On a class of Sasakian 5-manifolds

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Introduction

Sasakian structures are the analogues in odd dimensions of Kähler structures, and they can be defined in terms of the Riemannian cone.

Given a Riemannian manifold (M, g), its Riemannian cone is the product $M \times \mathbb{R}^+$ equipped with the cone metric $t^2 g + \mathrm{d}\, t^2$.

A manifold M^{2n+1} equipped with a 1-form α is *contact* if the 2-form $t^2 \,\mathrm{d}\,\alpha + 2t \,\mathrm{d}\, t \wedge \alpha$ is symplectic on the cone. Equivalently, $\alpha \wedge (\mathrm{d}\,\alpha)^n \neq 0$.

If, moreover, this 2-form is Kähler, then (M, g) is called *Sasakian*.

This is the characterization given by Boyer-Galicki ('99) of Sasakian manifolds.

The original definition, given by Sasaki in the '60s, involves a quadruple (Φ, α, ξ, g), where Φ is a $(1,1)$-tensor, α is a 1-form and ξ is a nowhere vanishing vector field on M such that

$$
\begin{aligned}
&\alpha(\xi) = 1, \qquad \Phi^2 = -\mathrm{I} + \xi \otimes \alpha \\
&g(\Phi X, \Phi Y) = g(X, Y) - \alpha(X)\alpha(Y)\,, \\
&2g(X, \Phi Y) = \mathrm{d}\alpha(X, Y) \\
&N_\Phi = -\mathrm{d}\alpha \otimes \xi\,,
\end{aligned}
$$

where N_Φ, the Nijenhuis tensor associated to Φ, is given by

$$
N_\Phi(X, Y) = \Phi^2[X, Y] + [\Phi X, \Phi Y] - \Phi[\Phi X, Y] - \Phi[X, \Phi Y].
$$

In the '90s Boyer, Galicki and their co-authors established relationships between Sasakian structures (or related structures such as Sasaki-Einstein, 3-Sasakian) and string theory and other geometries such as algebraic or quaternionic-Kähler geometry.

Some properties:

- $\Phi(\xi) = 0, \quad \alpha \circ \Phi = 0, \quad \mathrm{d}\,\alpha(\xi, X) = 0, \quad g(\xi, X) = \alpha(X) \quad \forall X;$
- ξ is a Killing vector field of unit length;
- $R(X,Y)\xi = \alpha(Y)X - \alpha(X)Y$; therefore $\mathrm{Ric}_g(\xi,\xi) = 2n$.
- The sectional curvature of all plane sections containing ξ are equal to 1.
- Let M^{2n+1} be a compact Sasakian manifold, then the Betti numbers b_p are *even*, for p odd, $p \leq n$ or p even, $p \geq n+1$ [Blair-Goldberg '67, Fujitani '66].

Some examples of Sasakian manifolds:

• $\mathbb{R}^{2n+1}$ with the contact form $\alpha = \mathrm{d}\, z - \sum_{i=1}^{n} y_i \,\mathrm{d}\, x_i$, $\xi = \frac{\partial}{\partial z}$ and

$$2g = \alpha \otimes \alpha + \sum_{i=1}^{n} (\mathrm{d}\, x_i^2 + \mathrm{d}\, y_i^2),$$

$$\Phi\left(\frac{\partial}{\partial x_i}\right) = \frac{\partial}{\partial y_i} + x_i \frac{\partial}{\partial z}, \quad \Phi\left(\frac{\partial}{\partial y_i}\right) = -\frac{\partial}{\partial x_i}, \quad \Phi\left(\frac{\partial}{\partial z}\right) = 0.$$

• S^{2n+1}: considering the Hopf fibration $\pi : S^{2n+1} \to \mathbb{C}P^n$ as a special case of the Boothby-Wang fibration.

In dimension 3 a homogeneous Sasakian manifold is a Lie group endowed with a left-invariant Sasakian structure [Perrone '98].

By [Perrone-Vanhecke '91] a compact, simply connected, 5-dimensional homogeneous contact manifold is diffeomorphic to S^5 or to the product $S^2 \times S^3$. Moreover, both S^5 and $S^2 \times S^3$ carry Sasaki-Einstein structures.

[Conti '07] classified Sasaki-Einstein 5-manifolds of cohomogeneity 1.

[Diatta '08] classified 5-dimensional Lie groups equipped with left-invariant contact structures.

Left-invariant Sasakian structures on Lie groups

We aim to classify 5-dimensional Lie groups endowed with left-invariant Sasakian structures. This is equivalent to determining all 5-dimensional Sasakian Lie algebras.

A *Sasakian structure* on a Lie algebra $\mathfrak{g}$ is a quadruple (Φ, α, ξ, g), where $\Phi \in \mathrm{End}(\mathfrak{g})$, $\alpha \in \mathfrak{g}^*$, $\xi \in \mathfrak{g}$ and g is an inner product on $\mathfrak{g}$ such that

$$\alpha(\xi) = 1, \qquad \Phi^2 = -\mathrm{I} + \xi \otimes \alpha\,, \quad g(\Phi X, \Phi Y) = g(X, Y) - \alpha(X)\alpha(Y)\,,$$
$$2g(X, \Phi Y) = \mathrm{d}\alpha(X, Y)\,, \quad N_\Phi = -\mathrm{d}\alpha \otimes \xi\,,$$

where N_Φ is defined as before. A Lie algebra equipped with a Sasakian structure will be called a *Sasakian Lie algebra.* The vector ξ will be called the *Reeb vector*.

Example

The classical example of a Sasakian Lie algebra is given by the $(2n+1)$-dimensional real Heisenberg Lie algebra $\mathfrak{h}_{2n+1}$. We recall that

$$\mathfrak{h}_{2n+1} = \mathrm{span}\{X_1, \ldots, X_n, Y_1, \ldots, Y_n, Z\},$$

$$[X_i, Y_i] = Z\,, \quad i = 1, \ldots, n;$$

in this case, a Sasakian structure is defined by

$$\Phi(X_i) = Y_i\,,\ \Phi(Y_i) = -X_i\,,\ \Phi(Z) = 0\,,\ i = 1, \ldots, n\,,$$

the inner product g is obtained by $||X_i||^2 = ||Y_i||^2 = 1/2$, $||Z|| = 1$, $\xi = Z$ and α is the dual 1-form of Z.

Fundamental property

In general for a Lie algebra $\mathfrak{g}$ with a contact structure α we can prove the following property for its center $\mathfrak{z}(\mathfrak{g})$.

$\alpha \in \mathfrak{g}^*$ is called a *contact form* if $\alpha \wedge (\mathrm{d}\,\alpha)^n \neq 0$; there always exists a unique $\xi \in \mathfrak{g}$ such that $\alpha(\xi) = 1$ and $\alpha([\xi, x]) = 0$ for all $x \in \mathfrak{g}$.

Proposition: Let $(\mathfrak{g}, \alpha)$ be a contact Lie algebra with ξ its Reeb vector and let $\mathfrak{z}(\mathfrak{g})$ be the center of $\mathfrak{g}$. Then

1. $\dim \mathfrak{z}(\mathfrak{g}) \leq 1\,;$

2. if $\dim \mathfrak{z}(\mathfrak{g}) = 1$, then $\mathfrak{z}(\mathfrak{g}) = \mathbb{R}\,\xi$.

Case with non-trivial center

Proposition: (A.-Fino-Vezzoni) Let $(\mathfrak{g}, \Phi, \alpha, \xi, g)$ be a Sasakian Lie algebra with $\mathfrak{z}(\mathfrak{g}) = \mathbb{R}\xi$. Then the quadruple $(\ker\alpha, \theta, \Phi, g)$ is a Kähler Lie algebra, where θ is the component of the Lie bracket of $\mathfrak{g}$ on $\ker\alpha$.

Corollary: Let $(\mathfrak{g}, \Phi, \alpha, \xi, g)$ be a Sasakian Lie algebra with $\mathfrak{z}(\mathfrak{g}) = \mathbb{R}\xi$. Then $\mathfrak{g}/\mathfrak{z}(\mathfrak{g})$ inherits a natural Kähler structure.

Conversely, let $(\mathfrak{h}, [\,,\,]_{\mathfrak{h}}, \omega, g)$ be a Kähler Lie algebra and set $\mathfrak{g} = \mathfrak{h} \oplus \mathbb{R}\,\xi$. Then defining

$$[X, Y] = [X, Y]_{\mathfrak{h}} - \omega(X, Y)\,\xi, \qquad [\xi, \mathfrak{h}] = 0$$

for $X, Y \in \mathfrak{h}$ we obtain a new Lie algebra $(\mathfrak{g}, [\,,\,])$ equipped with a natural Sasakian structure.

Particular case: nilpotent Lie algebras

It is known that in dimensions 3 and 5 the only nilpotent Sasakian Lie algebras are the real Heisenberg algebras $\mathfrak{h}_3$ and $\mathfrak{h}_5$, respectively ([Geiges '97] and [Ugarte '07]). We show next that this still holds in any dimension.

Proposition: (A.-Fino-Vezzoni) Let $\mathfrak{g}$ be a $(2n+1)$-dimensional nilpotent Lie algebra admitting a Sasakian structure. Then $\mathfrak{g}$ is isomorphic to the $(2n+1)$-dimensional Heisenberg Lie algebra.

Proof: Let (Φ, α, ξ, g) be a Sasakian structure on $\mathfrak{g}$. Since $\mathfrak{g}$ is nilpotent it has non-trivial center $\mathfrak{z}(\mathfrak{g}) = \mathbb{R}\,\xi$. The quotient $\mathfrak{g}/\mathfrak{z}(\mathfrak{g})$ is a Kähler nilpotent Lie algebra and, as a consequence, it is abelian. This implies immediately that $\mathfrak{g}$ is isomorphic to the Heisenberg Lie algebra.

Case with trivial center

Proposition: Let $(\mathfrak{g}, \Phi, \alpha, \xi, g)$ be a Sasakian Lie algebra. Then

1. $\mathrm{ad}_\xi \Phi = \Phi\, \mathrm{ad}_\xi$, and therefore $\ker \mathrm{ad}_\xi$ and $\mathrm{Im}\, \mathrm{ad}_\xi$ are Φ-invariant subspaces of $\mathfrak{g}$;

2. $\mathrm{ad}_\xi \Phi$ is symmetric with respect to g;

3. ad_ξ is skew-symmetric with respect to g, thus $(\mathrm{Im}\, \mathrm{ad}_\xi)^\perp = \ker \mathrm{ad}_\xi$.

Corollary: Let $(\mathfrak{g}, \Phi, \alpha, \xi, g)$ be a Sasakian Lie algebra. Then there is an orthogonal decomposition

$$\mathfrak{g} = \ker \mathrm{ad}_\xi \oplus \mathrm{Im}\ \mathrm{ad}_\xi\ .$$

Proposition: Let $(\mathfrak{g}, \Phi, \alpha, \xi, g)$ be a Sasakian Lie algebra with trivial center.

1. If $\dim \mathfrak{g} \geq 5$, then $\ker \mathrm{ad}_\xi$ is a Sasakian Lie subalgebra of $\mathfrak{g}$ with non-trivial center.

2. If $X \in \ker \mathrm{ad}_\xi, Y \in \mathrm{Im}\ \mathrm{ad}_\xi$, then $[X, Y] \in \mathrm{Im}\ \mathrm{ad}_\xi$.

With respect to the decomposition $\mathfrak{g} = \ker \mathrm{ad}_\xi \oplus \mathrm{Im}\ \mathrm{ad}_\xi$, we have

$$(\mathrm{ad}_\xi)_{|\ker\alpha} = \begin{pmatrix} 0 & 0 \\ 0 & U \end{pmatrix}, \quad \Phi_{|\ker\alpha} = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix},$$

where $U \colon \mathrm{Im}\ \mathrm{ad}_\xi \to \mathrm{Im}\ \mathrm{ad}_\xi$ is non-singular, and

$$A^2 = D^2 = -\mathrm{I} \qquad DU = UD.$$

In particular, if $\mathfrak{g}$ is solvable, then the Reeb vector ξ cannot belong to the commutator $\mathfrak{g}'$.

5-dimensional Sasakian Lie algebras with trivial center

In [Ovando '06] a classification of 4-dimensional Kähler Lie algebras was given. Using this with our previous we obtain the following result:

Theorem: (AFV) Any 5-dimensional Sasakian Lie algebra $\mathfrak{g}$ with non-trivial center is isomorphic to one of the following solvable Lie algebras (for $\delta > 0$):

$$\mathfrak{g}_1 = \left(0,0,0,0,e^{12}+e^{34}\right) \simeq \mathfrak{h}_5\,;$$

$$\mathfrak{g}_2 = \left(0,-e^{12},0,0,e^{12}+e^{34}\right) \simeq \mathfrak{aff}(\mathbb{R}) \times \mathfrak{h}_3\,;$$

$$\mathfrak{g}_3 = \left(0,-e^{13},e^{12},0,e^{14}+e^{23}\right) \simeq \mathbb{R} \ltimes (\mathfrak{h}_3 \times \mathbb{R})\,;$$

$$\mathfrak{g}_4 = \left(0,-e^{12},0,-e^{34},e^{12}+e^{34}\right) \simeq \mathfrak{aff}(\mathbb{R}) \times \mathfrak{aff}(\mathbb{R}) \times \mathbb{R}\,;$$

$$\mathfrak{g}_5 = \left(\frac{1}{2}\,e^{14},\frac{1}{2}\,e^{24},-e^{12}+e^{34},0,e^{12}-e^{34}\right) \simeq \mathbb{R} \times (\mathbb{R} \ltimes \mathfrak{h}_3)\,;$$

$$\mathfrak{g}_6 = \left(2e^{14},-e^{24},-e^{12}+e^{34},0,e^{23}\right) \simeq \mathbb{R} \ltimes \mathfrak{n}_4\;;$$

$$\mathfrak{g}_7^\delta = \left(\frac{\delta}{2}e^{14}+e^{24},-e^{14}+\frac{\delta}{2}e^{24},-e^{12}+\delta e^{34},0,e^{12}-\delta e^{34}\right) \simeq \mathbb{R} \times (\mathbb{R} \ltimes \mathfrak{h}_3);$$

$$\mathfrak{g}_8^\delta = \left(e^{14},\delta\,e^{34},-\delta\,e^{24},0,e^{14}+e^{23}\right) \simeq \mathbb{R} \ltimes (\mathfrak{h}_3 \times \mathbb{R}).$$

Corollary: A unimodular Sasakian Lie algebra with non-trivial center is isomorphic either to the nilpotent Heisenberg Lie algebra $\mathfrak{h}_5$ or the solvable Lie algebra $\mathfrak{g}_3$. The simply connected Lie group G_3 with Lie algebra $\mathfrak{g}_3$ admits a co-compact discrete subgroup Γ.

The group G_3 is isomorphic to $\mathbb{R}^5$ with a certain product, and it can be checked that the subset

$$\Gamma = \left\{ \left(2\pi m_1, m_2, m_3, m_4, \frac{1}{2\pi} m_5 \right) \mid m_i \in \mathbb{Z} \right\}$$

is a discrete subgroup that acts freely and properly discontinuously on G_3. Moreover, the quotient manifold $\Gamma \backslash G_3$ is compact.

The solvmanifold $\Gamma \backslash G_3$ is by construction the total space of an S^1-bundle over a 4-dimensional non-completely solvable Kähler solvmanifold (this Kähler solvmanifold was found by Hasegawa in 2006).

5-dimensional Sasakian Lie algebras with trivial center

Let $(\mathfrak{g}, \Phi, \alpha, \xi, g)$ be a 5-dimensional Sasakian Lie algebra with trivial center.

First, if $\mathfrak{g}' = \mathfrak{g}$ then the only contact Lie algebra is $\mathfrak{sl}(2, \mathbb{R}) \ltimes \mathbb{R}^2$, according to [Diatta '08]. However, we can prove the following

Proposition: The Lie algebra $\mathfrak{sl}(2, \mathbb{R}) \ltimes \mathbb{R}^2$ does not admit any Sasakian structure.

Now we can consider the case of 5-dimensional Sasakian Lie algebras with trivial center and such that $\mathfrak{g}' \neq \mathfrak{g}$. In this case

$$\dim \ker(\mathrm{ad}_\xi)_{|\ker \alpha} = \dim \mathrm{Im}(\mathrm{ad}_\xi) = 2\,.$$

There exists an orthonormal basis $\{e_1, \dots, e_4\}$ of $\ker \alpha$ with respect to which $\Phi_{| \ker \alpha}$ can be written as

$$\Phi_{| \ker \alpha} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

and $\ker \mathrm{ad}_\xi = \text{span}\{\xi, e_1, e_2\}$, $\mathrm{Im}\, \mathrm{ad}_\xi = \text{span}\{e_3, e_4\}$. Moreover in this basis

$$(\mathrm{ad}_\xi)_{| \ker \alpha} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -b \\ 0 & 0 & b & 0 \end{pmatrix}.$$

Note that in terms of $\{e_1, \dots, e_4\}$ the 2-form $\mathrm{d}\alpha$ takes the standard form $\mathrm{d}\alpha = 2\,(e^{12} + e^{34})$.

Set $e_5 = \xi$ and denote by $\{e^1, \dots, e^5\}$ the dual basis of $\{e_1, \dots, e_5\}$. Since $b \neq 0$, we may assume $b = \pm 1$.

Case A: $b = 1$. The Maurer-Cartan equations are given by

$$\begin{aligned}
\mathrm{d}e^1 &= a_1\, e^{12} + a_6\, e^{34}\,, \\
\mathrm{d}e^2 &= b_1 e^{12} + b_6\, e^{34}\,, \\
\mathrm{d}e^3 &= -e^{45} + c_2\, e^{13} + c_3\, e^{14} + c_4\, e^{23} + c_5\, e^{24}\,, \\
\mathrm{d}e^4 &= e^{35} + f_2\, e^{13} + f_3\, e^{14} + f_4\, e^{23} + f_5\, e^{24}\,, \\
\mathrm{d}e^5 &= 2(e^{12} + e^{34})\,.
\end{aligned} \tag{1}$$

Case B: $b=-1$. The Maurer-Cartan equations are given by

$$\begin{aligned} de^1 &= a_1\, e^{12} + a_6\, e^{34}\,, \\ de^2 &= b_1 e^{12} + b_6\, e^{34}\,, \\ de^3 &= e^{45} + c_2\, e^{13} + c_3\, e^{14} + c_4\, e^{23} + c_5\, e^{24}\,, \\ de^4 &= -e^{35} + f_2\, e^{13} + f_3\, e^{14} + f_4\, e^{23} + f_5\, e^{24}\,, \\ de^5 &= 2(e^{12} + e^{34})\,. \end{aligned} \tag{2}$$

Now, imposing the conditions $d^2 = 0$ and $N_\Phi = -de^5 \otimes e_5$, we obtain in each case a system of equations, whose solutions give rise to the Lie algebras with trivial center admitting Sasakian structures.

Theorem: (AFV) If a 5-dimensional Sasakian Lie algebra $\mathfrak{g}$ has trivial center, then it is isomorphic to one of the following Lie algebras:

- the direct product $\mathfrak{sl}(2,\mathbb{R}) \times \mathfrak{aff}(\mathbb{R})$,

- the direct product $\mathfrak{su}(2) \times \mathfrak{aff}(\mathbb{R})$,

- the solvable (non-unimodular) Lie algebra $\mathfrak{g}_0$, with Lie bracket given by

$$\begin{aligned} &[e_1, e_3] = e_3\,, \quad [e_1, e_4] = \tfrac{1}{2}e_4\,, \quad [e_1, e_5] = \tfrac{1}{2}e_5, \\ &[e_2, e_4] = e_5\,, \quad [e_2, e_5] = -e_4, \quad [e_4, e_5] = -e_3, \end{aligned}$$

5-dimensional Sasaki α-Einstein Lie algebras

In [Diatta '08] it was proved that no left-invariant Sasakian structure on a Lie group can be Sasaki-Einstein. Thus, we will look for left-invariant Sasaki α-Einstein structures.

When the Ricci tensor of a Sasakian manifold $(M, \Phi, \alpha, \xi, g)$ satisfies the equation $\mathrm{Ric}_g = \lambda g + \nu\, \alpha \otimes \alpha$ for some constants λ, $\nu \in \mathbb{R}$, the Sasakian structure is called α-*Einstein*.

Sasaki α-Einstein metrics are natural analogues of Kähler-Einstein metrics.

Theorem: (Boyer-Galicki-Matzeu '99) Every Sasaki α-Einstein manifold is of constant scalar curvature equal to $s = 2n(\lambda + 1)$, and $\lambda + \nu = 2n$.

A Sasakian Lie algebra $(\mathfrak{g}, \Phi, \alpha, \xi, g)$ is called α-*Einstein* if the Ricci tensor Ric_g of the metric g satisfies $\mathrm{Ric}_g = \lambda g + \nu\, \alpha \otimes \alpha$ for some λ, $\nu \in \mathbb{R}$.

Some known facts in 5 dimensions:

• The canonical Sasakian structure on $\mathfrak{h}_5$ is α-Einstein [Tomassini-Vezzoni '08].

• The Lie algebra $\mathfrak{g}_0$ from the previous theorem is the only solvable (non nilpotent) 5-dimensional Lie algebra admitting a Sasaki α-Einstein structure [de Andrés-Fernández-Fino-Ugarte '08].

Thus, we only have to consider the non-solvable Sasakian Lie algebras, which are $\mathfrak{sl}(2, \mathbb{R}) \times \mathfrak{aff}(\mathbb{R})$ and $\mathfrak{su}(2) \times \mathfrak{aff}(\mathbb{R})$.

Proposition: The Lie algebra $\mathfrak{sl}(2, \mathbb{R}) \times \mathfrak{aff}(\mathbb{R})$ admits Sasaki α-Einstein structures, while no Sasakian structure on $\mathfrak{su}(2) \times \mathfrak{aff}(\mathbb{R})$ is α-Einstein.

To sum up, we can now state the following

Theorem: (AFV) The only 5-dimensional Lie algebras admitting a Sasaki α-Einstein structure are $\mathfrak{h}_5$, $\mathfrak{g}_0$ and $\mathfrak{sl}(2, \mathbb{R}) \times \mathfrak{aff}(\mathbb{R})$.

Corollary: The nilmanifolds $\Gamma \backslash H_5$ are the only compact manifolds of the form $\Gamma \backslash G$ (with G a simply connected Lie group and $\Gamma \subset G$ a lattice) which admit invariant Sasaki α-Einstein structures.