APPENDIX B: PROOFS OF REMARKABLE IDENTITIES

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In this appendix to Agricola's review article *The Srní lectures on non-integrable geometries with torsion*, we compile the proofs of the identities cited in Appendix B as well as of the first Bianchi identity for connections with skew-symmetric and vectorial torsion (Theorem 2.6).

In definition A.1, the 4-form $\sigma_T = \frac{1}{2}\sum_{i=1}^{n}(e_i \lrcorner T) \wedge (e_i \lrcorner T)$ was introduced and said to have the alternative global expression

$$(1) \quad \sigma_T(X_1, X_2, X_3, X_4) = g(T(X_1, X_2), T(X_3, X_4)) + g(T(X_2, X_3), T(X_1, X_4)) + g(T(X_3, X_1), T(X_2, X_4)).$$

Indeed, if X_1, X_2, X_3 and X_4 are vector fields, we have

$$\begin{aligned} \sigma_T(X_1, X_2, X_3, X_4) &= \frac{1}{2}\sum_{i=1}^{n}\frac{1}{4}\sum_{\sigma\in S_4} \operatorname{sgn}\sigma \cdot (e_i \lrcorner T)(X_{\sigma(1)}, X_{\sigma(2)})(e_i \lrcorner T)(X_{\sigma(3)}, X_{\sigma(4)}) \\ &= \frac{1}{8}\sum_{i=1}^{n}\sum_{\sigma\in S_4} \operatorname{sgn}\sigma \cdot g(T(e_i, X_{\sigma(1)}), X_{\sigma(2)})g(T(e_i, X_{\sigma(3)}), X_{\sigma(4)}) \\ &= \frac{1}{8}\sum_{i=1}^{n}\sum_{\sigma\in S_4} \operatorname{sgn}\sigma \cdot g(T(X_{\sigma(1)}, X_{\sigma(2)}), e_i)g(T(X_{\sigma(3)}, X_{\sigma(4)}), e_i), \end{aligned}$$

where the last identity holds because of the skew-symmetry of T. Now, we observe that

$$\sum_{i=1}^{n} g(T(X_{\sigma(1)}, X_{\sigma(2)}), e_i)g(T(X_{\sigma(3)}, X_{\sigma(4)}), e_i) = g(T(X_{\sigma(1)}, X_{\sigma(2)}), T(X_{\sigma(3)}, X_{\sigma(4)}))$$

and the claim follows with an easy combinatorial argument. As a consequence, we are able to prove Corollary A.1.

Proof of Corollary A.1. We want to check this identity in every point $x \in M$. Therefore, let $X, Y, Z, V \in T_xM$ be four tangent vectors in x. We extend X, Y, Z and V to local vector fields by parallel transport along geodesics (with respect to ∇). Without using a new notation for these local vector fields, the torsion form T is given by $[X, Y] = -T(X, Y)$. In particular, the global expression (1) for σ_T is reduced to

$$\sigma_T(X, Y, Z, V) = g([X, Y], [Z, V]) + g([Y, Z], [X, V]) + g([Z, X], [Y, V]).$$

Now, we define the 4-form Ω to be the right hand side of the equation we want to prove,

$$\Omega(X, Y, Z, V) := \overset{X,Y,Z}{\mathfrak{S}} (\nabla_X T)(Y, Z, V) - (\nabla_V T)(X, Y, Z) + 2\sigma_T(X, Y, Z, V).$$

Writing out the cyclic sum explicitely, we obtain

$$\Omega(X, Y, Z, V) = X(T(Y, Z, V)) - Y(T(X, Z, V)) + Z(T(X, Y, V)) - V(T(X, Y, Z)) + 2\sigma_T(X, Y, Z, V).$$

Together with the previous expression for σ_T, this yields

$$\begin{aligned}\Omega(X,Y,Z,V) &= X(T(Y,Z,V)) - Y(T(X,Z,V)) + Z(T(X,Y,V)) - V(T(X,Y,Z)) \\ &\quad +2g([X,Y],[Z,V]) + 2g([Y,Z],[X,V]) + 2g([Z,X],[Y,V]).\end{aligned}$$

This is easily seen to be the exterior derivative dT, since by definition we have

$$\begin{aligned}dT(X,Y,Z,V) &= X(T(Y,Z,V)) - Y(T(X,Z,V)) + Z(T(X,Y,V)) - V(T(X,Y,Z)) \\ &\quad -T([X,Y],Z,V) + T([X,Z],Y,V) - T([X,V],Y,Z) \\ &\quad -T([Y,Z],X,V) + T([Y,V],X,Z) - T([Z,V],X,Y) \\ &= X(T(Y,Z,V)) - Y(T(X,Z,V)) + Z(T(X,Y,V)) - V(T(X,Y,Z)) \\ &\quad +2g([X,Y],[Z,V]) + 2g([Y,Z],[X,V]) + 2g([Z,X],[Y,V]). \quad \square\end{aligned}$$

We turn to the proof of the relation between the curvature tensor of a metric connection with skew-symmetric torsion and the Riemannian curvature tensor in the induced identity between Ricci tensors.

Proof of Theorem A.1. We first discuss curvature tensors. Let $x \in M$ be a fixed point and $X, Y, Z, V \in T_xM$ four tangent vectors that we extend to local vector fields by parallel transport along ∇-geodesics. Then we have

$$g(\nabla_X Y, Z) \;=\; g(\nabla^g_X Y, Z) - \frac{1}{2}T(X,Y,Z) = -\frac{1}{2}T(X,Y,Z)$$

and

$$\mathcal{R}^g(X,Y,Z,V) \;=\; g(\mathcal{R}^g(X,Y)Z,V) \;=\; g(\nabla^g_X\nabla^g_Y Z - \nabla^g_Y\nabla^g_Z Z - \nabla^g_{[X,Y]}Z, V) \;=:\; A+B+C.$$

We compute the terms A, B and C separately. First we have

$$A \;:=\; g(\nabla^g_X\nabla^g_Y Z, V) \;=\; g(\nabla_X\nabla^g_Y Z, V) - \frac{1}{2}T(X, \nabla^g_Y Z, V).$$

Since ∇ is a metric connection, we can write

$$\begin{aligned}A &= X(g(\nabla^g_Y Z, V)) - g(\nabla^g_Y Z, \nabla_X V) + \frac{1}{2}g(T(X,V), \nabla^g_Y Z) \\ &= X(g(\nabla_Y Z, V)) - \frac{1}{2}X(T(Y,Z,V)) + \frac{1}{2}g(\nabla_Y Z, T(X,V)) - \frac{1}{4}T(Y,Z,T(X,V)) \\ &= g(\nabla_X\nabla_Y Z, V) - \frac{1}{2}X(g(T(Y,Z),V)) - \frac{1}{4}T(Y,Z,T(X,V)).\end{aligned}$$

B is obtained from A by interchanging X and Y. Finally, we have

$$\begin{aligned}C &:= -g(\nabla^g_{[X,Y]}Z, V) \;=\; -g(\nabla_{[X,Y]}Z, V) + \frac{1}{2}T([X,Y],Z,V) \\ &= -g(\nabla_{[X,Y]}Z, V) + \frac{1}{2}g(T(Z,V),[X,Y]).\end{aligned}$$

Summing up A, B and C we obtain

$$\begin{aligned}A+B+C &= g(\mathcal{R}^\nabla(X,Y)Z,V) - \frac{1}{2}X(T(Y,Z,V)) + \frac{1}{2}Y(T(X,Z,V)) \\ &\quad -\frac{1}{4}g(T(Y,Z),T(X,V)) - \frac{1}{4}g(T(Z,X),T(Y,V)) + \frac{1}{2}g(T(Z,V),[X,Y]) \\ &= \mathcal{R}^\nabla(X,Y,Z,V) - \frac{1}{2}(\nabla_X T)(Y,Z,V) + \frac{1}{2}(\nabla_Y T)(X,Z,V) \\ &\quad -\frac{1}{4}\sigma^T(X,Y,Z,V) + \frac{1}{4}g(T(X,Y),T(Z,V)) + \frac{1}{2}g(T(Z,V),[X,Y]).\end{aligned}$$

By $[X,Y]=-T(X,Y)$, the claim follows. Contracting the respective curvature tensors yields then the relation between Ricci-tensors $\mathrm{Ric}^g(X,Y):=\sum_i \mathcal{R}^g(e_i,X,Y,e_i)$,

$$\begin{aligned}\mathrm{Ric}^g(X,Y) &= \sum_{i=1}^n \mathcal{R}^\nabla(e_i,X,Y,e_i)-\frac{1}{2}\sum_{i=1}^n(\nabla_{e_i}T)(X,Y,e_i)+\frac{1}{2}\sum_{i=1}^n\underbrace{(\nabla_X T)(e_i,Y,e_i)}_{=0}\\ &\quad -\frac{1}{4}g(T(e_i,X),T(Y,e_i))\\ &= \mathrm{Ric}^\nabla(X,Y)-\frac{1}{2}\sum_{i=1}^n(e_i\lrcorner\nabla_{e_i}T)(X,Y)-\frac{1}{4}g(T(e_i,X),T(Y,e_i))\\ &= \mathrm{Ric}^\nabla(X,Y)+(\delta^\nabla T)(X,Y)-\frac{1}{4}g(T(e_i,X),T(Y,e_i)).\end{aligned}$$

But on T the codifferentials δ^g and δ^∇ coincide, i. e.

$$\mathrm{Ric}^g(X,Y) = \mathrm{Ric}^\nabla(X,Y)+(\delta^g T)(X,Y)-\frac{1}{4}g(T(e_i,X),T(Y,e_i)). \qquad \square$$

We finally prove the first Bianchi identity for vectorial torsion (1) and for skew-symmetric torsion (2).

Proof of Theorem 2.6 (1). The problem is reduced to the first Bianchi identity for the Levi-Civita connection. We pick again three tangent vectors $X,Y,Z\in T_xM$ and extend them locally by parallel transport along ∇^g-geodesics. Recall that the torsion of a metric connection ∇ with vectorial torsion is given by $T(X,Y)=g(X,Y)V-g(V,Y)X$, where V is the generating vector field. In order to express the curvature of ∇ by the curvature of ∇^g, we compute

$$\begin{aligned}&\nabla_X\nabla_Y Z =\\ &= \nabla_X(\nabla^g_Y Z+g(Y,Z)V-g(V,Z)Y)-\nabla_Y(\nabla^g_X Z+g(X,Z)V-g(V,Z)X)\\ &\quad -\nabla^g_{[X,Y]}Z-g([X,Y],Z)V+g(V,Z)[X,Y]\\ &= \nabla^g_X\nabla^g_Y Z+\left[g(\nabla^g_X Y,Z)+g(Y,\nabla^g_X Z)\right]V-\left[g(\nabla^g_X V,Z)+g(V,\nabla^g_X Z)\right]Y\\ &\quad +g(Y,Z)\nabla^g_X V-g(V,Z)\nabla^g_X Y+\big[g(X,\nabla^g_Y Z)+g(Y,Z)g(X,V)\\ &\quad -g(V,Z)g(X,Y)\big]V-\left[g(V,\nabla^g_Y Z)+g(V,V)g(Y,Z)-g(V,Y)g(V,Z)\right]X\\ &= \nabla^g_X\nabla^g_Y Z+g(Y,Z)\nabla^g_X V-g(\nabla^g_X V,Z)Y+\left[g(Y,Z)g(X,V)-g(V,Z)g(X,Y)\right]V\\ &\quad -\left[g(V,V)g(Y,Z)-g(V,Y)g(V,Z)\right]X.\end{aligned}$$

Therefore we have

$$\begin{aligned}&\mathcal{R}(X,Y)Z =\\ &= \mathcal{R}^g(X,Y)Z+g(Y,Z)\nabla^g_X V-g(\nabla^g_X V,Z)Y+(g(Y,Z)g(X,V)-g(V,Z)g(X,Y))V\\ &\quad -\left[g(V,V)g(Y,Z)-g(V,Y)g(V,Z)\right]X-g(X,Z)\nabla^g_Y V+g(\nabla^g_Y V,Z)X\\ &\quad -\left[g(X,Z)g(Y,V)-g(V,Z)g(Y,X)\right]V+\left[g(V,V)g(X,Z)-g(V,X)g(V,Z)\right]Y.\end{aligned}$$

Cyclic permutation of X,Y and Z yields similar expressions for $\mathcal{R}(Y,Z)X$ and $\mathcal{R}(Z,X)Y$. Together with the first Bianchi identity for the Levi-Civita connection (and observing that most of the terms above cancel out), we obtain

$$\begin{aligned}\overset{X,Y,Z}{\mathfrak{S}}\ \mathcal{R}(X,Y)Z &= g(\nabla^g_X V,Y)Z-g(\nabla^g_Y V,X)Z+g(\nabla^g_Y V,Z)X-g(\nabla^g_Z V,Y)X\\ &\quad +g(\nabla^g_Z V,X)Y-g(\nabla^g_X V,Z)Y.\end{aligned}$$

But the exterior derivative of V can be expressed by ∇^g as follows:

$$\begin{aligned} dV(X,Y) &= X(V(Y)) - Y(V(X)) - V([X,Y]) \\ &= \langle \nabla^g_X V, Y\rangle + \langle V, \nabla^g_X Y\rangle - \langle \nabla^g_Y V, X\rangle - \langle V, \nabla^g_Y X\rangle \\ &= \langle \nabla^g_X V, Y\rangle - \langle \nabla^g_Y V, X\rangle + \langle V, [X,Y]\rangle \;=\; \langle \nabla^g_X V, Y\rangle - \langle \nabla^g_Y V, X\rangle, \end{aligned}$$

which completes the proof. ☐

Proof of Theorem 2.6 (2). First we set $K := \overset{X,Y,Z}{\mathfrak{S}} \mathcal{R}^\nabla(X,Y,Z,V)$. Using Theorem A.1, we compute

$$\begin{aligned} K &= \overset{X,Y,Z}{\mathfrak{S}} \mathcal{R}^g(X,Y,Z,V) + \frac{1}{2} \overset{X,Y,Z}{\mathfrak{S}} (\nabla_X T)(Y,Z,V) - \frac{1}{2} \overset{X,Y,Z}{\mathfrak{S}} (\nabla_Y T)(X,Z,V) \\ &\quad + \frac{1}{4} \overset{X,Y,Z}{\mathfrak{S}} g(T(X,Y),T(Z,V)) + \frac{1}{4} \overset{X,Y,Z}{\mathfrak{S}} \sigma_T(X,Y,Z,V). \end{aligned}$$

The first Bianchi identity for the Levi-Civita connection $\overset{X,Y,Z}{\mathfrak{S}} \mathcal{R}^g(X,Y,Z,V) = 0$ together with Corollary A.1 gives

$$\begin{aligned} K &= dT(X,Y,Z,V) - \frac{1}{2} \overset{X,Y,Z}{\mathfrak{S}} (\nabla_X T)(Y,Z,V) + (\nabla_V T)(X,Y,Z) - 2\sigma_T(X,Y,Z,V) \\ &\quad - \frac{1}{2} \overset{X,Y,Z}{\mathfrak{S}} (\nabla_Y T)(X,Z,V) + \underbrace{\frac{1}{4} \overset{X,Y,Z}{\mathfrak{S}} g(T(X,Y),T(Z,V))}_{\frac{1}{4}\sigma_T(X,Y,Z,V)} + \underbrace{\frac{1}{4} \overset{X,Y,Z}{\mathfrak{S}} \sigma_T(X,Y,Z,V)}_{3\sigma_T(X,Y,Z,V)} \\ &= dT(X,Y,Z,V) + (\nabla_V T)(X,Y,Z) - \sigma_T(X,Y,Z,V) \\ &\quad - \frac{1}{2}\left[\overset{X,Y,Z}{\mathfrak{S}} (\nabla_X T)(Y,Z,V) + \overset{X,Y,Z}{\mathfrak{S}} (\nabla_Y T)(X,Z,V) \right]. \end{aligned}$$

To finish the proof, we set

$$S_1 := \overset{X,Y,Z}{\mathfrak{S}} (\nabla_X T)(Y,Z,V), \quad S_2 := \overset{X,Y,Z}{\mathfrak{S}} (\nabla_Y T)(X,Z,V)$$

and show that $S_1 + S_2 = 0$. Taking the covariant derivative of T we see that

$$\begin{aligned} (\nabla_Y T)(X,Z,V) &= Y(T(X,Z,V)) - T(\nabla_Y X, Z, V) - T(X, \nabla_Y Z, V) - T(X,Z,\nabla_Y V) \\ &= g(\nabla_Y T(X,Z), V) + g(T(X,Z), \nabla_Y V) \;=\; g(\nabla_Y T(X,Z), V). \end{aligned}$$

This implies

$$S_1 = g(\nabla_Y T(X,Z),V) + g(\nabla_Z T(Y,X),V) + g(\nabla_X T(Z,Y),V).$$

Similarly, $S_2 := (\nabla_X T)(Y,Z,V) + (\nabla_Y T)(Z,X,V) + (\nabla_Z T)(X,Y,V)$ can be expressed as

$$S_2 = g(\nabla_X T(Y,Z),V) + g(\nabla_Y T(Z,X),V) + g(\nabla_Z T(X,Y),V).$$

The skew-symmetry of T then implies $S_1 = -S_2$. ☐

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