DGA 2019

Generalizations of 3-Sasakian manifolds and skew torsion

Ilka Agricola

Philipps Universität Marburg

Joint work with Giulia Dileo (Bari) and Leander Stecker (Marburg)

A few classical facts

- 1960: Sasaki introduces Sasakian manifolds
- 1970: 3-Sasakian manifolds defined (Kuo, Udriste)
- Quick definition:
  (M^{4n+3}, g) is 3-Sasakian if its metric cone $(\mathbb{R}^+ \times M, dr^2 + r^2 g)$ has holonomy inside $\mathrm{Sp}(n+1)$, i. e. it is hyperkähler.
  - odd Betti numbers up to middle dimension are divisible by 4, structure group is $\mathrm{Sp}(n) \times \mathrm{Id}_3$, it's spin (Kuo)
  - they are Einstein (Kashiwada, 1971)
  - relation to quaternionic Hopf fibration $S^3 \to S^7 \to S^4$ (Tanno, 1971) and quaternionic Kähler manifolds (Ishihara, 1974; Salamon, 1982)
  - $\Leftrightarrow$ there exist three Killing spinors (Friedrich-Kath, 1990)
  - Many examples, classification of homogeneous case (Boyer-Galicki, $\geq$ 1993)
- Berger's holonomy Theorem: Does not cover any contact manifolds, meaning that the Levi-Civita connection is not adapted for investigating such geometries

Context: Geometry of almost 3-contact metric manifolds

Goals

Define and investigate new classes of such manifolds:

- what geometric quantities are best suited for capturing their key geometric properties – in particular, the relative behaviour of the 3 almost contact structures?
- should admit 'good' metric connections with skew torsion

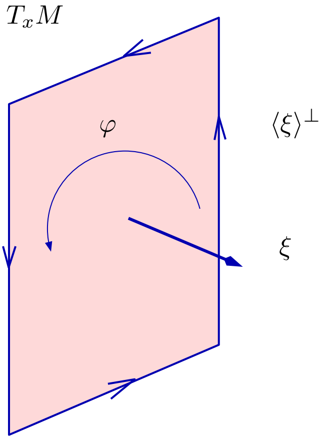
In particular,

- introduce 'Reeb commutator function' and 'Reeb Killing function',
- define the new class of 3-(α, δ)-Sasaki manifolds,
- introduce notion of φ-compatible connections,
- make them unique by a certain extra condition $\rightarrow$ canonical connection,
- compute torsion, holonomy, curvature of this connection,
- provide lots of examples, classify the homogeneous ones, further applications (metric cone, generalized Killing spinors. . .),

Almost contact metric mnfds

$(M^{2n+1}, g, \eta, \xi, \varphi)$ almost contact metric mnfd if

- η: 1-form (dual to vector field ξ)
- $\langle \xi \rangle^{\perp}$ admits an almost complex structure φ compatible with g.



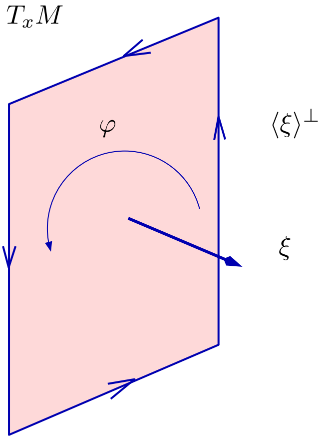
Almost contact metric mnfds

$(M^{2n+1}, g, \eta, \xi, \varphi)$ almost contact metric mnfd if

- η: 1-form (dual to vector field ξ)
- $\langle \xi \rangle^\perp$ admits an almost complex structure φ compatible with g.

Then,

- the structure group is reducible to $U(n) \times \{1\}$,
- the fundamental 2-form is defined by
$$\Phi(X, Y) = g(X, \varphi Y),$$
- it is called normal if $N_\varphi := [\varphi, \varphi] + d\eta \otimes \xi \equiv 0$,
- α-Sasakian, $\alpha \in \mathbb{R}^*$, if $\mathrm{d}\eta = 2\alpha\Phi, \quad N_\varphi \equiv 0 \quad (\Rightarrow \xi$ Killing)
- Sasakian if 1-Sasakian.



Special geometries via connections with (skew) torsion

Given a mnfd M^n with G-structure ($G \subset \mathrm{SO}(n)$), replace ∇^g by a *metric connection ∇ with torsion that preserves the geometric structure!*

$$\text{torsion: } T(X,Y,Z) := g(\nabla_X Y - \nabla_Y X - [X,Y], Z)$$

Special case: require $T \in \Lambda^3(M^n)$ ($\Leftrightarrow$ same geodesics as ∇^g)

$$\Rightarrow \quad g(\nabla_X Y, Z) = g(\nabla^g_X Y, Z) + \tfrac{1}{2} T(X,Y,Z)$$

If existent and unique it is called 'characteristic connection'.

Theorem (Friedrich-Ivanov, 2002)

An almost contact metric manifold (M, ϕ, ξ, η, g) admits a unique metric connection ∇ with skew torsion satisfying $\nabla\eta = \nabla\xi = \nabla\varphi = 0$ iff

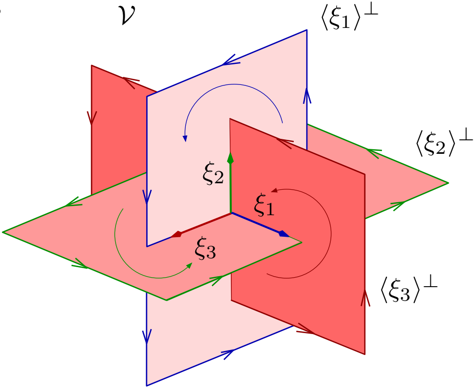
1. *the tensor $N_\varphi := [\varphi, \varphi] + d\eta \otimes \xi$ is totally skew-symmetric,*
2. *ξ is a Killing vector field.*

In particular, it exists for α-Sasaki mnfds and its torsion $T = \eta \wedge d\eta$ is parallel. [Kowalski-Wegrzynowski, 1987]

Almost 3-contact metric mnfds

$(M^{4n+3}, g, \eta_i, \xi_i, \varphi_i), i = 1, 2, 3$
almost 3-contact metric mnfd if

- each triple $(\eta_i, \xi_i, \varphi_i)$ defines an a.c.m. str. on M^{4n+3}
- $TM = \mathcal{H} \oplus \mathcal{V}$ with
  $\mathcal{H} := \bigcap_{i=1}^{3} \ker \eta_i$,
  $\mathcal{V} := \langle \xi_1, \xi_2, \xi_3 \rangle$
- Compatibility conditions:
  $\xi_1 \times \xi_2 = \xi_3$ on $\mathcal{V}$
  $\varphi_1 \circ \varphi_2 = \varphi_3$ in $\mathcal{H}$
  $\varphi_1(\xi_2) = \xi_3$ + cyclic perm.
- structure group reducible to $\mathrm{Sp}(n) \times \{1_3\}$



The manifold is said to be hypernormal if $N_{\varphi_i} \equiv 0$, $i = 1, 2, 3$.

Some remarkable classes:

$$\forall i = 1, 2, 3 :$$

3-α-Sasakian (3-Sasakian)	$(\varphi_i, \xi_i, \eta_i, g)$ is α-Sasakian ($\alpha = 1$)	$\}\Rightarrow$ Einstein!
3-cosymplectic	$(\varphi_i, \xi_i, \eta_i, g)$ is cosymplectic	
3-quasi-Sasakian	$(\varphi_i, \xi_i, \eta_i, g)$ is quasi Sasakian	

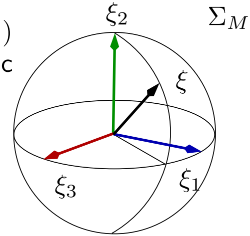
Observe: No new conditions on the relative 'behaviour' of the three single a.c.m. structures, just for each single structure!

Theorem (Kashiwada, 2001)

If $d\eta_i = 2\Phi_i$, $i = 1, 2, 3$, then the manifold is hypernormal (and thus 3-Sasakian).

The associated sphere of a.c.m. structures Σ_M

Any almost 3-contact metric mnfd $(M^{4n+3}, g, \eta_i, \xi_i, \varphi_i)$ comes with a sphere $\Sigma_M \cong S^2$ of almost contact metric structures:



$\forall a = (a_1, a_2, a_3) \in S^2 \subset \mathbb{R}^3$ put

$$\varphi_a = \sum_{i=1}^{3} a_i \varphi_i, \quad \xi_a = \sum_{i=1}^{3} a_i \xi_i, \quad \eta_a = \sum_{i=1}^{3} a_i \eta_i.$$

Then $(\varphi_a, \xi_a, \eta_a, g)$ defines an almost contact metric structure on M^{4n+3}.

Theorem (Cappelletti Montano - De Nicola - Yudin, 2016)

If $N_{\varphi_i} = 0$ for all $i = 1, 2, 3$, then $N_\varphi = 0$ for all $\varphi \in \Sigma_M$.

Theorem

If each N_{φ_i} is skew symmetric on $\mathcal{H}$ (resp. on TM), then for all $\varphi \in \Sigma_M$, N_φ is skew symmetric on $\mathcal{H}$ (resp. on TM).

Proposition

Let $(M, \varphi_i, \xi_i, \eta_i, g)$ *be a almost* 3*-contact metric manifold. If each* $(\varphi_i, \xi_i, \eta_i, g)$*,* $i = 1, 2, 3$ *admits a characteristic connection, the same holds for every structure in the sphere.*

Proposition

Let $(M, \varphi_i, \xi_i, \eta_i, g)$ *be a almost* 3*-contact metric manifold. If each* $(\varphi_i, \xi_i, \eta_i, g)$, $i = 1, 2, 3$ *admits a characteristic connection, the same holds for every structure in the sphere.*

Do these connections coincide?

Is it possible to find a metric connection with skew torsion parallelizing ALL the structure tensor fields?

Proposition

Let $(M, \varphi_i, \xi_i, \eta_i, g)$ *be a almost* 3*-contact metric manifold. If each* $(\varphi_i, \xi_i, \eta_i, g)$, $i = 1, 2, 3$ *admits a characteristic connection, the same holds for every structure in the sphere.*

Do these connections coincide?

Is it possible to find a metric connection with skew torsion parallelizing ALL the structure tensor fields?

! For a 3-Sasakian manifold the characteristic connection of the structure $(\varphi_i, \xi_i, \eta_i, g)$ is

$$\nabla^i = \nabla^g + \frac{1}{2} T_i, \qquad T_i = \eta_i \wedge d\eta_i.$$

For $i \neq j$, $T_i \neq T_j$ and thus $\nabla^i \neq \nabla^j$
$\Rightarrow$ No characteristic connection for 3-Sasakian manifolds!

Canonical connection for 7-dimensional 3-Sasaki manifolds (Agricola-Friedrich, 2010)

Let $(M, \varphi_i, \xi_i, \eta_i, g)$ be a 7-dimensional 3-Sasakian manifold.

The 3-form

$$\omega := \frac{1}{2} \sum_i \eta_i \wedge d\eta_i + 4\,\eta_{123} \qquad \eta_{123} := \eta_1 \wedge \eta_2 \wedge \eta_3$$

defines a *cocalibrated G_2-structure* and hence admits a characteristic connection ∇; its torsion is

$$T = \sum_{i=1}^{3} \eta_i \wedge d\eta_i$$

∇ is called the canonical connection, and verifies the following:

- it preserves $\mathcal{H}$ and $\mathcal{V}$,
- $\nabla T = 0$,
- ∇ admits a parallel spinor ψ, called *canonical spinor*, such that the Clifford products $\xi_i \cdot \psi$ are exactly the 3 Riemannian Killing spinors.

Canonical connection for quaternionic Heisenberg groups

$N_p \cong \mathbb{R}^{4p+3}$ connected, simply connected Lie group, with commutators depending on a parameter $\lambda > 0$.

N_p admits an almost 3-contact metric structure $(\varphi_i, \xi_i, \eta_i, g_\lambda)$ which is hypernormal but not 3-quasi-Sasakian. None of the metrics g_λ is Einstein.

The canonical connection is the metric connection ∇ with skew torsion (Agricola-Ferreira-Storm, 2015)

$$T = \sum_{i=1}^{3} \eta_i \wedge d\eta_i \; - 4\,\lambda\,\eta_{123}$$

It satisfies:

- $\nabla T = \nabla R = 0 \rightsquigarrow$ naturally reductive homogeneous space,
- $\mathfrak{hol}(\nabla) \simeq \mathfrak{su}(2)$, acting irreducibly on $\mathcal{V}$ and $\mathcal{H}$.

Canonical connection for quaternionic Heisenberg groups

$N_p \cong \mathbb{R}^{4p+3}$ connected, simply connected Lie group, with commutators depending on a parameter $\lambda > 0$.

N_p admits an almost 3-contact metric structure $(\varphi_i, \xi_i, \eta_i, g_\lambda)$ which is hypernormal but not 3-quasi-Sasakian. None of the metrics g_λ is Einstein.

The canonical connection is the metric connection ∇ with skew torsion (Agricola-Ferreira-Storm, 2015)

$$T = \sum_{i=1}^{3} \eta_i \wedge d\eta_i \; - 4\,\lambda\,\eta_{123}$$

It satisfies:

- $\nabla T = \nabla R = 0 \rightsquigarrow$ naturally reductive homogeneous space,
- $\mathfrak{hol}(\nabla) \simeq \mathfrak{su}(2)$, acting irreducibly on $\mathcal{V}$ and $\mathcal{H}$.

In the 7-dim. case, ∇ is the *characteristic connection* of a cocalibrated G_2 structure $\Rightarrow \exists$ parallel spinor field ψ and $\psi_i := \xi_i \cdot \psi$, $i = 1, 2, 3$, are generalised Killing spinors:

$$\nabla^g_{\xi_i} \psi_i = \frac{\lambda}{2}\, \xi_i \cdot \psi_i, \quad \nabla^g_{\xi_j} \psi_i = -\frac{\lambda}{2}\, \xi_j \cdot \psi_i \ (i \neq j), \quad \nabla^g_X \psi_i = \frac{5\lambda}{4}\, X \cdot \psi_i, X \in \mathcal{H}$$

Well-known:

- the metric cone of a 3-Sasakian manifold is hyper-Kähler
- the metric cone of the quaternionic Heisenberg group is a hyper-Kähler manifold with torsion ('HKT manifold')

Agricola-Höll, 2015: Criterion when the metric cone (for suitable $a > 0$)

$$(\bar{M}, \bar{g}) = (M \times \mathbb{R}^+, a^2 r^2 g + dr^2)$$

of an almost 3-contact metric manifold M admits a hyper-Hermitian structure, and when it is a HKT manifold (but unclear what a 'good' large class of manifolds satisfying the criterion could be)

Well-known:

- the metric cone of a 3-Sasakian manifold is hyper-Kähler
- the metric cone of the quaternionic Heisenberg group is a hyper-Kähler manifold with torsion ('HKT manifold')

Agricola-Höll, 2015: Criterion when the metric cone (for suitable $a > 0$)

$$(\bar{M}, \bar{g}) = (M \times \mathbb{R}^+, a^2 r^2 g + dr^2)$$

of an almost 3-contact metric manifold M admits a hyper-Hermitian structure, and when it is a HKT manifold (but unclear what a 'good' large class of manifolds satisfying the criterion could be)

Is it possible to find a larger class of almost 3-contact metric manifolds with similar properties?

3-(α, δ)-Sasaki manifolds

Definition

A 3-(α, δ)-*Sasaki manifold is an almost* 3*-contact metric manifold* $(M, \varphi_i, \xi_i, \eta_i, g)$ *such that*

$$d\eta_i = 2\alpha\Phi_i + 2(\alpha - \delta)\eta_j \wedge \eta_k,$$

$\alpha \in \mathbb{R}^*, \delta \in \mathbb{R}$, (i, j, k) *even permutation of* $(1, 2, 3)$.

3-(α, δ)-Sasaki manifolds

Definition

A 3-(α, δ)*-Sasaki manifold is an almost* 3*-contact metric manifold* $(M, \varphi_i, \xi_i, \eta_i, g)$ *such that*

$$d\eta_i = 2\alpha\Phi_i + 2(\alpha - \delta)\eta_j \wedge \eta_k,$$

$\alpha \in \mathbb{R}^*, \delta \in \mathbb{R}$, (i, j, k) *even permutation of* $(1, 2, 3)$.

- 3-α-Sasakian manifolds: $d\eta_i = 2\alpha\Phi_i \rightsquigarrow \alpha = \delta$
- quat. Heisenberg groups: $d\eta_i = \lambda(\Phi_i + \eta_j \wedge \eta_k) \rightsquigarrow 2\alpha = \lambda, \delta = 0$

3-(α, δ)-Sasaki manifolds

Definition

A 3-(α, δ)-Sasaki manifold is an almost 3-contact metric manifold $(M, \varphi_i, \xi_i, \eta_i, g)$ such that

$$d\eta_i = 2\alpha\Phi_i + 2(\alpha - \delta)\eta_j \wedge \eta_k,$$

$\alpha \in \mathbb{R}^*, \delta \in \mathbb{R}$, (i, j, k) *even permutation of* $(1, 2, 3)$.

- 3-α-Sasakian manifolds: $d\eta_i = 2\alpha\Phi_i \rightsquigarrow \alpha = \delta$
- quat. Heisenberg groups: $d\eta_i = \lambda(\Phi_i + \eta_j \wedge \eta_k) \rightsquigarrow 2\alpha = \lambda, \delta = 0$

We call the structure degenerate if $\delta = 0$ and nondegenerate otherwise.

Theorem

For every 3-(α, δ)-Sasaki manifold:

- *the structure is hypernormal (generalization of Kashiwada's thm),*
- *the distribution $\mathcal{V}$ is integrable with totally geodesic leaves,*
- *each ξ_i is a Killing vector field, and $[\xi_i, \xi_j] = 2\delta\xi_k$.*

Definition

An $\mathcal{H}$-homothetic deformation of an almost 3-contact metric strucure $(\varphi_i, \xi_i, \eta_i, g)$ is given by

$$\eta_i' = c\eta_i, \quad \xi_i' = \frac{1}{c}\xi_i, \qquad \varphi_i' = \varphi_i, \qquad g' = ag + b\sum_{i=1}^{3} \eta_i \otimes \eta_i,$$

$a, b, c \in \mathbb{R}$, $a > 0$, $c^2 = a + b > 0$.

If $(\varphi_i, \xi_i, \eta_i, g)$ is 3-(α, δ)-Sasaki, then $(\varphi_i', \xi_i', \eta_i', g')$ is 3-(α', δ')-Sasaki with

$$\alpha' = \alpha\frac{c}{a}, \qquad \delta' = \frac{\delta}{c}.$$

- the class of degenerate 3-(α, δ)-Sasaki structures is preserved
- in the non-degenerate case, the sign of $\alpha\delta$ is preserved.

Definition

We say that a 3-(α, δ)-Sasaki manifold is positive (resp. negative) if $\alpha\delta > 0$ (resp. $\alpha\delta < 0$).

Proposition

$\alpha\delta > 0 \iff M$ is $\mathcal{H}$-homothetic to a 3-Sasakian manifold ($\alpha = \delta = 1$)
$\alpha\delta < 0 \iff M$ is $\mathcal{H}$-homothetic to one with $\alpha = -1$, $\delta = 1$.

Do there exist 3-(α, δ)-Sasaki manifolds with $\alpha\delta < 0$?

YES – here is a construction:

Definition

A negative 3-Sasakian manifold is a normal almost 3-contact manifold $(M, \varphi_i, \xi_i, \eta_i)$ endowed with a compatible semi-Riemannian metric $\tilde{g}$ of signature $(3, 4n)$ and s. t. $d\eta_i(X, Y) = 2\tilde{g}(X, \varphi_i Y)$.

Proposition

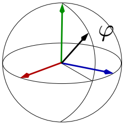
If $(M, \varphi_i, \xi_i, \eta_i, \tilde{g})$ is a negative 3-Sasakian manifold, take

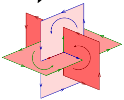
$$g = -\tilde{g} + 2\sum_{i=1}^{3} \eta_i \otimes \eta_i.$$

Then $(\varphi_i, \xi_i, \eta_i, g)$ is a 3-(α, δ)-Sasaki structure with $\alpha = -1$ and $\delta = 1$.

It is known that quat. Kähler (not hK) mnfds with neg. scalar curvature admit a canonically associated principal $\mathrm{SO}(3)$-bundle which is endowed with a negative 3-Sasakian structure (Konishi, 1975/Tanno, 1996).

Overview: Hierarchy of ‘good’ connections





φ-compatible connections $\supset$ **canonical connection**

φ-compatible connections:

- depend only on $\varphi \in \Sigma_M$
- main defining condition: $(\nabla_X \varphi)Y = 0 \qquad \forall X, Y \in \Gamma(\mathcal{H})$
- not unique: depends on a parameter function γ
- exist under very weak assumptions

canonical connection:

- depends on the whole a. 3-contact m. str. $(\beta := 2(\delta - 2\alpha))$
- main defining condition: $\nabla_X \varphi_i = \beta(\eta_k(X)\varphi_j - \eta_j(X)\varphi_k) \quad \forall X \in \mathfrak{X}(M)$
- unique: corresponds to $\gamma = 2(\beta - \delta)$
- exists on all 3-(α, δ)-Sasaki manifolds (and again some weaker assumptions)

φ-compatible connections

Definition

Let $(M, \varphi_i, \xi_i, \eta_i, g)$ *be an almost* 3*-contact metric manifold,* φ *a structure in the associated sphere* Σ_M*. Let* ∇ *be a metric connection with skew torsion on* M*. We say that* ∇ *is a* *φ-compatible connection* *if*

1) ∇ *preserves the splitting* $TM = \mathcal{H} \oplus \mathcal{V}$,
2) $(\nabla_X \varphi)Y = 0 \qquad \forall X, Y \in \Gamma(\mathcal{H})$.

φ-compatible connections

Definition

Let $(M, \varphi_i, \xi_i, \eta_i, g)$ be an almost 3-contact metric manifold, φ a structure in the associated sphere Σ_M. Let ∇ be a metric connection with skew torsion on M. We say that ∇ is a φ-compatible connection if

1) ∇ *preserves the splitting* $TM = \mathcal{H} \oplus \mathcal{V}$,
2) $(\nabla_X \varphi)Y = 0 \qquad \forall X, Y \in \Gamma(\mathcal{H})$.

Theorem

M admits a φ-compatible connection if

1) N_φ *is skew-symmetric on* $\mathcal{H}$,
2) *each* ξ_i *is Killing.*

Remark This is a special case of an iff criterion. φ-compatible connections are parametrized by their parameter function

$$\gamma := T(\xi_1, \xi_2, \xi_3) \in C^\infty(M).$$

φ-compatible connections

Definition

Let $(M, \varphi_i, \xi_i, \eta_i, g)$ be an almost 3-contact metric manifold, φ a structure in the associated sphere Σ_M. Let ∇ be a metric connection with skew torsion on M. We say that ∇ is a φ-compatible connection if

1) ∇ *preserves the splitting* $TM = \mathcal{H} \oplus \mathcal{V}$,
2) $(\nabla_X \varphi)Y = 0 \qquad \forall X, Y \in \Gamma(\mathcal{H})$.

Theorem

M admits a φ-compatible connection if

1) N_φ *is skew-symmetric on* $\mathcal{H}$,
2) *each* ξ_i *is Killing.*

Remark This is a special case of an iff criterion. φ-compatible connections are parametrized by their parameter function

$$\gamma := T(\xi_1, \xi_2, \xi_3) \in C^\infty(M).$$

The canonical connection

$\nabla\varphi_i \equiv 0$ is too strong $\rightsquigarrow$ suppose ∇ preserves the 3-dim. distribution in $\mathrm{End}(TM)$ spanned by φ_i as do quaternionic connections (qK case):

$$\nabla_X\varphi_i = \beta(\eta_k(X)\varphi_j - \eta_j(X)\varphi_k) \quad \forall X \in \mathfrak{X}(M)$$

The canonical connection

Theorem

Let $(M, \varphi_i, \xi_i, \eta_i, g)$ be a 3-(α, δ)-Sasakian manifold. Then M admits a metric connection ∇ with skew torsion such that for a smooth function β,

$$\nabla_X \varphi_i = \beta(\eta_k(X)\varphi_j - \eta_j(X)\varphi_k) \quad \forall X \in \mathfrak{X}(M)$$

for every even permutation (i, j, k) of $(1, 2, 3)$.

Such a connection ∇ is unique, preserves the splitting $TM = \mathcal{V} \oplus \mathcal{H}$ and the φ_i are parallel along $\mathcal{H}$.

∇ is called the canonical connection of M. The function β is a constant given by

$$\beta = 2(\delta - 2\alpha).$$

The canonical connection ∇ satisfies

$$\nabla_X \varphi_i = \beta(\eta_k(X)\varphi_j - \eta_j(X)\varphi_k),$$
$$\nabla_X \xi_i = \beta(\eta_k(X)\xi_j - \eta_j(X)\xi_k),$$
$$\nabla_X \eta_i = \beta(\eta_k(X)\eta_j - \eta_j(X)\eta_k),$$

and also

$$\nabla \Psi = 0, \qquad \nabla \eta_{123} = 0,$$

$\Psi := \Phi_1 \wedge \Phi_1 + \Phi_2 \wedge \Phi_2 + \Phi_3 \wedge \Phi_3$, fundamental 4-form. In particular

$$\mathfrak{hol}(\nabla) \subset (\mathfrak{sp}(n) \oplus \mathfrak{sp}(1)) \oplus \mathfrak{so}(3) \subset \mathfrak{so}(4n) \oplus \mathfrak{so}(3).$$

For parallel canonical manifolds ($\beta = 0$):

$$\nabla \varphi_i = 0,\ \nabla \xi_i = 0,\ \nabla \eta_i = 0,\ \text{and } \mathfrak{hol}(\nabla) \subset \mathfrak{sp}(n)$$

$\Rightarrow$ canonical conn. $=$ characteristic conn. of all 3 a.c.m. str.
[first known examples where this happens!]

The canonical connection ∇ satisfies

$$\nabla_X \varphi_i = \beta(\eta_k(X)\varphi_j - \eta_j(X)\varphi_k),$$
$$\nabla_X \xi_i = \beta(\eta_k(X)\xi_j - \eta_j(X)\xi_k),$$
$$\nabla_X \eta_i = \beta(\eta_k(X)\eta_j - \eta_j(X)\eta_k),$$

and also

$$\nabla \Psi = 0, \qquad \nabla \eta_{123} = 0,$$

$\Psi := \Phi_1 \wedge \Phi_1 + \Phi_2 \wedge \Phi_2 + \Phi_3 \wedge \Phi_3$, fundamental 4-form. In particular

$$\mathfrak{hol}(\nabla) \subset (\mathfrak{sp}(n) \oplus \mathfrak{sp}(1)) \oplus \mathfrak{so}(3) \subset \mathfrak{so}(4n) \oplus \mathfrak{so}(3).$$

For parallel canonical manifolds ($\beta = 0$):

$$\nabla \varphi_i = 0,\ \nabla \xi_i = 0,\ \nabla \eta_i = 0,\ \text{and } \mathfrak{hol}(\nabla) \subset \mathfrak{sp}(n)$$

$\Rightarrow$ canonical conn. = characteristic conn. of all 3 a.c.m. str.
[first known examples where this happens!]

The metric cone

Given an almost 3-contact metric manifold $(M, \varphi_i, \xi_i, \eta_i, g)$, on the metric cone

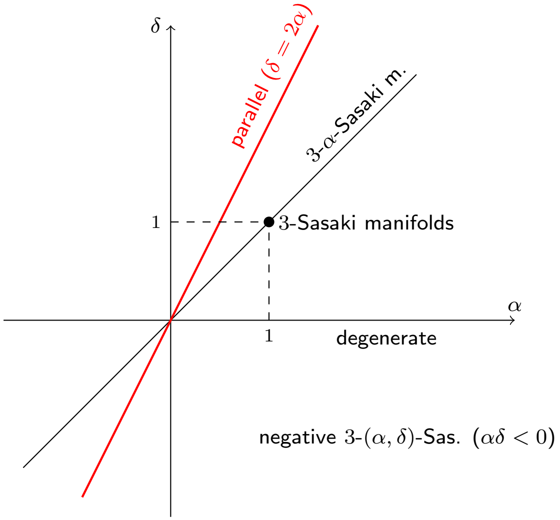
$$(\bar{M}, \bar{g}) = (M \times \mathbb{R}^+, a^2 r^2 g + dr^2), \quad a > 0,$$

one can define an almost hyperHermitian structure $(\bar{g}, J_1, J_2, J_3)$ (Agricola-Höll, 2015).

Theorem

If $(M, \varphi_i, \xi_i, \eta_i, g)$ is 3-(α, δ)-Sasakian, the metric cone is hyper-Kähler with torsion (HKT manifold).

Overview: 3-(α, δ)-Sasakian structures



The canonical connection of 3-(α, δ)-Sasaki manifolds

Theorem

The canonical connection of a 3-(α, δ)-Sasaki manifold has torsion

$$T = \sum_{i=1}^{3} \eta_i \wedge d\eta_i + 8(\delta - \alpha)\, \eta_{123}$$

and satisfies $\nabla T = 0$.

Moreover, every 3-(α, δ)-Sasakian manifold admits an underlying quaternionic contact structure, and the canonical connection turns out to be a quaternionic contact connection. In fact, it is qc-Einstein (Ivanov - Minchev - Vassilev, 2016) and this allows to determine the Riemannian Ricci curvature:

Theorem

The Riemannian Ricci curvature of a 3-(α, δ)*-Sasaki manifold is*

$$\mathrm{Ric}^g = 2\alpha\big(2\delta(n+2) - 3\alpha\big)g + 2(\alpha - \delta)\big((2n+3)\alpha - \delta\big)\sum_{i=1}^{3} \eta_i \otimes \eta_i$$

Theorem

The Riemannian Ricci curvature of a 3-(α, δ)*-Sasaki manifold is*

$$\mathrm{Ric}^g = 2\alpha\big(2\delta(n+2) - 3\alpha\big)g + 2(\alpha - \delta)\big((2n+3)\alpha - \delta\big)\sum_{i=1}^{3} \eta_i \otimes \eta_i$$

The ∇*-Ricci curvature is*

$$\mathrm{Ric} = 4\alpha\{\delta(n+2) - 3\alpha\}\, g + 4\alpha\{\delta(2-n) - 5\alpha\}\sum_{i=1}^{3} \eta_i \otimes \eta_i.$$

The property of being symmetric follows for Ric from $\nabla T = 0$.

Theorem

The Riemannian Ricci curvature of a 3-(α,δ)*-Sasaki manifold is*

$$\mathrm{Ric}^g = 2\alpha\big(2\delta(n+2)-3\alpha\big)g + 2(\alpha-\delta)\big((2n+3)\alpha-\delta\big)\sum_{i=1}^{3}\eta_i\otimes\eta_i$$

The ∇*-Ricci curvature is*

$$\mathrm{Ric} = 4\alpha\{\delta(n+2)-3\alpha\}\,g + 4\alpha\{\delta(2-n)-5\alpha\}\sum_{i=1}^{3}\eta_i\otimes\eta_i .$$

The property of being symmetric follows for Ric from $\nabla T = 0$.

- M is Riemannian Einstein iff $\alpha=\delta$ or $\delta=(2n+3)\alpha$.
- The manifold is ∇-Einstein iff $\delta(2-n)=5\alpha$.
- The manifold is both Riemannian Einstein and ∇-Einstein if and only if $\dim M = 7$ and $\delta = 5\alpha$ (happens for example for 'compatible' nearly parallel G_2-str., see next result).

Spinors on 7-dimensional 3-(α, δ)-Sasaki manifolds

Theorem

Any 7-dimensional 3-(α, δ)-Sasaki manifold admits a a cocalibrated G_2-structure (Fernandez-Gray type $W_1 \oplus W_3$) such that its characteristic connection ∇ coincides with the canonical connection.

Because G_2 is the stabilizer of a generic spinor in dim. 7, this G_2-structure defines a unique parallel spinor field ψ_0, called the canonical spinor field.

Theorem

1) *The canonical spinor field ψ_0 is a generalized Killing spinor, Killing iff $\delta = 5\alpha$ (nearly parallel G_2-structure).*

2) *The Clifford products $\psi_i := \xi_i \cdot \psi_0$, $i = 1, 2, 3$, are generalized Killing spinors; any two of the generalized Killing numbers coincide iff $\alpha = \delta$, i. e. if M^7 is 3-α-Sasakian.*

Homogeneous 3-Sasakian manifolds

Theorem (Boyer, Galicki, Mann, 1994)

Let $(M, g, \eta_i, \xi_i, \varphi_i)$ be a homogeneous 3-Sasakian manifold. Then M is one of the following homogeneous spaces:

$$\frac{\mathrm{Sp}(n+1)}{\mathrm{Sp}(n)}, \quad \frac{\mathrm{Sp}(n+1)}{\mathrm{Sp}(n)\times\mathbb{Z}_2}, \quad \frac{\mathrm{SU}(m+2)}{S(\mathrm{U}(m)\times\mathrm{U}(1))}, \quad \frac{\mathrm{SO}(k+4)}{\mathrm{SO}(k)\times\mathrm{Sp}(1)},$$

$$\frac{\mathrm{G}_2}{\mathrm{Sp}(1)}, \quad \frac{\mathrm{F}_4}{\mathrm{Sp}(3)}, \quad \frac{\mathrm{E}_6}{\mathrm{SU}(6)}, \quad \frac{\mathrm{E}_7}{\mathrm{Spin}(12)}, \quad \frac{\mathrm{E}_8}{\mathrm{E}_7}.$$

Here $n \geq 0$, $m \geq 1$ and $k \geq 3$.

- They are all simply connected except for $\mathbb{R}P^{4n+3} \simeq \frac{\mathrm{Sp}(n+1)}{\mathrm{Sp}(n)\times\mathbb{Z}_2}$
- 1-1 correspondence between simply connected 3-Sasakian homogeneous manifolds and compact simple Lie algebras

Uniform description of homogeneous 3-Sasakian manifolds

(Draper, Ortega, Palomo, 2018)

Definition

A 3-Sasakian data is a triple (G, G_0, H) of Lie groups such that

- G *is a compact, simple Lie Group*
- $H \subset G_0 \subset G$ *connected Lie subgroups*

Uniform description of homogeneous 3-Sasakian manifolds

(Draper, Ortega, Palomo, 2018)

Definition

A 3-Sasakian data is a triple (G, G_0, H) of Lie groups such that

- G *is a compact, simple Lie Group*
- $H \subset G_0 \subset G$ *connected Lie subgroups*

and the Lie algebras $\mathfrak{h} \subset \mathfrak{g}_0 \subset \mathfrak{g}$ satisfy:

- $\mathfrak{g}_0 = \mathfrak{h} \oplus \mathfrak{sp}(1)$ *with $\mathfrak{sp}(1)$ and $\mathfrak{h}$ commuting subalgebras,*

Uniform description of homogeneous 3-Sasakian manifolds

(Draper, Ortega, Palomo, 2018)

Definition

A 3-Sasakian data is a triple (G, G_0, H) of Lie groups such that

- G *is a compact, simple Lie Group*
- $H \subset G_0 \subset G$ *connected Lie subgroups*

and the Lie algebras $\mathfrak{h} \subset \mathfrak{g}_0 \subset \mathfrak{g}$ satisfy:

- $\mathfrak{g}_0 = \mathfrak{h} \oplus \mathfrak{sp}(1)$ *with $\mathfrak{sp}(1)$ and $\mathfrak{h}$ commuting subalgebras,*
- $(\mathfrak{g}, \mathfrak{g}_0)$ *form a symmetric pair, $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$,*

Uniform description of homogeneous 3-Sasakian manifolds

(Draper, Ortega, Palomo, 2018)

Definition

A 3-Sasakian data is a triple (G, G_0, H) of Lie groups such that

- G *is a compact, simple Lie Group*
- $H \subset G_0 \subset G$ *connected Lie subgroups*

and the Lie algebras $\mathfrak{h} \subset \mathfrak{g}_0 \subset \mathfrak{g}$ satisfy:

- $\mathfrak{g}_0 = \mathfrak{h} \oplus \mathfrak{sp}(1)$ *with $\mathfrak{sp}(1)$ and $\mathfrak{h}$ commuting subalgebras,*
- $(\mathfrak{g}, \mathfrak{g}_0)$ *form a symmetric pair, $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$,*
- *the complexification $\mathfrak{g}_1^{\mathbb{C}} = \mathbb{C}^2 \otimes_{\mathbb{C}} W$ for some $\mathfrak{h}^{\mathbb{C}}$-module of $\dim_{\mathbb{C}} W = 2n$,*

Uniform description of homogeneous 3-Sasakian manifolds

(Draper, Ortega, Palomo, 2018)

Definition

A 3-Sasakian data is a triple (G, G_0, H) of Lie groups such that

- G *is a compact, simple Lie Group*
- $H \subset G_0 \subset G$ *connected Lie subgroups*

and the Lie algebras $\mathfrak{h} \subset \mathfrak{g}_0 \subset \mathfrak{g}$ satisfy:

- $\mathfrak{g}_0 = \mathfrak{h} \oplus \mathfrak{sp}(1)$ *with $\mathfrak{sp}(1)$ and $\mathfrak{h}$ commuting subalgebras,*
- $(\mathfrak{g}, \mathfrak{g}_0)$ *form a symmetric pair, $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$,*
- *the complexification $\mathfrak{g}_1^{\mathbb{C}} = \mathbb{C}^2 \otimes_{\mathbb{C}} W$ for some $\mathfrak{h}^{\mathbb{C}}$-module of $\dim_{\mathbb{C}} W = 2n$,*
- $\mathfrak{h}^{\mathbb{C}}, \mathfrak{sp}(1)^{\mathbb{C}} \subset \mathfrak{g}_0^{\mathbb{C}}$ *act on $\mathfrak{g}_1^{\mathbb{C}}$ by their action on W and $\mathbb{C}^2$.*

Uniform description of homogeneous 3-Sasakian manifolds

(Draper, Ortega, Palomo, 2018)

Definition

A 3-Sasakian data is a triple (G, G_0, H) of Lie groups such that

- G *is a compact, simple Lie Group*
- $H \subset G_0 \subset G$ *connected Lie subgroups*

and the Lie algebras $\mathfrak{h} \subset \mathfrak{g}_0 \subset \mathfrak{g}$ satisfy:

- $\mathfrak{g}_0 = \mathfrak{h} \oplus \mathfrak{sp}(1)$ *with $\mathfrak{sp}(1)$ and $\mathfrak{h}$ commuting subalgebras,*
- $(\mathfrak{g}, \mathfrak{g}_0)$ *form a symmetric pair, $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$,*
- *the complexification $\mathfrak{g}_1^{\mathbb{C}} = \mathbb{C}^2 \otimes_{\mathbb{C}} W$ for some $\mathfrak{h}^{\mathbb{C}}$-module of $\dim_{\mathbb{C}} W = 2n$,*
- $\mathfrak{h}^{\mathbb{C}}, \mathfrak{sp}(1)^{\mathbb{C}} \subset \mathfrak{g}_0^{\mathbb{C}}$ *act on $\mathfrak{g}_1^{\mathbb{C}}$ by their action on W and $\mathbb{C}^2$.*

Remark In total the Lie algebra decomposes as

$$\mathfrak{g} = \overbrace{\mathfrak{h} \oplus \mathfrak{sp}(1)}^{\mathfrak{g}_0} \oplus \mathfrak{g}_1 = \mathfrak{h} \oplus \underbrace{\mathfrak{sp}(1) \oplus \mathfrak{g}_1}_{\mathfrak{m}}$$

($\mathfrak{m}$ is a reductive complement for $M = G/H$)

$$\mathfrak{g} = \overbrace{\mathfrak{h} \oplus \mathfrak{sp}(1)}^{\mathfrak{g}_0} \oplus \mathfrak{g}_1$$

- The subspaces $\mathfrak{sp}(1)$ and $\mathfrak{g}_1$ will play the role of the vertical and horizontal subspace $\mathcal{V}, \mathcal{H}$ of the 3-(α, δ)-Sasakian structure on $M = G/H$
- M fibers over the compact quaternion Kähler symmetric space G/G_0

Homogeneous 3-Sasakian model

Theorem (Draper, Ortega, Palomo, 2018)

Let (G, G_0, H) be 3-Sasakian data. On $M = G/H$ consider the G-invariant structure defined by the $\mathrm{Ad}(H)$-invariant tensors on $\mathfrak{m}$:

- *the inner product g*

$$g\big|_{\mathfrak{sp}(1)} = \frac{-\kappa}{4(n+2)}, \qquad g\big|_{\mathfrak{g}_1} = \frac{-\kappa}{8(n+2)}, \qquad g\big|_{\mathfrak{sp}(1)\times\mathfrak{g}_1} = 0$$

  κ the Killing form on G.

- *$\xi_i = \sigma_i$, $i = 1, 2, 3$, σ_i standard basis of $\mathfrak{sp}(1) = \mathcal{V} \subset \mathfrak{g}_0$, $\eta_i = g(\xi_i, \cdot)$*
- *the endomorphisms φ_i as*

$$\varphi_i\big|_{\mathfrak{sp}(1)} = \frac{1}{2}\,\mathrm{ad}(\xi_i), \qquad \varphi_i\big|_{\mathfrak{g}_1} = \mathrm{ad}(\xi_i).$$

Then $(M, \varphi_i, \xi_i, \eta_i, g)$ defines a homogeneous 3-Sasakian manifold. Conversely every homogeneous 3-Sasakian manifold $M \neq \mathbb{R}P^{4n+3}$ is obtained by this construction.

Homogeneous positive 3-(α, δ)-Sasakian model

Idea: Use $\mathcal{H}$-homothetic deformation to obtain 3-(α, δ)-Sasakian mnfds for $\alpha\delta > 0$

Homogeneous positive 3-(α, δ)-Sasakian model

Idea: Use $\mathcal{H}$-homothetic deformation to obtain 3-(α, δ)-Sasakian mnfds for $\alpha\delta > 0$

Theorem

Let (G, G_0, H) be 3-Sasakian data, $\alpha\delta > 0$. On $M = G/H$ consider the G-invariant structure by the $\mathrm{Ad}(H)$-invariant tensors on $\mathfrak{m}$:

$$g\big|_{\mathfrak{sp}(1)} = \frac{-\kappa}{4\delta^2(n+2)}, \qquad g\big|_{\mathfrak{g}_1} = \frac{-\kappa}{8\alpha\delta(n+2)}, \qquad g\big|_{\mathfrak{sp}(1)\times\mathfrak{g}_1} = 0$$

$$\xi_i = \delta\sigma_i, \qquad \eta_i = g(\xi_i, \cdot)$$

$$\varphi_i\big|_{\mathfrak{sp}(1)} = \frac{1}{2\delta}\,\mathrm{ad}(\xi_i), \qquad \varphi_i\big|_{\mathfrak{g}_1} = \frac{1}{\delta}\,\mathrm{ad}(\xi_i).$$

Then $(M, \varphi_i, \xi_i, \eta_i, g)$ defines a homogeneous 3-(α, δ)-Sasakian mnfd.
Conversely every homogeneous 3-(α, δ)-Sasakian manifold $M \neq \mathbb{R}P^{4n+3}$ with $\alpha\delta > 0$ is obtained by this construction.

Remark: $(G/H, g)$ is naturally reductive $\Leftrightarrow \delta = 2\alpha \Leftrightarrow$ parallel 3-(α, δ).

Generalized setup

Definition

A generalized 3-Sasakian data is a triple (G, G_0, H) of Lie groups such that

- *G is a real simple Lie Group*
- *$H \subset G_0 \subset G$ connected Lie subgroups*

and the Lie algebras $\mathfrak{h} \subset \mathfrak{g}_0 \subset \mathfrak{g}$ satisfy:

- *$\mathfrak{g}_0 = \mathfrak{h} \oplus \mathfrak{sp}(1)$ with $\mathfrak{sp}(1)$ and $\mathfrak{h}$ commuting subalgebras,*
- *$(\mathfrak{g}, \mathfrak{g}_0)$ form a symmetric pair, $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$,*
- *the complexification $\mathfrak{g}_1^{\mathbb{C}} = \mathbb{C}^2 \otimes_{\mathbb{C}} W$ for some $\mathfrak{h}^{\mathbb{C}}$-module of $\dim_{\mathbb{C}} W = 2n$,*
- *$\mathfrak{h}^{\mathbb{C}}, \mathfrak{sp}(1)^{\mathbb{C}} \subset \mathfrak{g}_0^{\mathbb{C}}$ act on $\mathfrak{g}_1^{\mathbb{C}}$ by their action on W and $\mathbb{C}^2$.*

Generalized setup

Definition

A generalized 3-Sasakian data is a triple (G, G_0, H) of Lie groups such that

- *G is a real simple Lie Group*
- *$H \subset G_0 \subset G$ connected Lie subgroups*

and the Lie algebras $\mathfrak{h} \subset \mathfrak{g}_0 \subset \mathfrak{g}$ satisfy:

- *$\mathfrak{g}_0 = \mathfrak{h} \oplus \mathfrak{sp}(1)$ with $\mathfrak{sp}(1)$ and $\mathfrak{h}$ commuting subalgebras,*
- *$(\mathfrak{g}, \mathfrak{g}_0)$ form a symmetric pair, $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$,*
- *the complexification $\mathfrak{g}_1^{\mathbb{C}} = \mathbb{C}^2 \otimes_{\mathbb{C}} W$ for some $\mathfrak{h}^{\mathbb{C}}$-module of $\dim_{\mathbb{C}} W = 2n$,*
- *$\mathfrak{h}^{\mathbb{C}}, \mathfrak{sp}(1)^{\mathbb{C}} \subset \mathfrak{g}_0^{\mathbb{C}}$ act on $\mathfrak{g}_1^{\mathbb{C}}$ by their action on W and $\mathbb{C}^2$.*

If $(\mathfrak{g}, \mathfrak{g}_0)$ is a compact symmetric pair such that (G, G_0, H) is 3-Sasakian data, then (G^*, G_0, H) is generalized 3-Sasakian data, where $(\mathfrak{g}^*, \mathfrak{g}_0)$ is the dual non-compact symmetric pair.

Negative homogeneous 3-(α, δ)-Sasakian manifolds

Theorem

Let (G^*, G_0, H) be non-compact generalized 3-Sasakian data, $\alpha\delta < 0$. On $M = G^*/H$ consider the G^*-invariant structure defined by the $\mathrm{Ad}(H)$-invariant tensors on $\mathfrak{m}$

$$g\big|_{\mathfrak{sp}(1)} = \frac{-\kappa}{4\delta^2(n+2)}, \qquad g\big|_{\mathfrak{g}_1} = \frac{-\kappa}{8\alpha\delta(n+2)}, \qquad g\big|_{\mathfrak{sp}(1)\times\mathfrak{g}_1} = 0,$$

$$\xi_i = \delta\sigma_i, \qquad \eta_i = g(\xi_i, \cdot),$$

$$\varphi_i\big|_{\mathfrak{sp}(1)} = \frac{1}{2\delta}\,\mathrm{ad}(\xi_i), \qquad \varphi_i\big|_{\mathfrak{g}_1} = \frac{1}{\delta}\,\mathrm{ad}(\xi_i),$$

κ the Killing form on G^*, σ_i standard basis $\mathfrak{sp}(1) = \mathcal{V} \subset \mathfrak{g}_0$.
Then $(M, g, \xi_i, \eta_i, \varphi_i)$ defines a homogeneous 3-(α, δ)-Sasakian manifold.

In total we obtain homogeneous 3-(α,δ)-Sasakian structures on the following list of homogeneous spaces (G/H compact, G^*/H non-compact):

G	G^*	H	G_0	dim
$\mathrm{Sp}(n+1)$	$\mathrm{Sp}(n,1)$	$\mathrm{Sp}(n)$	$\mathrm{Sp}(n)\mathrm{Sp}(1)$	$4n+3$
$\mathrm{SU}(n+2)$	$\mathrm{SU}(n,2)$	$S(\mathrm{U}(n)\times\mathrm{U}(1))$	$S(\mathrm{U}(n)\mathrm{U}(2))$	$4n+3$
$\mathrm{SO}(n+4)$	$\mathrm{SO}(n,4)$	$\mathrm{SO}(n)\times\mathrm{Sp}(1)$	$\mathrm{SO}(n)\mathrm{SO}(4)$	$4n+3$
G_2	G_2^2	$\mathrm{Sp}(1)$	$\mathrm{SO}(4)$	11
F_4	F_4^{-20}	$\mathrm{Sp}(3)$	$\mathrm{Sp}(3)\mathrm{Sp}(1)$	31
E_6	E_6^2	$\mathrm{SU}(6)$	$\mathrm{SU}(6)\mathrm{Sp}(1)$	43
E_7	E_7^{-5}	$\mathrm{Spin}(12)$	$\mathrm{Spin}(12)\mathrm{Sp}(1)$	67
E_8	E_8^{-24}	E_7	$\mathrm{E}_7\mathrm{Sp}(1)$	115

.

Remark: $\mathbb{R}P^{4n+3} = \frac{\mathrm{Sp}(n+1)}{\mathrm{Sp}(n)\times\mathbb{Z}_2}$ and non compact dual $\frac{\mathrm{Sp}(n,1)}{\mathrm{Sp}(n)\times\mathbb{Z}_2}$ also admit 3-(α,δ)-Sasaki structures, as the quotient of $S^{4n+3} = \frac{\mathrm{Sp}(n+1)}{\mathrm{Sp}(n)}$, resp. $\frac{\mathrm{Sp}(n,1)}{\mathrm{Sp}(n)}$ by $\mathbb{Z}_2$ inside the fiber.

Question: Are these all homogenous negative 3-(α, δ)-Sasaki manifolds?

Question: Are these all homogenous negative 3-(α, δ)-Sasaki manifolds?

NO!

Question: Are these all homogenous negative 3-(α, δ)-Sasaki manifolds?

NO!

Idea: Start with V. Cortes, *A New Construction of Homogeneous Quaternionic Manifolds and Related Geometric Structures*, Mem. AMS 147 (2000) and previous work of $\subset$ $\{$Alekseevsky, Cortes$\}$

Question: Are these all homogenous negative 3-(α, δ)-Sasaki manifolds?

NO!

Idea: Start with V. Cortes, *A New Construction of Homogeneous Quaternionic Manifolds and Related Geometric Structures*, Mem. AMS 147 (2000) and previous work of $\subset$ {Alekseevsky, Cortes}

The construction is highly algebraic!

- Obtain examples over bases not included in previous construction (for example, Alekseevsky spaces of negative scalar curvature)
- First such example not covered by previous theorem: dimension $n = 19 = 4 \cdot 4 + 3$

Difficulty: Pick the positive definite examples, discard redundancies, give a more geometric description...

Definiteness of curvature operators

Consider the Riemannian curvature as a symmetric operator

$$\mathcal{R}^g : \Lambda^2 M \to \Lambda^2 M \qquad \langle \mathcal{R}^g(X \wedge Y), Z \wedge V \rangle = -g(R^g(X, Y)Z, V).$$

Definition

A Riemannian manifold (M, g) is said to have strongly positive curvature if there exists a 4-form ω such that $\mathcal{R}^g + \omega$ is positive-definite at every point $x \in M$ (Thorpe, 1971).

For every 2-plane σ, being $\langle \omega(\sigma), \sigma \rangle = 0$, one has

$$\sec(\sigma) = \langle \mathcal{R}^g(\sigma), \sigma \rangle = \langle (\mathcal{R}^g + \omega)(\sigma), \sigma \rangle.$$

Then,

$\mathcal{R}^g > 0 \Longrightarrow$ strongly positive curvature $\Longrightarrow$ positive sectional curvature

$\mathcal{R}^g \geq 0 \Longrightarrow$ strongly non-negative curvature $\Longrightarrow$ non-negative sec. curv.

On a 3-(α, δ)-Sasakian manifold the symmetric operators defined by the Riemannian curvature and the curvature of the canonical connection:

$$\mathcal{R}^g : \Lambda^2 M \to \Lambda^2 M \qquad \mathcal{R} : \Lambda^2 M \to \Lambda^2 M$$

are related by

$$\mathcal{R}^g - \frac{1}{4}\sigma_T = \mathcal{R} + \frac{1}{4}\mathcal{G}_T$$

with

$$\langle \mathcal{G}_T(X \wedge Y), Z \wedge V \rangle := g(T(X,Y), T(Z,V)),$$
$$\langle \sigma_T(X \wedge Y), Z \wedge V \rangle := \frac{1}{2} dT(X,Y,Z,V).$$

(M,g) is strongly non-negative with 4-form $-\frac{1}{4}\sigma_T$ if and only if

$$\mathcal{R} + \frac{1}{4}\mathcal{G}_T \geq 0.$$

Being $\mathcal{G}_T \geq 0$, if $\mathcal{R} \geq 0$ we directly have strong non-negativity.

Theorem

Let M be a homogeneous 3-(α, δ)-Sasakian manifold obtained from a generalized 3-Sasakian data.

- *If $\alpha\delta < 0$ then $\mathcal{R} \leq 0$.*
- *If $\alpha\delta > 0$ then*

$$\mathcal{R} \geq 0 \text{ if and only if } \alpha\beta \geq 0$$

.

Then, on a positive homogeneous 3-(α, δ)-Sasaki manifold with $\alpha\beta \geq 0$:

$$\mathcal{R}^g - \frac{1}{4}\sigma_T = \mathcal{R} + \frac{1}{4}\mathcal{G}_T \geq 0.$$

The converse also holds, i.e.

Theorem

A positive homogeneous 3-(α, δ)-Sasaki manifold is strongly non-negative with 4-form $-\frac{1}{4}\sigma_T$ if and only if $\alpha\beta \geq 0$.

Strong positivity is much more restrictive than strong non-negativity.

Strong positivity implies strict positive sectional curvature.

Homogeneous manifolds with strictly positive sectional curvature have been classified (Wallach 1972, Bérard Bergery 1976).

Only the 7-dimensional Aloff-Wallach-space $W^{1,1}$, the spheres S^{4n+3} and real projective spaces $\mathbb{R}P^{4n+3}$ admit homogeneous 3-(α,δ)-Sasaki structures.

Theorem

The 3-(α,δ)-Sasakian spaces

- $W^{1,1} = \mathrm{SU}(3)/S^1$ *with 4-form* $-(\frac{1}{4}+\varepsilon)\sigma_T$ *for small* $\varepsilon > 0$,
- S^{4n+3}, $\mathbb{R}P^{4n+3}$, $n \geq 1$, *with 4-form* $\frac{\delta}{8\alpha}\sigma_T|_{\Lambda^4\mathcal{H}} - (\frac{1}{4}+\varepsilon)\sigma_T$ *for small* $\varepsilon > 0$

are strongly positive if and only if $\alpha\beta > 0$.

Some open questions

- Investigate the geometry of the new homogeneous negative 3-(α, δ)-Sasakian manifolds
- 3-Sasakian manifolds admit Riemannian Killing spinors. They correspond to pseudo-Riemannian Killing spinors on the non-compact duals when equipped with an indefinite metric. How does this translate to the negative 3-Sasakian case? Are there special spinors?
- 3-(α, δ)-Sasakian manifolds are ∇-Einstein if $(2 - n)\delta = 5\alpha$. How do these geometries look like for $n > 2$?

Further reading

I. Agricola, *The Srní lectures on non-integrable geometries with torsion*, Arch. Math.(Brno) **42** (2006), suppl., 5–84.

I. Agricola, G. Dileo, *Generalizations of 3-Sasakian manifolds and skew torsion*, Adv. Geom. 2020. Online since 4/2019.

I. Agricola, T. Friedrich, *3-Sasakian manifolds in dimension seven, their spinors and G_2-structures*, J. Geom. Phys. **60** (2010), 326–332.

R. Bettiol, R. Mendes, *Strongly positive curvature*, Ann. Global Anal. Geom. **53** (2018), 287–309.

B. Cappelletti-Montano, A. De Nicola, I. Yudin, *Hard Lefschetz theorem for Sasakian manifolds*, J. Differ. Geom. **101** (2015), 47–66.

D. Conti, Th. B. Madsen, *The odd side of torsion geometry*, Ann. Mat. Pura Appl. **193** (2014), 1041-1067.

T. Houri, H. Takeuchi, Y. Yasui, *A Deformation of Sasakian Structure in the Presence of Torsion and Supergravity Solutions*, Class.Quant.Grav. **30** (2013), 135008.

S. Ivanov, I. Minchev, D. Vassilev, *Quaternionic contact Einstein structures and the quaternionic contact Yamabe problem*, Mem. AMS **231** (2014)

‘This page intentionally left blank’

‘This page intentionally left blank’