_2, \quad \mathfrak{n}^\mu = \mathbb{R} \cdot \bar{e}_1^\mu, \quad \mathfrak{m}_1 = \langle \bar{e}_2, \bar{e}_3 \rangle, \quad \mathfrak{m}_2 = \langle \bar{e}_4, \bar{e}_5 \rangle,$$

is a reductive complement of \mathfrak{h}_μ in \mathfrak{g} , and each if its summands is isotropy invariant. We can thus define a homogeneous metric on G/H_μ by $g_{\alpha\beta\gamma} = \mathrm{diag}(\alpha, \beta, \beta, \gamma, \gamma)$, $\alpha, \beta, \gamma > 0$. We drop the bar to denote the rescaled elements that form an orthonormal basis for this metric, and agree to write e_0^μ for \bar{e}_0^μ as well. All non-vanishing commutator relations between these new base vectors are

$$\begin{aligned} [e_1^\mu, e_2] &= -\frac{\mu}{\sqrt{\alpha}} e_3, & [e_1^\mu, e_3] &= \frac{\mu}{\sqrt{\alpha}} e_2, & [e_1^\mu, e_4] &= -\frac{2\mu}{\sqrt{\alpha}} e_5, & [e_1^\mu, e_5] &= \frac{2\mu}{\sqrt{\alpha}} e_4 \\ [e_2, e_4] &= -\frac{1}{\sqrt{\gamma}} e_3, & [e_2, e_5] &= \frac{1}{\sqrt{\gamma}} e_2, & [e_3, e_4] &= -\frac{1}{\sqrt{\gamma}} e_2, & [e_3, e_5] &= -\frac{1}{\sqrt{\gamma}} e_3 \\ [e_4, e_5] &= \frac{2}{\gamma(\mu^2 + 1)} (\mu\sqrt{\alpha} e_1 - e_0). \end{aligned}$$

Observe that these do not only depend on the metric, but also on the embedding parameter μ .

Theorem 4.5 (Properties of $W^{\mathrm{ir}} = \mathbb{R} \times (\mathrm{SL}(2, \mathbb{R}) \ltimes \mathbb{R}^2) / \mathrm{SO}(2)_{\mathrm{ir}}$). *The reductive homogeneous space $(W^{\mathrm{ir}}, g_{\alpha\beta\gamma})$ has the following properties:*

- (1) For any $\beta > 0$ and parameters $\alpha, \gamma > 0$ satisfying $\alpha \geq 12\gamma$, the $\mathrm{SO}(3)_{\mathrm{ir}}$ structure is nearly integrable for the two embeddings of $\mathrm{SO}(2) \cong H_\mu \rightarrow \mathrm{SO}(5)$ defined by

$$\mu = \frac{\sqrt{\alpha} \pm \sqrt{\alpha - 12\gamma}}{2\sqrt{3\gamma}}$$

and the torsion $T^{\alpha\beta\gamma}$ of its characteristic connection $\nabla^{\alpha\beta\gamma}$ is, in a suitable orthonormal basis, given by

$$T^{\alpha\beta\gamma} = -\frac{2\sqrt{3}}{\sqrt{\gamma}} (e_1 \wedge e_2 \wedge e_3 + e_1 \wedge e_4 \wedge e_5).$$

Its holonomy is $\mathrm{SO}(3)_{\mathrm{ir}} \subset \mathrm{SO}(5)$. Its torsion is not parallel, $\nabla^{\alpha\beta\gamma} T^{\alpha\beta\gamma} \neq 0$, but it is divergence-free, $\delta T^{\alpha\beta\gamma} = 0$.

- (2) The metric of the nearly integrable $\mathrm{SO}(3)_{\mathrm{ir}}$ structure is never naturally reductive and never Einstein.

Proof. In contrast to the spaces treated before, we choose the more appropriate basis of $\mathfrak{so}(3)_{\text{ir}} \subset \mathfrak{so}(5)$

$$Y_3 := h = E_{23} + 2E_{45}, \quad Y_1 := -\sqrt{3}E_{12} + E_{35} + E_{24}, \quad Y_2 := -\sqrt{3}E_{13} + E_{25} - E_{34}.$$

These elements satisfy the same commutator rules as the previously used basis X_1, X_2, X_3 . We describe a metric connection $\Lambda_{\mathfrak{m}} : \mathfrak{h}_{\mu} \rightarrow \mathfrak{so}(3)_{\text{ir}}$ and its invariant torsion T via the standard Ansatz ($a, b, c, m, n \in \mathbb{R}$)

$$\Lambda_{\mathfrak{m}}(e_1^{\mu}) = aY_3, \quad \Lambda_{\mathfrak{m}}(e_2^{\mu}) = bY_1 - cY_2, \quad \Lambda_{\mathfrak{m}}(e_3) = cY_1 + bY_2, \quad \Lambda_{\mathfrak{m}}(e_4) = \Lambda_{\mathfrak{m}}(e_5) = 0, \\ T = m e_1^{\mu} \wedge e_2^{\mu} \wedge e_3 + n e_1^{\mu} \wedge e_4 \wedge e_5.$$

Using the relation (*) between T and $\Lambda_{\mathfrak{m}}$, one checks that the coefficients have to satisfy $b = 0$, $c = 1/\sqrt{\gamma}$ and

$$m = a - \frac{\sqrt{3}}{\sqrt{\gamma}} + \frac{\mu}{\sqrt{\alpha}}, \quad n = 2a + 2\frac{\mu}{\sqrt{\alpha}}, \quad m = -\frac{2\sqrt{3}}{\sqrt{\gamma}}, \quad n = -\frac{2\mu\sqrt{\alpha}}{\gamma(\mu^2 + 1)}.$$

Thus, a short calculation yields $a = -\sqrt{3/\gamma} - \mu/\sqrt{\alpha}$ and the quadratic equation for the embedding parameter μ

$$\sqrt{3}\gamma\mu^2 - \sqrt{\alpha\gamma}\mu + \gamma\sqrt{3} = 0.$$

It has real solutions if and only if $\alpha \geq 12\gamma$; for further use, we give the final expression for the non-vanishing parts of the connection map $\Lambda_{\mathfrak{m}}$:

$$\Lambda_{\mathfrak{m}}(e_1) = -\left[\frac{\sqrt{3}}{\sqrt{\gamma}} + \frac{\mu}{\sqrt{\alpha}}\right]Y_3, \quad \Lambda_{\mathfrak{m}}(e_2) = -\frac{1}{\sqrt{\gamma}}Y_2, \quad \Lambda_{\mathfrak{m}}(e_3) = \frac{1}{\sqrt{\gamma}}Y_2,$$

where it is understood that μ takes one of the two admissible values. We observe that $\Lambda_{\mathfrak{m}} = 0$ is not in the admissible parameter range, hence the nearly integrable $\text{SO}(3)_{\text{ir}}$ structure is never naturally reductive. Using formula (**) for the action of $\nabla^{\alpha\beta\gamma}$ on 2-forms, one checks that T has the non vanishing covariant derivatives

$$\nabla_{e_2}^{\alpha\beta\gamma}T = \Lambda_{\mathfrak{m}}(e_2)T = -\frac{6}{\gamma}e_3 \wedge e_4 \wedge e_5, \quad \nabla_{e_3}^{\alpha\beta\gamma}T = \Lambda_{\mathfrak{m}}(e_3)T = \frac{6}{\gamma}e_2 \wedge e_4 \wedge e_5.$$

One next computes the map $\Lambda_{\mathfrak{m}}^g$ representing the Levi-Civita connection,

$$\Lambda_{\mathfrak{m}}^g(e_1) = -\frac{\mu}{\sqrt{\alpha}}E_{23} - \left[\frac{2\mu}{\sqrt{\alpha}} + \frac{\sqrt{3}}{\sqrt{\gamma}}\right]E_{45}, \quad \Lambda_{\mathfrak{m}}^g(e_2) = \frac{1}{\gamma}(E_{34} - E_{25}), \\ \Lambda_{\mathfrak{m}}^g(e_3) = \frac{1}{\gamma}(E_{24} + E_{35}), \quad \Lambda_{\mathfrak{m}}^g(e_4) = -\frac{\sqrt{3}}{\sqrt{\gamma}}E_{15}, \quad \Lambda_{\mathfrak{m}}^g(e_5) = \frac{\sqrt{3}}{\sqrt{\gamma}}E_{14}.$$

One deduces that the only non-vanishing ∇^g -derivatives of the elementary invariant forms e_{123} and e_{145} are

$$\nabla_{e_2}^g e_{123} = \frac{1}{\gamma}(e_{135} + e_{124}), \quad \nabla_{e_3}^g e_{123} = \frac{1}{\gamma}(e_{125} - e_{134}), \quad \nabla_{e_4}^g e_{123} = -\frac{\sqrt{3}}{\sqrt{\gamma}}e_{235}, \quad \nabla_{e_4}^g e_{123} = \frac{\sqrt{3}}{\sqrt{\gamma}}e_{234}$$

as well as

$$\nabla_{e_2}^g e_{145} = -\frac{1}{\gamma}(e_{135} + e_{124}), \quad \nabla_{e_3}^g e_{145} = \frac{1}{\gamma}(e_{134} - e_{125}),$$

One thus checks that

$$de_{123} = -\frac{2\sqrt{3}}{\sqrt{\gamma}}e_{2345}, \quad de_{145} = 0, \quad dT = \frac{6}{\gamma}e_{2345} \neq 0, \quad \delta T = -\sum_{i=1}^5 e_i \lrcorner \nabla_{e_i}^g T = 0.$$

A computation of the Riemannian curvature tensor and its contraction shows that the Ricci tensor is diagonal, with

$$\text{Ric}^g(e_1, e_1) = \frac{2\mu^2\alpha}{\gamma^2(\mu^2 + 1)}, \quad \text{Ric}^g(e_2, e_2) = \text{Ric}^g(e_3, e_3) = 0, \quad \text{Ric}^g(e_4, e_4) = \text{Ric}^g(e_5, e_5),$$

and with the somehow uninspiring expression

$$\operatorname{Ric}^g(e_4, e_4) = -\frac{(\alpha + 6\gamma)(\mu^4 + 2\mu^2) + 6\gamma}{\gamma^2(\mu^2 + 1)^2}.$$

However, it is plain that the space will never be Einstein. \square

Remark 4.5. It is a natural question to ask in this case, as before for the twisted Stiefel manifold, about compatible contact structures. It turns out that the picture is rather different. As explained in Theorem 4.3 and its proof,

$$\xi \cong \eta = e_1, \quad \varphi_{\pm} = -E_{23} \pm E_{45},$$

is a natural choice for an almost contact structure (basically, because the isotropy representation is the same). However, one checks that the Nijenhuis tensor of φ_+ is not a 3-form, hence, by [FrI02, Thm 8.2], it does not admit an invariant metric connection with skew-symmetric torsion. For φ_- , the Nijenhuis tensor vanishes and $dF_- = 0$, but the corresponding contact connection has torsion

$$T^- = \eta \wedge d\eta = -\frac{2\sqrt{3}}{\sqrt{\gamma}} e_{145} \neq T^{\alpha\beta\gamma}.$$

Thus, the almost metric contact structure defined by (η, φ_-) is not compatible with the $\mathrm{SO}(3)_{\mathrm{ir}}$ structure.

Remark 4.6. In [BN07, p.77 and p.88], it was claimed that W^{ir} with the standard embedding of $\mathrm{SO}(2) \rightarrow \mathrm{SL}(2, \mathbb{R})$ (corresponding to $\mu = 0$ in our notation) carries an $\mathrm{SO}(3)_{\mathrm{ir}}$ structure (that was not further investigated). The results above correct this point.

REFERENCES

- [Ag06] I. Agricola, *The Srní lectures on non-integrable geometries with torsion*, Arch. Math. (Brno) 42 (2006), 5–84. With an appendix by Mario Kassuba.
- [Agr03] ———, *Connections on naturally reductive spaces, their Dirac operator and homogeneous models in string theory*, Comm. Math. Phys. 232 (2003), 535–563.
- [AF06] I. Agricola and Th. Friedrich, *Geometric structures of vectorial type*, J. Geom. Phys. 56 (2006), 2403–2414.
- [AFS05] B. Alexandrov, Th. Friedrich, Nils Schoemann, *Almost Hermitian 6-manifolds revisited*, J. Geom. Phys. 53 (2005), 1–30.
- [At70] M. F. Atiyah, *Vector fields on manifolds*, Arbeitsgemeinschaft Forsch. Nordrhein-Westfalen Heft 200, 26 S. (1970).
- [BFGK91] H. Baum, Th. Friedrich, R. Grunewald, I. Kath, *Twistors and Killing spinors on Riemannian manifolds*, Teubner-Texte zur Mathematik, Band 124, Teubner-Verlag Stuttgart / Leipzig, 1991.
- [BB09] J. Becker-Bender, *SO(3)-Strukturen auf 5-dimensionalen Mannigfaltigkeiten*, diploma thesis, Humboldt-Universität zu Berlin, April 2009.
- [Bob06] M. Bobieński, *The topological obstructions to the existence of an irreducible SO(3)-structure on a five manifold*, math.DG/0601066.
- [BN07] M. Bobieński and P. Nruowski, *Irreducible SO(3)-geometries in dimension five*, J. Reine Angew. Math. 605 (2007), 51–93.
- [CF07] S. Chiossi and A. Fino, *Nearly integrable SO(3) structures on 5-dimensional Lie groups*, J. Lie Theory 17 (2007), 539–562.
- [DW59] A. Dold, H. Whitney, *Classification of oriented sphere bundles over 4-complexes*, Ann. of Math. 69, (1959), 667–677.
- [Fr02] Th. Friedrich, *Dirac operators in Riemannian geometry*, Graduate Studies in Mathematics 25, AMS, Providence, Rhode Island, 2000.
- [Fri03b] ———, *On types of non-integrable geometries*, Suppl. Rend. Circ. Mat. di Palermo Ser. II, 71 (2003), 99–113.
- [Fri80] ———, *Der erste Eigenwert des Dirac-Operators einer kompakten, Riemannschen Mannigfaltigkeit nichtnegativer Skalarkrümmung*, Math. Nachr. 97 (1980), 117–146.
- [FrI02] Th. Friedrich and S. Ivanov, *Parallel spinors and connections with skew-symmetric torsion in string theory*, Asian Journ. Math. 6 (2002), 303–336.
- [Fu01] M. Furuta, *Monopole equation and the $\frac{11}{8}$ -conjecture*, Math. Res. Lett. 8 (2001), 279–291.
- [Jen75] G. Jensen, *Imbeddings of Stiefel manifolds into Grassmannians*, Duke Math. J. 42 (1975), 397–407.
- [Kir77] V. F. Kirichenko, *K-spaces of maximal rank*, Mat. Zam. 22 (1977), 465–476.

- [KN91] S. Kobayashi and K. Nomizu, *Foundations of differential geometry I*, Wiley Classics Library, Wiley Inc., Princeton, 1963, 1991.
- [KN96] ———, *Foundations of differential geometry II*, Wiley Classics Library, Wiley Inc., Princeton, 1969, 1996.
- [LMP69] G. Lusztig, J. M. Milnor, G. Peterson, *Semi-characteristics and cobordism*, *Topology* 8 (1969), 357-359.
- [MS74] John W. Milnor and James D. Stasheff, *Characteristic classes*, Princeton University Press, 1974.
- [Pae56] G. Paechter, *The groups $\pi_r(V_{n,m})$* , *Quart. J. Math. Oxford* 7 (1956), 249-268.
- [Serre70] J.-P. Serre, *Cours d'arithmétique*, Paris 1970.
- [Span66] E. Spanier, *Algebraic topology*, Springer 1966.
- [Th68] E. Thomas, *Vector fields on low dimensional manifolds*, *Math. Zeitschr.* 103 (1968), 85-93.

ILKA AGRICOLA, JULIA BECKER-BENDER
FACHBEREICH MATHEMATIK UND INFORMATIK
PHILIPPS-UNIVERSITÄT MARBURG
HANS-MEERWEIN-STRASSE
D-35032 MARBURG, GERMANY
agricola@mathematik.uni-marburg.de
beckbend@mathematik.uni-marburg.de

THOMAS FRIEDRICH
INSTITUT FÜR MATHEMATIK
HUMBOLDT-UNIVERSITÄT ZU BERLIN
SITZ: WBC ADLERSHOF
D-10099 BERLIN, GERMANY
friedric@mathematik.hu-berlin.de