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study: kernel and cokernel

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$s_n$ - integral version of $n!\mathbf{ch}_n$

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2. $R(\hat{c}_i(\mathbf{V})) = c_i(\nabla^V) \in \Omega^{2i}(B)$ - Chern-Weil representative
3. $\hat{c}_i(\mathbf{V}') - \hat{c}_i(\mathbf{V}) = a(\tilde{c}_i(\nabla^{V'}, \nabla^V))$ - transgression formula

# Geometric aspects of index theory

Assume: $\mathtt{ker}(D(\mathcal{E}))$ is a vector bundle

geometry of $\mathcal{E}$ induces metric and connection on $\mathtt{ker}(D(\mathcal{E}))$

Question: Calculate $\mathtt{ker}(D(\mathcal{E}))$ as a geometric bundle or at least some invariants!

Invariants of geometric vector bundles $\mathbf{V}$:

Cheeger-Simons Chern classes

$\hat{c}_i(\mathbf{V}) \in \widehat{H\mathbb{Z}}^{2i}(B;\mathbb{Z})$ - differential integral cohomology :

Rational version : $\widehat{\mathbf{ch}}(\mathbf{V}) \in \widehat{H\mathbb{Q}}^{ev}(B)$

# differentiable $K$-theory formalism

joint with Thomas Schick

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introduce differential $K$-theory $\hat{K}(B)$

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many other models: Hopkins-Freed, Freed-Morgan, Hopkins-Singer (2005), Ortiz (2009)

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1. technical work $\rightsquigarrow$ drop assumption that $\mathtt{ker}(D(\mathcal{E}))$ is vector bundle
2. exists unique differential lift of the Chern character
$$\widehat{\mathbf{ch}} : \hat{K}^0(B) \to \widehat{H\mathbb{Q}}^{ev}(B)$$
3. $T^v\pi$ - *spin*$^c$ $\rightsquigarrow$ write $\mathcal{V} = S^c(T^v\pi) \otimes \mathbf{F}$, **F** - twisting bundle
$$\widehat{\mathbf{ch}}([\mathcal{E}]) = \int_{E/B} \hat{\hat{\mathbf{A}}}(\nabla^{T^v\pi}) \cup \widehat{\mathbf{ch}}(\mathbf{F})$$

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Freed-Lott (09) - define $\widehat{\mathtt{index}}^{top}$ and show

$$\widehat{\mathtt{index}}^{an} = \widehat{\mathtt{index}}^{top}$$

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This has geometric content! Differential version of Freed's theorem.

# categorial aspect

Describe $\texttt{ker}(D(\mathcal{E}))$ as an object in a category of vector bundles over $B$, e.g.:

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objects - families, morphisms - bordisms

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this can be done, but for interesting applications need $\infty$-categorial version
1-morphisms $\rightsquigarrow$ secondary invariants (Chern-Simons, Adams' $e$ )
2-morphisms $\rightsquigarrow$ tertiary invariants (higher torsion, Laures $f$)

$\vdots$

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String theory needs trivializations of $\mathtt{det}(\mathcal{E})$ (or of $\mathtt{Pfaff}(\mathcal{E})$)

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Categorial version does not help much:

reason: want to trivialize $\mathtt{det}(\mathcal{E})$ but $\mathcal{E}$ is not trivial (e.g. zero bordant).

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string structure $\rightsquigarrow$ trivialization

# topological string structures

Stolz/Teichner: string structure is lift to homotopy fibre

$$\begin{array}{ccccc} & & BString(n) & & \\ & \nearrow & \downarrow & & \\ B & \xrightarrow{V} & BSpin(n) & \xrightarrow{\frac{p_1}{2}} & K(\mathbb{Z},4) \end{array}$$

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(refines the notion of a topological string structure in the same way as a geometric vector bundle refines the notion of an isomorphism class of a vector bundle)

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(B 09): A geometric string structure *str* on **V** induces a natural unit-norm section $s_{str}$ of $\mathtt{Pfaff}(\mathcal{E} \otimes_{\mathbb{R}} \mathbf{V})$ with

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$H_{str}$ - canonical 3-form of the string structure

$$dH_{str} = \frac{p_1}{2}(\nabla^V) \ .$$