Classification of abelian complex structures on 6-dimensional Lie algebras

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Plan

- Basic definitions.
- A motivating example.
- Relation to HKT geometry.
- Generalities on abelian complex structures.
- Affine Lie algebras and their standard complex structure.
- The 4-dimensional case.
- Outline of the classification in dimension 6.

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Basic definitions

- A complex structure on a real Lie algebra $\mathfrak{g}$ is $J \in \operatorname{End}(\mathfrak{g})$ satisfying:

$$J^2 = -I, \qquad J[x,y] - [Jx,y] - [x,Jy] - J[Jx,Jy] = 0, \quad (1)$$

for any $x, y \in \mathfrak{g}$.

- Complex Lie algebras are those for which J is bi-invariant:

$$J[x,y] = [x,Jy], \quad \forall\, x,y \in \mathfrak{g}. \qquad (2)$$

- A complex structure J on $\mathfrak{g}$ is called *abelian* when it satisfies:

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- Two complex structures J_1 and J_2 on $\mathfrak{g}$ are said to be *equivalent* if there exists $\alpha \in \mathrm{Aut}(\mathfrak{g})$ satisfying:

$$J_2\,\alpha = \alpha\, J_1.$$

- Two pairs $(\mathfrak{g}_1, J_1)$ and $(\mathfrak{g}_2, J_2)$ are *holomorphically isomorphic* if there exists a Lie algebra isomorphism $\alpha : \mathfrak{g}_1 \to \mathfrak{g}_2$ such that:

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- Given a complex structure J on $\mathfrak{g}$, set $\boxed{\mathfrak{g}'_J := \mathfrak{g}' + J\mathfrak{g}'}$. We will say that J is *proper* when

$$\mathfrak{g}'_J \subsetneq \mathfrak{g}.$$

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A motivating example: $\mathfrak{aff}(\mathbb{C})$

$$\mathfrak{aff}(\mathbb{C}) = \left\{ \begin{pmatrix} a & -b & c & -d \\ b & a & d & c \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} : a, b, c, d \in \mathbb{R} \right\}$$

$\mathfrak{aff}(\mathbb{C})$ has a basis $\{e_1, e_2, e_3, e_4\}$ with Lie brackets:

$$[e_1, e_3] = e_3, \quad [e_1, e_4] = e_4, \quad [e_2, e_3] = e_4, \quad [e_2, e_4] = -e_3$$

- There are two abelian complex structures on $\mathfrak{aff}(\mathbb{C})$ up to equivalence:

$$J_1 = \begin{pmatrix} 0 & 1 & & \\ -1 & 0 & & \\ & & 0 & -1 \\ & & 1 & 0 \end{pmatrix} \qquad J_2 = \begin{pmatrix} & & -1 & 0 \\ & & 0 & -1 \\ 1 & 0 & & \\ 0 & 1 & & \end{pmatrix}$$

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- J_1 anticommutes with J_2.
- For $x = (x_1, x_2, x_3) \in S^2$,

$$J_x := x_1 J_1 + x_2 J_2 + x_3 J_1 J_2$$

  is an abelian complex structure on $\mathfrak{aff}(\mathbb{C})$.
- $J_x \sim J_1 \quad \text{for} \quad x = (\pm 1, 0, 0)$.
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Relation to HKT geometry

- A *hyperhermitian* structure on a smooth manifold M is $(\{J_\alpha\}_{\alpha=1,2,3}, g)$, where
  1. $\{J_\alpha\}_{\alpha=1,2,3}$ are complex structures such that
  $$J_1 J_2 = -J_2 J_1 = J_3,$$
  2. g is a Riemannian metric which is Hermitian with respect to $J_\alpha,\ \alpha = 1, 2, 3.$

- Given a hyperhermitian structure $(\{J_\alpha\}_{\alpha=1,2,3}, g)$ on M, g is called *hyper-Kähler with torsion* (HKT) if there exists a connection ∇ on M satisfying
  1. $\nabla g = 0, \quad \nabla J_\alpha = 0, \alpha = 1, 2, 3,$
  2. the torsion tensor $c(X, Y, Z) = g(X, T(Y, Z))$ is skew-symmetric.

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Relation to HKT geometry

This class of metrics has been introduced by P.S. Howe - G.Papadopoulos (1996).

- A left invariant hyperhermitian metric on a Lie group G is HKT if and only if

$$g([J_1x, J_1y], z) + g([J_1y, J_1z], x) + g([J_1z, J_1x], y)$$
$$= g([J_2x, J_2y], z) + g([J_2y, J_2z], x) + g([J_2z, J_2x], y)$$
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  for all $x, y, z \in \mathfrak{g}$, the Lie algebra of G.

- Given an abelian hypercomplex structure, any hyperhermitian metric is HKT.

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Relation to HKT geometry

Theorem (Dotti - Fino, 2002)

If G is a 2-step nilpotent Lie group with a left invariant HKT structure $(\{J_\alpha\}_{\alpha=1,2,3}, g)$, then the hypercomplex structure is abelian.

- **Question.** Does the above result hold for any nilpotent Lie group?

Theorem (B - I. Dotti - M. Verbitsky, 2007)

Let $(N, \{J_\alpha\}_{\alpha=1,2,3}, g)$ be an HKT nilmanifold such that $\{J_\alpha\}$ is left invariant. Then the hypercomplex structure $\{J_\alpha\}$ is abelian.

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Abelian complex structures

An abelian complex structure J satisfies:

- $(1,0)$-vectors in $\mathfrak{g}^{\mathbb{C}}$ commute;
- The center $\mathfrak{z}$ of $\mathfrak{g}$ is J-stable;
- For any $x \in \mathfrak{g}$, $\mathrm{ad}_{Jx} = -\mathrm{ad}_x J$.

Examples.

1. Let $\mathfrak{h}_{2n+1} = \mathrm{span}\{e_1, \dots, e_{2n}, z_0\}$ be the Heisenberg algebra:
$$[e_{2i-1}, e_{2i}] = z_0, \quad 1 \le i \le n,$$
and $\{z_1, \dots, z_{2k+1}\}$ a basis of $\mathbb{R}^{2k+1}$. An abelian complex structure on $\mathfrak{h}_{2n+1} \times \mathbb{R}^{2k+1}$ is given by:
$$Je_{2i-1} = \pm e_{2i}, \qquad Jz_{2j} = z_{2j+1}, \quad 1 \le i \le n, \ 0 \le j \le k.$$
2. Let $\mathfrak{aff}(\mathbb{R}) = \mathrm{span}\{e_1, e_2\}$ with Lie bracket: $[e_1, e_2] = e_2$. It has a unique abelian complex structure up to equiv.:
$$Je_1 = e_2.$$

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A general result

Proposition

If $\mathfrak{g}$ is an even dimensional real Lie algebra with 1-dimensional commutator $\mathfrak{g}'$, then:

1. $\mathfrak{g}$ *is isomorphic to either* $\mathfrak{h}_{2n+1} \times \mathbb{R}^{2k+1}$ *or* $\mathfrak{aff}(\mathbb{R}) \times \mathbb{R}^{2k}$;
2. *All these Lie algebras carry abelian complex structures and every complex structure on* $\mathfrak{g}$ *is abelian;*
3. *There are* $\left[\frac{n}{2}\right] + 1$ *equivalence classes of complex structures on* $\mathfrak{h}_{2n+1} \times \mathbb{R}^{2k+1}$;
4. *There is a unique complex structure on* $\mathfrak{aff}(\mathbb{R}) \times \mathbb{R}^{2k}$ *up to equivalence.*

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If $\mathfrak{g}$ is an even dimensional real Lie algebra with 1-dimensional commutator $\mathfrak{g}'$, then:

1. *$\mathfrak{g}$ is isomorphic to either $\mathfrak{h}_{2n+1} \times \mathbb{R}^{2k+1}$ or $\mathfrak{aff}(\mathbb{R}) \times \mathbb{R}^{2k}$;*
2. *All these Lie algebras carry abelian complex structures and every complex structure on $\mathfrak{g}$ is abelian;*
3. *There are $\left[\frac{n}{2}\right] + 1$ equivalence classes of complex structures on $\mathfrak{h}_{2n+1} \times \mathbb{R}^{2k+1}$;*
4. *There is a unique complex structure on $\mathfrak{aff}(\mathbb{R}) \times \mathbb{R}^{2k}$ up to equivalence.*

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Obstructions

- Petravchuk (1988): If $\mathfrak{g}$ is a real Lie algebra admitting an abelian complex structure, then $\mathfrak{g}$ is 2-step solvable.

- B - Dotti (2004): If $\mathfrak{g}$ is solvable, $\operatorname{codim} \mathfrak{g}' = 1$ and $\dim \mathfrak{g} > 2$, then $\mathfrak{g}$ does not admit abelian complex structures.

- If $\mathfrak{g}$ is k-step nilpotent with an abelian complex structure J, set $\boxed{\mathfrak{g}_J^i := \mathfrak{g}^i + J\mathfrak{g}^i}$. Then

$$\mathfrak{g}_J^i \subsetneq \mathfrak{g}_J^{i-1} \qquad \text{for all } i \leq k.$$

In particular, if $\dim \mathfrak{g} = 2m$, $\mathfrak{g}$ is at most m-step nilpotent.

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Affine Lie algebras

- Let $(A, \cdot)$ be a finite dimensional associative, commutative algebra. Set $\mathfrak{aff}(A) := A \oplus A$ with Lie bracket:

$$[(a, a'), (b, b')] = (0, a \cdot b' - b \cdot a'), \qquad a, b, a', b' \in A,$$

In particular, when $A = \mathbb{R}$ or $A = \mathbb{C}$, we obtain the Lie algebra of the group of affine motions of either $\mathbb{R}$ or $\mathbb{C}$.

- Let J be the endomorphism of $\mathfrak{aff}(A)$ defined by

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The four dimensional case

Theorem (J.E. Snow, 1990)

Let $\mathfrak{g}$ be a 4-dimensional Lie algebra admitting an abelian complex structure. Then $\mathfrak{g}$ is isomorphic to $\mathfrak{aff}(A_i)$ for some $1 \leq i \leq 6$, where A_i are given by:

$$A_1 = \left\{ \begin{pmatrix} 0 & a & & \\ 0 & 0 & & \\ & & 0 & b \\ & & 0 & 0 \end{pmatrix} \right\}, \qquad A_2 = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & a \\ 0 & 0 & 0 \end{pmatrix} \right\},$$

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The 6-dimensional case

Proposition

If $\dim \mathfrak{s} = 6$ *and* J *is an abelian complex structure on* $\mathfrak{s}$ *such that* $\mathfrak{s}'_J$ *is nilpotent, then* $\mathfrak{s}'_J$ *is abelian.*

To carry out the classification, we consider separately the following cases:

1. $\mathfrak{s}$ is nilpotent,
2. $\mathfrak{s}$ is not nilpotent and J is proper,
3. $\mathfrak{s}$ is not nilpotent and J is not proper.

- We start by classifying the 6-dim. nilpotent Lie algebras carrying abelian complex structures.
- This can also be obtained as a consequence of results of Salamon (2001) and Cordero - Fernández - Ugarte (2002).

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The nilpotent case

Theorem

Let $\mathfrak{n}$ be a non-abelian 6-dimensional nilpotent Lie algebra with an abelian complex structure J. Then $\mathfrak{n}$ is isomorphic to one (and only one) of the following Lie algebras:

$$\mathfrak{n}_1 := \mathfrak{h}_3 \times \mathbb{R}^3,$$

$$\mathfrak{n}_2 := \mathfrak{h}_5 \times \mathbb{R},$$

$$\mathfrak{n}_3 := \mathfrak{h}_3 \times \mathfrak{h}_3,$$

$$\mathfrak{n}_4 := \mathfrak{h}_3(\mathbb{C}),$$

$$\mathfrak{n}_5 : \ [e_1, e_2] = e_5, \quad [e_1, e_4] = [e_2, e_3] = e_6,$$

$$\mathfrak{n}_6 : \ [e_1, e_2] = e_5, \quad [e_1, e_4] = [e_2, e_5] = e_6,$$

$$\mathfrak{n}_7 : \ [e_1, e_2] = e_4, \quad [e_1, e_3] = -[e_2, e_4] = e_5, \quad [e_1, e_4] = [e_2, e_3] = e_6.$$

Idea of proof

$\mathfrak{n}$ is k-step nilpotent with $k = 2$ or 3.

- If $k = 2$, then:

$$\mathfrak{n} \cong \begin{cases} \mathfrak{n}_1 \text{ or } \mathfrak{n}_2, & \text{if } \dim \mathfrak{n}' = 1 \\ \mathfrak{n}_3, \mathfrak{n}_4 \text{ or } \mathfrak{n}_5, & \text{if } \dim \mathfrak{n}' = 2 \end{cases}$$

- If $k = 3$, we obtain:

$$\mathfrak{n} \cong \begin{cases} \mathfrak{n}_6, & \text{if } \dim \mathfrak{n}^2 = 1 \\ \mathfrak{n}_7, & \text{if } \dim \mathfrak{n}^2 = 2 \end{cases}$$

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Equivalence classes of abelian complex structures

$\mathcal{C}_a(\mathfrak{n}) := \{$ abelian complex structures on $\mathfrak{n}\}$

$\mathcal{C}_a(\mathfrak{n})/\mathrm{Aut}\,(\mathfrak{n}) =$ moduli space of abelian complex structures on $\mathfrak{n}$.

Theorem (A-B-D, 2009)

- *The Lie algebras $\mathfrak{n}_1, \mathfrak{n}_5$ and $\mathfrak{n}_6$ have a unique abelian complex structure up to equivalence.*
- *The Lie algebra $\mathfrak{n}_2$ has two abelian complex structures up to equivalence.*
- *The moduli space of abelian complex structures on $\mathfrak{n}_3$ is homeomorphic to $\mathbb{R}$.*
- *The moduli space of abelian complex structures on $\mathfrak{n}_4$ is homeomorphic to $(0,1] \times \mathbb{Z}_2$.*
- *The moduli space of abelian complex structures on $\mathfrak{n}_7$ is homeomorphic to $[-1,0) \cup (0,1]$.*

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The Lie algebra $\mathfrak{n}_3 = \mathfrak{h}_3 \times \mathfrak{h}_3$

$$[e_1, e_2] = e_5, \qquad [e_3, e_4] = e_6$$

$$\mathcal{C}_a(\mathfrak{n}_3) = \left\{ \begin{pmatrix} 0 & -1 & & & & \\ 1 & 0 & & & & \\ & & 0 & -1 & & \\ & & 1 & 0 & & \\ & & & & s & (-s^2-1)/t \\ & & & & t & -s \end{pmatrix} : t \neq 0 \right\}$$

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$$[e_1, e_2] = e_4, \quad [e_1, e_3] = -[e_2, e_4] = e_5, \quad [e_1, e_4] = [e_2, e_3] = e_6$$

$$\mathcal{C}_a(\mathfrak{n}_7) = \left\{ \begin{pmatrix} 0 & -1 & & & & \\ 1 & 0 & & & & \\ & & 0 & 1 & & \\ & & -1 & 0 & & \\ & & & & s & (-s^2-1)/t \\ & & & & t & -s \end{pmatrix} : t \neq 0 \right\}$$

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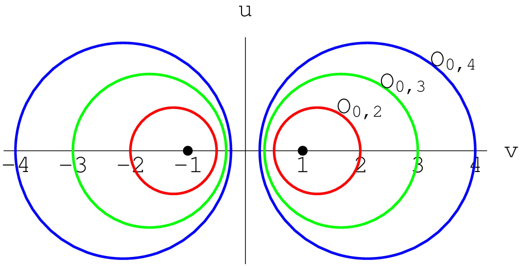
Orbits in $\mathcal{C}_a(\mathfrak{n}_7)$



For $t_0 \neq 0, \pm 1$:

$$O_{(0,t_0)} = \left\{ (u,v) : \; u^2 + \left(v - \frac{c}{2}\right)^2 = \left(\frac{c}{2}\right)^2 - 1 \right\} = F^{-1}(c),$$

where $F(u,v) = v + \dfrac{1+u^2}{v}$ and $c = t_0 + \dfrac{1}{t_0}$.

$$O_{(0,-1)} = \{(0,-1)\}, \qquad O_{(0,1)} = \{(0,1)\}.$$

The Lie algebra $\mathfrak{n}_4 = \mathfrak{h}_3(\mathbb{C})$

$$[e_1, e_3] = -[e_2, e_4] = e_5, \qquad [e_1, e_4] = [e_2, e_3] = e_6$$

$$\mathcal{C}_a(\mathfrak{n}_4) = \left\{ \begin{pmatrix} J_k & & \\ & s & (-s^2-1)/t \\ & t & -s \end{pmatrix} : k = 1 \text{ or } 2,\ t \neq 0 \right\}$$

where

$$J_1 = \begin{pmatrix} & & -1 & 0 \\ & & 0 & -1 \\ 1 & 0 & & \\ 0 & 1 & & \end{pmatrix}, \quad J_2 = \begin{pmatrix} 0 & -1 & & \\ 1 & 0 & & \\ & & 0 & 1 \\ & & -1 & 0 \end{pmatrix}$$

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Non-nilpotent $\mathfrak{s}$, proper J

- If $\dim \mathfrak{s}'_J = 2$, or
- $\dim \mathfrak{s}'_J = 4$ and $\mathfrak{s}'_J$ is non-abelian,

$\boxed{\text{then } (\mathfrak{s}, J) \text{ is decomposable}}$.

- If $\mathfrak{s}'_J = \mathbb{R}^4$, we obtain:

1. A non-standard complex structure on $\mathfrak{aff}(\mathbb{C}) \times \mathbb{R}^2$.
2. Two Lie algebras $\mathfrak{s}_1$, $\mathfrak{s}_2$: $\mathfrak{s}_1$ has two non-equivalent structures and $\mathfrak{s}_2$ has a unique structure.
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Theorem (A-B-D, 2009)

Let $\mathfrak{s}$ be a 6-dimensional Lie algebra with a non-proper abelian complex structure J. Then $\dim \mathfrak{s}' = 3$ and $(\mathfrak{s}, J)$ is holomorphically isomorphic to $\mathfrak{aff}(A)$ with its standard complex structure, where A is a 3-dimensional commutative associative algebra such that $A^2 = A$. $A = A_i$ for some $1 \leq i \leq 5$, where

$$A_1 = \left\{ \begin{pmatrix} a & & \\ & b & \\ & & c \end{pmatrix} \right\}, \qquad A_2 = \left\{ \begin{pmatrix} a & & \\ & b & -c \\ & c & b \end{pmatrix} \right\},$$

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